NAME: ....

ABBOTSLEIGH

4 UNIT MATHEMATICS

HSC ASSESSMENT # 2

March 1999

Time allowed: 1½ hours

Instructions to candidates

Attempt all questions.

Show all necessary working.

Approved calculators may be used.

Begin each question on a new page.

All questions are of equal value.

Question One (Start a new page) **Marks**

(a) Derive the equations of the tangent and normal at P(2,1) on the ellipse $4x^2 + 9y^2 = 25$. **3**

(b) Sketch the graph of $\frac{x^2}{16}-\frac{y^2}{9}=1$, showing clearly the foci, directrices, asymptotes and vertices. **3**

(c) In the expansion of $(1-2x)(1+ax)^{10}$ the coefficient of x^6 is 0. Find the value of a. **3**

(d) (i) Evaluate $\int_0^1 \frac{dx}{(x+1)(x+3)}$ **6**

(ii) Evaluate $\int_{-1}^{2} x\sqrt{2-x}\,dx$ using $u^2 = 2 - x$, for $u > 0$

Question Two (Start a new page)

(a) Assume $(3+2x)^{15} = \sum_{k=0}^{15} t_k x^k$ **6**

(i) Use the Binomial Theorem to write an expression for t_k, $0 \le k \le 15$

(ii) Show that $\frac{t_{k+1}}{t_k} = \frac{2(15-k)}{3(k+1)}$

(iii) Hence, or otherwise, find the largest coefficient t_k (you may leave your answer in the form ${}^{15}C_k 3^a 2^b$)

(b) (i) Sketch the graphs of $y = x^2$ and $y = (x - 2)^2$ on the same diagram. **3**

(ii) Find the area enclosed between the curves $y = x^2$ and $y = (x - 2)^2$ and the x-axis.

(c) Find the exact volume of the solid formed if the curve $y=\sqrt[4]{3x-1}$ is rotated about the x-axis from $x = 1$ to $x = 2$. **3**

(d) Find $\int \sin^2 x \cos^2 x\,dx$ **3**

Question Three (Start a new page) **Marks**

(a) $P(2at, at^2)$ is a point on the parabola $x^2 = 4ay$. S is the focus of the parabola. PQ is the perpendicular from P to the directrix d of the parabola. The tangent at P to the parabola cuts the axis of the parabola at the point R **9**

(i) Show that the tangent at the point P to the parabola has equation $tx - y - at^2 = 0$

(ii) Show that PR and QS bisect each other.

(iii) Show that $PR \perp QS$

(iv) State with reasons what type of quadrilateral PQRS is.

(b) (i) Find $\int \tan^4 x dx$ **6**

(ii) Use the method of integration by parts to find a recurrence formula for $I_n = \int \sec^n x dx$

Question Four (Start a new page)

(a) (i) Derive the equation of the tangent at the point $P(a\sec\theta, b\tan\theta)$ on the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$. **6**

(ii) The tangent at P meets the axes at Q and S. If OQRS is a rectangle, find the equation for the locus of R

(b) The points $P(2p, \frac{2}{p})$ and $Q(2q, \frac{2}{q})$ lie on the rectangular hyperbola $xy = 4$. **9**

M is the midpoint of the chord PQ. P and Q move on the rectangular hyperbola so that the chord PQ always passes through the point R(4, 2).

(i) Show that the chord PQ has equation $x + pqy = 2(p+q)$

(ii) Show that $pq = p + q - 2$

(iii) Hence show that the locus of M has equation $y = \frac{x}{x-2}$

(iv) On the same axes sketch the rectangular hyperbola $xy = 4$ and the locus of M, showing clearly the equations of any asymptotes and the point R.

End of paper

STANDARD INTEGRALS

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x}\,dx = \ln x, \quad x > 0$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax}, \quad a \neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax, \quad a \neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax, \quad a \neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax\,dx = \frac{1}{a}\sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a}, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \ln\left(x+\sqrt{x^2-a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \ln\left(x+\sqrt{x^2+a^2}\right)$$

NOTE : $\ln x = \log_e x, \quad x > 0$

Solutions: Assess # 1999 Abbotsleigh

Question One (15 marks)

(a). $4x^2+9y^2=25$

Differentiate w.r.t x

$8x+18y\frac{dy}{dx}=0$

$\frac{dy}{dx}=\frac{-8x}{18y}$ ✓

At $(2,1)$ $m=-\frac{8}{9}$

∴ Eqn of tangent is

$y-1=-\frac{8}{9}(x-2)$ ✓

or $8x+9y=25$

Eqn of normal is

$y-1=\frac{9}{8}(x-2)$ ✓

or $9x-8y=10$

(b). $\frac{x^2}{16}-\frac{y^2}{9}=1$ $(b^2=a^2(e^2-1))$

$a=4$, $b=3$, $e=\frac{5}{4}$

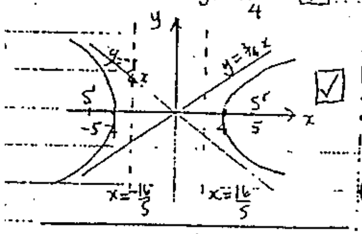
Foci are $(\pm ae,0)=(\pm5,0)$ ½

Directrices $x=\pm\frac{a}{e}=\pm\frac{4}{5/4}=\pm\frac{16}{5}$ ½

Vertices $(\pm4,0)$ ½

Asymptotes $y=\pm\frac{b}{a}x$

∴ $y=\pm\frac{3}{4}x$ ½



(c). $(1-2x)(1+ax)^{10}$

$=(1-2x)(1+\ldots+{}^{10}C_5(ax)^5+{}^{10}C_6(ax)^6+\ldots+(ax)^{10})$ ✓

∴ coeff $x^6={}^{10}C_6a^6-2\,{}^{10}C_5a^5$

∴ ${}^{10}C_6a^6-2\,{}^{10}C_5a^5=0$ ✓

∴ ${}^{10}C_6a^6=2\,{}^{10}C_5a^5$ ✓

$210\cdot a^6=2\cdot252\cdot a^5$

$a=\frac{504}{210}=\frac{12}{5}$ ✓ $2\cdot4$ $2\frac{2}{5}$

(d) (i) $\int_0^1\frac{dx}{(x+1)(x+3)}$

Use method of partial fractions

$\frac{1}{(x+1)(x+3)}=\frac{A}{x+1}+\frac{B}{x+3}$

$1=A(x+3)+B(x+1)$

When $x=-1$ $A=\frac{1}{2}$ ½

When $x=-3$ $B=-\frac{1}{2}$ ½

$\therefore I=\frac{1}{2}\int_0^1\frac{dx}{x+1}-\frac{1}{2}\int_0^1\frac{dx}{x+3}$

$=\frac{1}{2}\left[\ln(x+1)-\ln(x+3)\right]_0^1$ ✓

$=\frac{1}{2}\left[\ln2-\ln4-\ln1+\ln3\right]$

$=\frac{1}{2}\ln\frac{3}{2}$ $\ln(\frac{3}{2})^{\frac{1}{2}}$ ✓

d(ii) $\int_{-1}^{2}x\sqrt{2-x}\,dx$, $u^2=2-x$

when $x=2$, $u=0$ When $x=-1$, $u=\sqrt3$

$I=\int_{\sqrt3}^{0}(2-u^2)u\cdot-2u\,du$ $2u\frac{du}{dx}=-1$

$=\int_{\sqrt3}^{0}-4u^2+2u^4\,du$ ✓

$=\left[-\frac{4u^3}{3}+\frac{2u^5}{5}\right]_{\sqrt3}^{0}$ ✓

$=\frac{4}{3}\cdot3\sqrt3-\frac{2}{5}9\sqrt3$

$=4\sqrt3-\frac{18}{5}\sqrt3$ ✓

$=\frac{2\sqrt3}{5}$

Question Two (15 marks)

(a) $(3+2x)^{15}=\sum_{k=0}^{15}t_kx^k$

(i) $(3+2x)^{15}=\sum_{k=0}^{15}{}^{15}C_k\,3^{15-k}(2x)^k$ ✓

$=\sum_{k=0}^{15}{}^{15}C_k\,3^{15-k}\cdot2^k\cdot x^k$ ✓

$\therefore t_k={}^{15}C_k\,3^{15-k}2^k$

(ii) $\frac{t_{k+1}}{t_k}=\frac{{}^{15}C_{k+1}3^{15-(k+1)}2^{k+1}}{{}^{15}C_k\,3^{15-k}\,2^k}$ ✓

$=\frac{{}^{15}C_{k+1}\times2}{{}^{15}C_k\times3}$

$=\frac{15!\times2}{(15-(k+1))!(k+1)!}\times\frac{(15-k)!\,k!}{15!\times3}$

$=\frac{2(15-k)}{3(k+1)}$ ✓

(iii) $\frac{2}{3}\frac{(15-k)}{(k+1)}<1$ ✓

$30-2k<3k+3$

$27<5k$

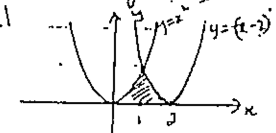
$5\frac{2}{5}<k$

∴ $k=6$

∴ $t_6={}^{15}C_6\,3^9\,2^6$ ✓

(b) $y=x^2$ $y=(x-2)^2$

(i)



(ii) $A=\int_0^1x^2dx+\int_1^2(x-2)^2dx$

$=\left[\frac{x^3}{3}\right]_0^1+\left[\frac{(x-2)^3}{3}\right]_1^2$ ✓

$=(\frac{1}{3}-0)+(0-\frac{-1}{3})$

$=\frac{2}{3}$ units² ✓

(c) $y=\sqrt[4]{3x-1}$

$V=\pi\int_a^b y^2\,dx$

$=\pi\int_1^2(3x-1)^{\frac{1}{2}}dx$ ✓

$=\pi\left[\frac{2(3x-1)^{3/2}}{3\cdot3}\right]_1^2$ ✓

$=\frac{2}{9}\pi\left(5^{3/2}-2^{3/2}\right)$

$=\frac{2\pi}{9}\left(\sqrt{125}-\sqrt8\right)$ ✓

$=\frac{2\pi}{9}\left(5\sqrt5-2\sqrt2\right)$ u^3

Question Two (cont)

(d) $\int\sin^2x\cos^2x\,dx$

$=\int\frac{1}{4}\sin^22x\,dx$ ✓

$=\int\frac{1}{8}(1-\cos4x)\,dx$ ✓

$=\frac{1}{8}\left[x-\frac{\sin4x}{4}\right]+C$ ✓

$=\frac{x}{8}-\frac{\sin4x}{32}+C$

Question Three (15 mks)

(a)(i) $P(2at,at^2)$ $x^2=4ay$

$y=\frac{x^2}{4a}$ ∴ $\frac{dy}{dx}=\frac{2x}{4a}=\frac{x}{2a}$

At $P(2at,at^2)$ ✓

$\frac{dy}{dx}=t$

∴ grad of tangent $=t$

Eqn of tangent is

$y-at^2=t(x-2at)$ ✓

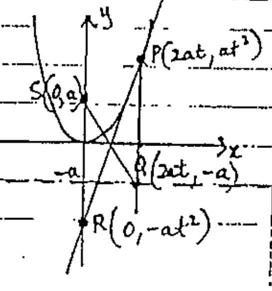
ie $tx-y-at^2=0$

(ii) $P(2at,at^2)$

Focus is $S(0,a)$

Directrix is $y=-a$

∴ Q is $(2at,-a)$ ½



R is where $tx-y-at^2=0$ intersects $x=0$

∴ $R(0,-at^2)$ ½

PR has midpt $(at,0)$ ½

QS has midpt $(at,0)$ ½

∴ PR + QS have same midpt ∴ PR + QS bisect each other. ✓

(iii) gradient $PR=t$.

gradient $QS=\frac{-1}{t}$ ✓

∴ $t\times\frac{-1}{t}=-1$ ∴ PR and QS are perpendicular to each other ✓

(iv) PQRS is a rhombus ✓ because the diagonals PR and QS bisect each other at 90°. ✓

(b) (i) $\int\tan^4x\,dx$ ½

$=\int\tan^2x(\sec^2x-1)\,dx$

$=\int(\tan^2x\sec^2x-\tan^2x)\,dx$ ½

$=\int\tan^2x\sec^2x-\sec^2x+1\,dx$

$=\frac{\tan^3x}{3}-\tan x+x+C$ ✓

$I_n=\int\sec^nx\,dx$

$=\int\sec^{n-2}x\sec^2x\,dx$ ✓

So $u=\sec^{n-2}x$

$\frac{du}{dx}=(n-2)\sec^{n-3}x\sec x\tan x$ ½ and $\frac{dv}{dx}=\sec^2x$, $v=\tan x$ ½

$\therefore I_n=\sec^{n-2}x\tan x-(n-2)\int\sec^{n-2}x\tan^2x\,dx$

$=\sec^{n-2}x\tan x-(n-2)\int\sec^{n-2}x(\sec^2x-1)\,dx$

$=\sec^{n-2}x\tan x-(n-2)\int\sec^nx\,dx+(n-2)\int\sec^{n-2}x\,dx$ ✓

$I_n=\sec^{n-2}x\tan x-(n-2)I_n+(n-2)I_{n-2}$ ✓

$\therefore(n-1)I_n=\sec^{n-2}x\tan x+(n-2)I_{n-2}$

$\therefore I_n=\frac{\sec^{n-2}x\tan x}{n-1}+\frac{n-2}{n-1}I_{n-2}$

$n>1$

(i) The tangent at P is $=\frac{b}{a\sin\theta}$

$\frac{\sec\theta\,x}{a}-\frac{\tan\theta\,y}{b}=1$ ✓

ii) (Show this)

It meets the x-axis at $Q(\frac{a}{\sec\theta},0)$; It meets the y-axis at $S(0,\frac{-b}{\tan\theta})$ ✓

If OQRS is a rectangle the co-ordinates of R are

$x=\frac{a}{\sec\theta}$, $y=\frac{-b}{\tan\theta}$ ✓

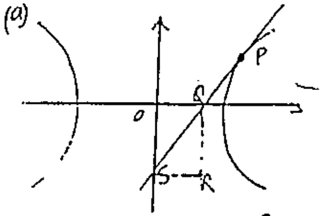
Hence $\sec\theta=\frac{a}{x}$, $\tan\theta=\frac{-b}{y}$

Since $\sec^2\theta-\tan^2\theta=1$ ✓

Then the locus of R is the curve $\frac{a^2}{x^2}-\frac{b^2}{y^2}=1$ ✓

Question Four

(a)



(b) The chord PQ has gradient

(i) $\frac{2/q-2/p}{2q-2p}=\frac{(p-q)/qp}{q-p}=-\frac{1}{pq}$ ✓

and equation

$y-\frac{2}{p}=-\frac{1}{pq}(x-2p)$ ✓

$pqy-2q=-1(x-2p)$

$x+pqy=2(p+q)$ ✓

(ii) $R(4,2)$ lies on PQ

∴ $(4)+pq(2)=2(p+q)$ ✓

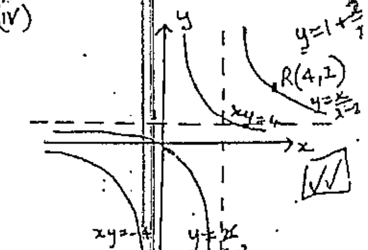
∴ $pq=p+q-2$

(iii) $M\left(\frac{1}{2}(2p+2q),\frac{1}{2}(\frac{2}{p}+\frac{2}{q})\right)$ ✓

∴ $M\left(p+q,\frac{p+q}{pq}\right)$ ✓

$x=p+q$ ✓ $y=\frac{p+q}{pq}$ ∴ $y=\frac{p+q}{p+q-2}=\frac{x}{x-2}$

(iv)



$xy=4$ has asymptotes $x=0$ $y=0$