BAULKHAM HILLS HIGH SCHOOL

YEAR 12 HALF YEARLY EXAMINATION

2000

MATHEMATICS

4 UNIT

Time allowed – three hours
(Plus 5 minutes reading time)

DIRECTIONS TO CANDIDATES

* Begin each Question on a fresh page.
* Show all working.
* Silent calculators may be used.

QUESTION 1:

(a) Simplify $\dfrac{6-4i}{2i}$

(b) If $z = -1 + i$ find

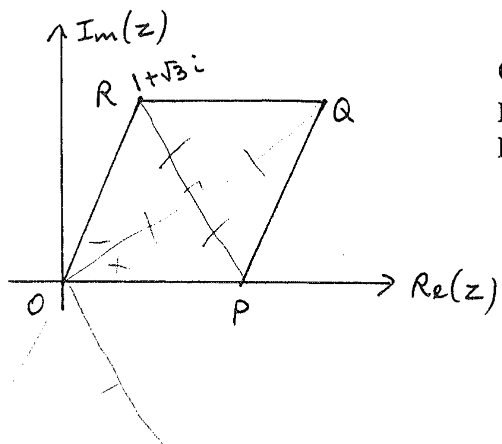
(i) $|z|$ (ii) $\arg z$ (iii) z^6 in the form of $x + iy$

(c) If $z = 3 + 2i$, $w = -1 + i$, find

(i) $\dfrac{2}{iw}$ (ii) $Im\,(\bar{z}w)$

(d) Sketch the region satisfied by $|z - 3 + i| \leq 5$ and $|z+1| \leq |z-1|$

(e)



OP QR is a rhombus.
R represents $1 + \sqrt{3i}$.
Find the complex number represented by Q.

QUESTION 2:

(a) Find the two square roots of $-2i$
Hence solve the equation $z^2 + (1-3i)z - (2+i) = 0$.

(b) Solve the equation $3z\bar{z} + 2(z - \bar{z}) = 39 + 12i$

(c) If w is a complex cube root of unity, show that $1 + w + w^2 = 0$ and find the value of $(1 + 2w + 3w^2)(1 + 2w^2 + 3w)$.

(d) If $z = a + ib$ is a complex number such that b is non-zero, and $z + \dfrac{1}{z}$ is purely real, find $|z|$.

QUESTION 3:

(a) Find the coordinates of the point on the curve $x^2y + xy^2 = 16$ at which the tangent is parallel to the x axis.

(b) Prove by mathematical induction that for $n \geq 1$

$$(\cos\theta + i\ \sin\theta)^n = \cos n\theta + i \sin n\theta$$

(c) The complex number z satisfies the equation $(z - \bar{z})^2 + 8(z + \bar{z}) = 16$. Show that the locus of z is a parabola and state the vertex and focus.

(d) Sketch the locus of z if $\arg(z+2) = \frac{\pi}{4} + \arg(z - 2i)$.

QUESTION 4:

(a) (i) P is any point on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ and S and S^1 are the foci. Prove that $PS + PS^1 = 2a$

(ii) Hence find the equation of the locus of the complex number z which satisfies $|z - 3i| + |z + 3i| = 12$.

(b) (i) Derive the equation of the tangent to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ at the $(a\cos\theta, b\sin\theta)$.

(ii) Show that this tangent cuts the x axis at $\left(\frac{a}{\cos\theta}, 0\right)$.

(iii) Hence find the coordinates of the point (s) on $\frac{x^2}{3} + \frac{y^2}{4} = 1$ whose tangent passes through (2, 0).

(iv) Find the eccentricity of the ellipse $\frac{x^2}{3} + \frac{y^2}{4} = 1$ and the coordinates of the foci.

QUESTION 5:

(a) Draw a large neat sketch of the hyperbola $\frac{x^2}{25} - \frac{y^2}{16} = 1$ labelling clearly the asymptotes, the foci, the directrices.

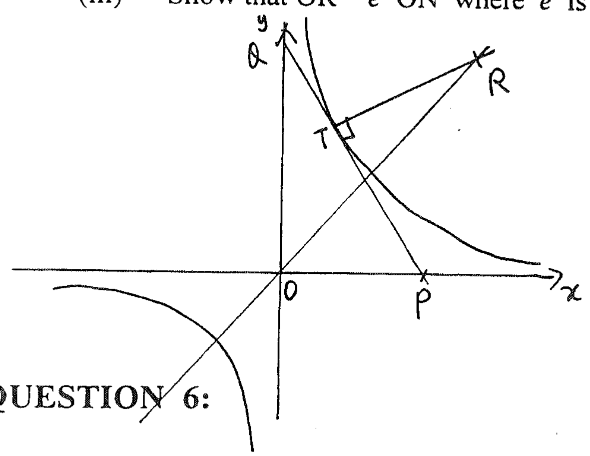
(b) Find the possible values of k if $\frac{x^2}{12-k} + \frac{y^2}{k+4} = 1$ represents a hyperbola

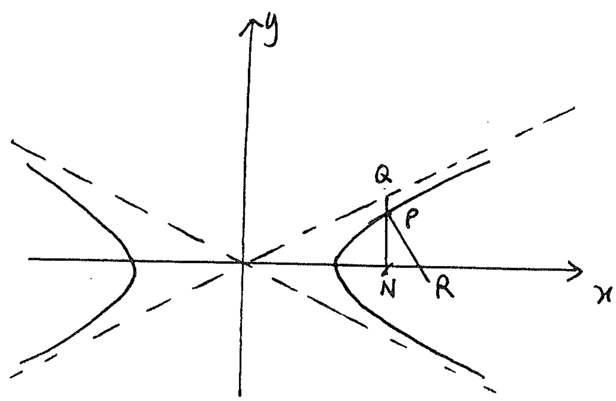
(c) $P(a\sec\theta, b\tan\theta)$ is any point on the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$. <u>Assume</u> the equation of the normal at P is given by $a\,x\tan\theta + b\,y\sec\theta = \left(a^2 + b^2\right)\sec\theta\tan\theta$.
A line through P parallel to the y axis meets an asymptote at Q and the x axis at N. The normal at P meets the x axis at R.

(i) Find the coordinates of Q, N, R.

(ii) Show that QR is perpendicular to the asymptote.

(iii) Show that $\mathrm{OR} = e^2\,\mathrm{ON}$ where e is the eccentricity.





QUESTION 6:

(a) The point $T\left(4t, \frac{4}{t}\right)$ lies on the hyperbola $x\,y = 16$. The tangent at T meets the x axis at P and y axis at Q. The normal at T meets the line $y = x$ at R.

(i) Show that the tangent at T has equation $x + t^2 y = 8t$.

(ii) Find the coordinates of P and Q.

(iii) Write down the equation of the normal at T (there is no need to simplify).

(iv) Show that the x coordinates of R is $x = \frac{4}{t}\left(t^2 + 1\right)$.

(v) Show that TQ = TP = TR = OT.

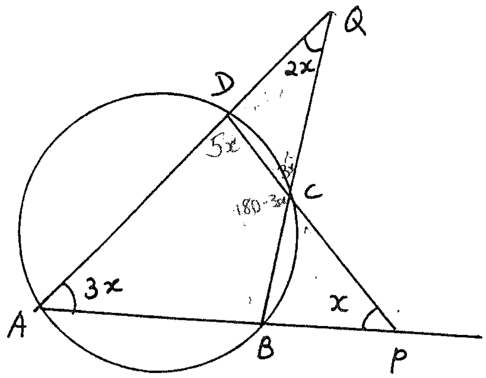
(vi) What can you deduce about the points O, P, R, Q?

QUESTION 7:

(a) If $y = \log_e\left(x + \sqrt{x^2 + 9}\right)$, Show that $\dfrac{dy}{dx} = \dfrac{1}{\sqrt{x^2 + 9}}$

Hence find $\displaystyle\int \frac{1}{\sqrt{x^2 + 9}}\,dx$

(b)



3x + 6x = 180
x = 20.

Find the value of x giving reasons.

(c) Given the ellipse $\dfrac{x^2}{9} + \dfrac{y^2}{4} = 1$

Show that the area enclosed by the ellipse is given by $A = \dfrac{8}{3}\displaystyle\int_0^3 \sqrt{9 - x^2}\,dx$.

Use the substitution $x = 3\sin\theta$ to evaluate this area.

QUESTION 8:

(a) Prove that $\operatorname{cosec}\theta - \cot\theta = \tan\frac{\theta}{2}$.

Hence solve the equation for $0 \le x \le 2\pi$.

$$\sqrt{3}\operatorname{cosec}2x = \sqrt{3}\cot 2x - 1$$

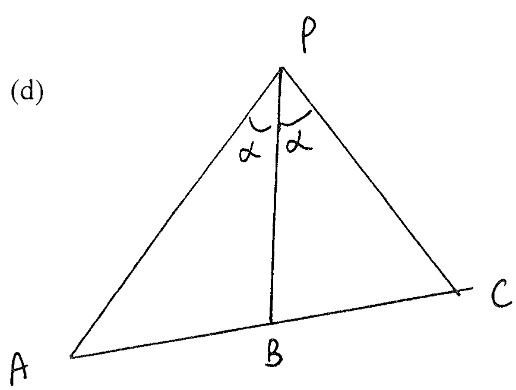
(b) (i) Show that $\sin(A+B) - \sin(A-B) = 2\cos A\sin B$.

(ii) Show that $2\sin x(\cos 2x + \cos 4x + \cos 6x) = \sin 7x - \sin x$.

(iii) Deduce that $\cos\frac{2\pi}{7} + \cos\frac{4\pi}{7} + \cos\frac{6\pi}{7} = -\frac{1}{2}$.

(c) If $p = \log_q r, q = \log_r p, r = \log_p q$ show that $pqr = 1$.

(d)



(i) Use the sine rule to show that $\frac{AB}{BC} = \frac{AP}{PC}$

(ii) Express the area of ΔBPA in terms of α.

(iii) By considering area of triangles or otherwise, show that if $\alpha = 60^0$ then $\frac{1}{PA} + \frac{1}{PC} = \frac{1}{PB}$