TEACHER'S NAME: ______________________

STUDENT'S NAME: ______________________

BAULKHAM HILLS HIGH SCHOOL

YEAR 12

YEAR 12 HALF YEARLY EXAMINATION

2007

MATHEMATICS
EXTENSION 2

Time allowed - Three hours
(Plus five minutes reading time)

DIRECTIONS TO CANDIDATES

- Attempt ALL questions.
- Begin each Question on a fresh page.
- All necessary working should be shown.
- Write your teacher's name and your name on the cover sheet provided.
- At the end of the exam, staple your answers in order behind the cover sheet provided.

QUESTION 1 (15 Marks) **Marks**

(a) The complex number z is given by $z = \frac{2+i}{2-i} - 2$. Find: 3

(i) $\bar{z}$
(ii) $\text{Im}(z)$
(iii) $|z|$

(b) Draw neat labelled sketches to show each of the following graphs on the Argand diagram. 4

(i) $|z - iz| = 2$
(ii) $|z - 1 - 2i| \le 3$ <u>and</u> $|z + i| \le |z - i|$

(c) Prove $|z_1 + z_2| \le |z_1| + |z_2|$ and explain when the equality holds. 3

(d) Solve for $z = x + iy$ the equation $z\bar{z} + 2z = \frac{1}{4} + i$. 3

(e) Use De Moivre's Theorem to express $Cos5\theta$ in terms of $Cos\theta$ and $Sin\theta$. 2

QUESTION 2 (15 Marks)

(a) Solve $z^2 - 5iz + 2 = 0$ 2

(b) Given $z = 3 - 3\sqrt{3}i$, 3
(i) Find z in mod-arg form
(ii) Find z^4 in the form $x + iy$.

(c) Find the square roots of $4 + 3i$. 3

(d) Describe in words and sketch the locus of z if $Arg(\frac{z-1}{z-2i}) = \frac{\pi}{2}$. 3

(e) Solve $z^5 = 1$ over the complex field and plot the solutions on an Argand diagram. 3

(f) Find i^{2009}. 1

QUESTION 3 (15 Marks) **Marks**

(a) The ellipse E is defined by E: $\frac{x^2}{9}+\frac{y^2}{4}=1$

(i) For E write down its eccentricity, co-ordinates of its foci and equations of its directrices, and sketch the curve, showing its features. 4

(ii) Show that the equation of the normal at $P(3Cos\theta, 2Sin\theta)$ is $\frac{3x}{Cos\theta}-\frac{2y}{Sin\theta}=5$. 3

(iii) If the normal at P cuts the major axis at G and the minor axis at H, then find M, the midpoint of GH. 3

(b)

(i) Show that for an ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, the sum of the focal distances is $2a$. 2

(ii) Hence find the equation of the locus of z such that $|z+2i|+|z-2i|=6$ 3

QUESTION 4 (15 Marks)

(a)

(i) Show that in the first quadrant the ellipse $4x^2+9y^2=36$ and the hyperbola $4x^2-y^2=4$ intersect at right angles. 4

(ii) Find the equation of the one circle through the points of intersection of the two conics in all four quadrants. 2

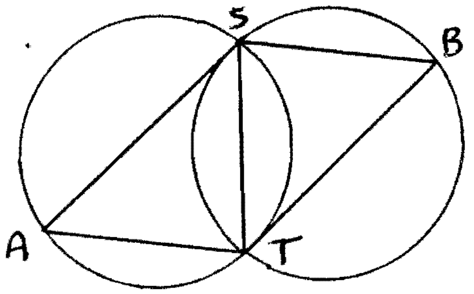
(b) Find the equation(s) of the normal to the curve $xy(x+y)+2=0$ at the point(s) where the gradient of the tangent is -1. 5

(c) AS and TB are tangents. Prove:

(i) $\Delta SAT\ |||\ \Delta BTS$ 2

(ii) $AT\times BS=ST^2$ 1

(iii) $SB \parallel AT$ 1



QUESTION 5 (15 Marks) **Marks**

(a) The rectangular hyperbola H has equation $x^2 - y^2 = 2$. Write down its eccentricity, the co-ordinates of its foci S and S', and vertices A and A', the equations of its directrices and asymptotes. 4

(b)

(i) Show the condition for the line $y = mx + c$ to be a tangent to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is $c^2 = a^2m^2 + b^2$. 3

(ii) Show that the pair of tangents drawn from (5,0) to the ellipse $\frac{x^2}{16} + \frac{y^2}{9} = 1$ are at right angles to each other. 2

(c)

(i) Prove that the equation of the tangent to $xy = c^2$ at the point $P(cp, \frac{c}{p})$ is $x + p^2y = 2cp$. 2

(ii) If the tangent at P passes through the foot of the ordinate of $Q(cq, \frac{c}{q})$ show that $q = 2p$. 2

(iii) Hence prove that the locus of the midpoint of PQ is $xy = \frac{9c^2}{8}$. 2

QUESTION 6 (15 Marks)

(a) Sketch the graph of $y = (2x+5)(x+1)$ showing clearly the intercepts on the axes and the co-ordinates of any turning points. 2

(b) Use the graph in part (a) to sketch the graph of $y = \frac{1}{(2x+5)(x+1)}$ showing clearly the intercepts on the axes, the co-ordinates of any turning points and the equations of any asymptotes. 4

(c) Use the graph in part (a) to sketch the graph of $y = \ln[(2x+5)(x+1)]$ showing the y intercept and the equations of any asymptotes. 4

(d) Find the equation of the tangent to the curve $y = \ln[(2x+5)(x+1)]$ at $x = 0$. 3

(e) Hence find the values of k such that exactly one of the solutions of the equation $\ln[(2x+5)(x+1)] = kx + \ln 5$ is a positive number. 2

QUESTION 7 (15 Marks) **Marks**

(a) Sketch the graph of $f(x)=\frac{3x^2-7}{(x-2)(x+3)}$, given that

$f'(x)=\frac{(3x-1)(x-7)}{(x-2)^2(x+3)^2}$ 5

(b) Hence draw neat half-page sketches of:

(i) $y=\sqrt{f(x)}$ 2

(ii) $y=(f(x))^2$ 2

(iii) $y=f(|x|)$ 2

(b) (i) Draw the graph of $y=|2x+1|-|x-2|$ in the domain $-6\le x\le 3$ 2

(ii) Use the graph to solve $|2x+1|-|x-2|\le 2$ 2

QUESTION 8 (15 Marks)

(a) For what values of a does $\int_1^a (x+\frac{1}{2})dx = 2\int_0^{\frac{\pi}{4}}\sec^2 x dx$? 3

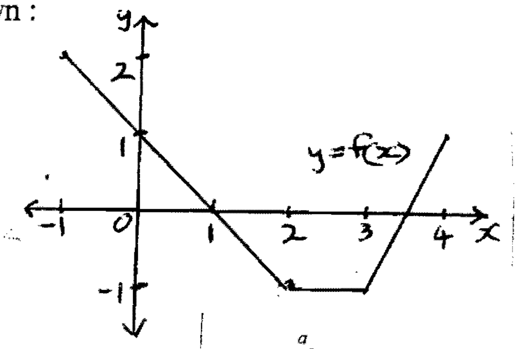
(b)

(i) Write an expression for Cos(A+B) and Cos(A-B). Hence express 2SinASinB as a difference of two functions. 2

(ii) Given that $SinxSin3x+SinxSin5x+SinxSin7x=SinaxSinbx$ then find the values of a and b. 3

(c) Given p and q are roots of $ax^2+bx+c=0$, show, without solving for p or q, that $\frac{q}{ap+b}+\frac{p}{aq+b}=\frac{-2}{a}$. 4

(d) For the graph of $y=f(x)$ shown :



(i) Find $\int_0^4 f(x)dx$ 2

(ii) For what value of a, where $0\le a\le 4$, does $\int_0^a f(x)dx$ have the smallest value? 1

Q1 (a) $z = \frac{2+i}{2-i} - 2 = \frac{2+i}{2-i} \times \frac{2+i}{2+i} - 2$

$= \frac{4+4i-1}{5} - 2 = \frac{3}{5} + \frac{4i}{5} - 2$

$= -\frac{7}{5} + \frac{4i}{5}$

(i) $\bar{z} = -\frac{7}{5} - \frac{4i}{5}$ (ii) $\operatorname{Im} z = \frac{4}{5}$

(iii) $|z| = \sqrt{\frac{49}{25} + \frac{16}{25}} = \frac{\sqrt{65}}{5}$

(b)(i) $|z - iz| = 2$

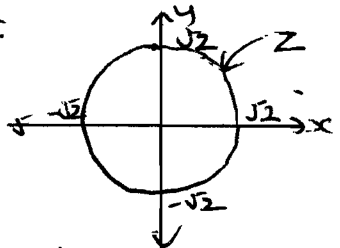
$|(x+iy) - i(x+iy)| = 2$

$|x+y + i(y-x)| = 2$

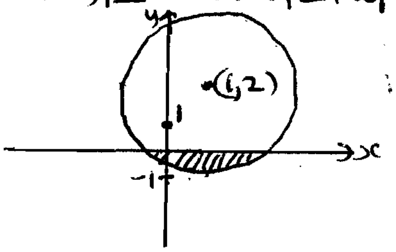
$(x+y)^2 + (y-x)^2 = 4$

$x^2 + 2xy + y^2 + y^2 - 2yx + x^2 = 4$

$\therefore x^2 + y^2 = 2$



(ii) $|z - (1+2i)| \le 3$ and $|z+i| \le |z-i|$



(c) In ΔOAC

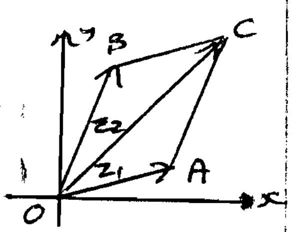
$OA = |z_1|$, $AC = OB = |z_2|$

$OC = |z_1 + z_2|$

In any Δ, $OC < OA + AC$

$\therefore |z_1 + z_2| < |z_1| + |z_2|$

Equality holds when $\arg z_1 = \arg z_2$ or O, A, B are collinear.



(d) $x^2 + y^2 + 2x + 2iy = \frac{1}{4} + i$

$\therefore 2y = 1 \quad \therefore y = \frac{1}{2}$

$x^2 + \frac{1}{4} + 2x = \frac{1}{4}$

$x(x+2) = 0 \quad \therefore x = 0$ or -2

$\therefore z = \frac{1}{2}i$ or $z = -2 + \frac{1}{2}i$

(e) $z^5 = (\cos\theta + i\sin\theta)^5 = \cos 5\theta + i\sin 5\theta$

or $z^5 = \cos^5\theta + 5\cos^4\theta\, i\sin\theta + 10\cos^3\theta\, i^2\sin^2\theta$
$+ 10\cos^2\theta\, i^3\sin^3\theta + 5\cos\theta\, i^4\sin^4\theta$
$+ i^5\sin^5\theta$

$= (\cos^5\theta - 10\cos^3\theta\sin^2\theta + 5\cos\theta\sin^4\theta)$
$+ i(5\cos^4\theta\sin\theta - 10\cos^2\theta\sin^3\theta + \sin^5\theta)$

Equating real parts give:

$\cos 5\theta = \cos^5\theta - 10\cos^3\theta\sin^2\theta + 5\cos\theta\sin^4\theta$

Q2 (a) $z = \frac{5i \pm \sqrt{-25-8}}{2} = \frac{5i \pm i\sqrt{33}}{2}$

$= \frac{(5 \pm \sqrt{33})i}{2}$

(b)(i) $r = \sqrt{9+27} = \sqrt{36} = 6$

$\tan(\arg z) = -\frac{3\sqrt{3}}{3} = -\sqrt{3} \quad \therefore \operatorname{Arg} z = -\frac{\pi}{3}$

$\therefore z = 6\left[\cos\left(-\frac{\pi}{3}\right) + i\sin\left(-\frac{\pi}{3}\right)\right]$

(ii) $z^4 = 6^4\left[\cos\left(-\frac{4\pi}{3}\right) + i\sin\left(-\frac{4\pi}{3}\right)\right]$

$= 1296\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right)$

$= 1296\left(-\frac{1}{2} + i\frac{\sqrt{3}}{2}\right)$

$= -648 + 648\sqrt{3}\,i$

(c) Let $a + ib = \sqrt{4+3i}$

$\therefore a^2 - b^2 + 2abi = 4 + 3i$

$\therefore a^2 - b^2 = 4$ and $2ab = 3 \therefore b = \frac{3}{2a}$

$\therefore a^2 - \frac{9}{4a^2} = 4$ or $4a^4 - 16a^2 - 9 = 0$

$(2a^2 - 9)(2a^2 + 1) = 0$

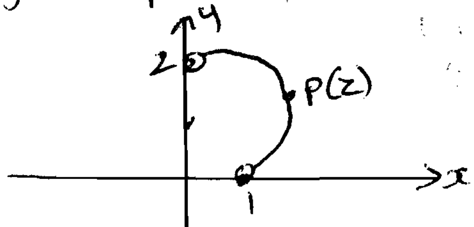
$a^2 = \frac{9}{2}$ or $a^2 = -\frac{1}{2} \therefore$ No soln.

$a = \pm\frac{3}{\sqrt{2}}$, $b = \pm\frac{3}{2} \times \frac{\sqrt{2}}{3} = \pm\frac{1}{\sqrt{2}}$

$\therefore \sqrt{4+3i} = \pm\left(\frac{3}{\sqrt{2}} + \frac{i}{\sqrt{2}}\right)$

(d) $\operatorname{Arg}(z-1) - \operatorname{Arg}(z-2i) = \frac{\pi}{2}$

Locus is a semi-circle with diameter endpoints (0,2) and (1,0) but not including endpoints.



(e) $z^5 = (\cos\theta + i\sin\theta)^5 = \cos 2k\pi + i\sin 2k\pi$

$\therefore \cos 5\theta = \cos 2k\pi$, for $k = 0, 1, 2, 3, 4$.

$\therefore \theta = \frac{2k\pi}{5}$, $k = 0, 1, 2, 3, 4$

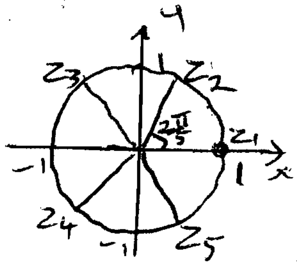
$\therefore z_1 = 1$

$z_2 = \cos\frac{2\pi}{5} + i\sin\frac{2\pi}{5}$

$z_3 = \cos\frac{4\pi}{5} + i\sin\frac{4\pi}{5}$

$z_4 = \cos\frac{6\pi}{5} + i\sin\frac{6\pi}{5}$

$z_5 = \cos\frac{8\pi}{5} + i\sin\frac{8\pi}{5}$



(f) $i^4 = 1$

$\therefore i^{2008} = 1$

$\therefore i^{2009} = i$

Q3 (a) $a=3,\ b=2$

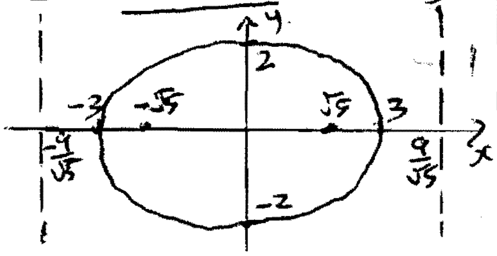
(i) $b^2=a^2(1-e^2)$

$4=9(1-e^2)$

$1-e^2=\frac{4}{9},\ e^2=\frac{5}{9},\ e=\frac{\sqrt5}{3}$

Foci $=(\pm ae,0)=(\pm\sqrt5,0)$

Dir. are $x=\pm\frac{a}{e}\ \therefore x=\pm\frac{9}{\sqrt5}$ or $x=\pm\frac{9\sqrt5}{5}$



(ii) Diff. Imp.

$\frac{2x}{9}+\frac{2y}{4}\cdot y'=0\ \therefore\ y'=-\frac{2x}{9}\times\frac{2}{y}=-\frac{4x}{9y}$

$m_{Tang}=\frac{-12\cos\theta}{18\sin\theta}=-\frac{2\cos\theta}{3\sin\theta}$

$\therefore m_{norm}=\frac{3\sin\theta}{2\cos\theta}$

Eqn of normal is

$y-2\sin\theta=\frac{3\sin\theta}{2\cos\theta}(x-3\cos\theta)$

$\frac{2y}{\sin\theta}-4=\frac{3x}{\cos\theta}-9$

$\frac{3x}{\cos\theta}-\frac{2y}{\sin\theta}=5$

(iii) At G, $y=0\ \therefore x=\frac{5\cos\theta}{3}$

At H, $x=0\ \therefore y=-\frac{5\sin\theta}{2}$

$\therefore M=\left(\frac{5}{6}\cos\theta,\ -\frac{5}{4}\sin\theta\right)$

(b)(i) $PS=e\,PM$ where M is on d

$=e\left(\frac{a}{e}-x_1\right)=a-ex_1$

$PS'=e\,PM'$ where M′ is on d′

$=e\left(x_1-\frac{-a}{e}\right)=ex_1+a$

$PS+PS'=a-ex_1+ex_1+a=2a$

(ii) $PS+PS'=2b$ where $S=(0,2)$

$S'=(0,-2)$, $b=3$

$\therefore be=2\ \therefore e=\frac{2}{3}$

$a^2=b^2(1-e^2)$

$=9\left(1-\frac{4}{9}\right)$

$=9\times\frac{5}{9}=5$

$\therefore a=\sqrt5$

$\therefore$ Eqn is

$\frac{x^2}{5}+\frac{y^2}{9}=1$

Q4 (a)(i) For $4x^2+9y^2=36$, diff implicitly

$8x+18yy'=0\ \therefore y_1'=-\frac{4x}{9y}$

For $4x^2-y^2=4$, diff imp.

$8x-2yy'=0\ \therefore y_2'=\frac{4x}{y}$

For $y^2=4x^2-4$ sub in $4x^2+9y^2=36$

$4x^2+36x^2-36=36$

$40x^2=72$

$x^2=\frac{9}{5}\quad x=\frac{3}{\sqrt5}$ in Q_1

$\therefore y^2=\frac{36}{5}-4=\frac{16}{5}\ \therefore y=\frac{4}{\sqrt5}$

$\therefore y_1'\times y_2'=-\frac{4}{9}\times\frac{3}{\sqrt5}\times\frac{\sqrt5}{4}\times4\times\frac{3}{\sqrt5}\times\frac{\sqrt5}{4}=-1$

$\therefore$ Intersect at right angles

(ii) By symmetry, pts of intersection are $\left(\pm\frac{3}{\sqrt5},\pm\frac{4}{\sqrt5}\right)$. Centre $=(0,0)$

$OP=\sqrt{\frac{9}{5}+\frac{16}{5}}=\sqrt5$

$\therefore$ Eqn is $x^2+y^2=5$

(b) $x^2y+xy^2+2=0$.

Diff. impl.

$y.2x+x^2.y'+y^2.1+x.2yy'=0$

$y'(x^2+2xy)=-y^2-2xy$

$y'=\frac{-y^2-2xy}{x^2+2xy}$

If $m_{Tang}=-1$, $-y^2-2xy=-x^2-2xy$

$\therefore x^2=y^2$

$x=\pm y$

If $x=y$, $x^3+x^3+2=0\ \therefore x=-1,\ y=-1$

If $-x=y$, $-x^3+x^3+2=0\ \therefore$ No soln.

$m_N=1$

$\therefore$ Eqn of normal is $y+1=1(x+1)$

$y=x$

(c) (i) In ΔSAT, ΔBTS

$\angle SAT=\angle BTS$ ($\angle$ bet tang & chord $=\angle$ in alt. seg)

$\angle AST=\angle SBT$ ($\angle$ bet tang & chord $=\angle$ in alt. seg)

$\therefore \angle ATS=\angle BST$ (3rd $\angle$'s in Δ's)

$\therefore \Delta SAT\ |||\ \Delta BTS$ (AAA)

(ii) $\frac{ST}{AT}=\frac{BS}{ST}$ (Match. sides in sim. Δ's are in same ratio)

(iii) $\angle ATS=\angle BST$ (Eq. $\angle$'s in sim. Δ's)

$\therefore SB\parallel AT$ (Alt. $\angle$'s equal)

Q5(a) $\frac{x^2}{2}-\frac{y^2}{2}=1 \therefore a=\sqrt{2}, b=\sqrt{2}$

$b^2=a^2(e^2-1) \therefore e^2-1=1 \therefore e^2=2 \therefore e=\sqrt{2}$

$\therefore$ Foci $=(\pm ae,0)=(\pm 2,0)$

Vertices $=(\pm a,0)=(\pm\sqrt{2},0)$

Directrices are $x=\pm\frac{a}{e} \therefore x=\pm 1$

Asymptotes are $y=\pm x$

(b)(i) Solve $y=mx+c$ with $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

$\therefore \frac{x^2}{a^2}+\frac{m^2x^2+2mxc+c^2}{b^2}=1$

$b^2x^2+a^2m^2x^2+2a^2mcx+a^2c^2=a^2b^2$

$(b^2+a^2m^2)x^2+2a^2mcx+a^2c^2-a^2b^2=0$

If tangent $\Delta=0$

$\therefore 4a^4m^2c^2=4(b^2+a^2m^2)a^2(c^2-b^2)$

$a^2m^2c^2=b^2c^2-b^4+a^2m^2c^2-a^2b^2m^2$

$b^4+a^2b^2m^2=b^2c^2$

$\therefore c^2=a^2m^2+b^2$

(ii) Since $y=mx+c$ then

$y=mx\pm\sqrt{16m^2+9}$

Sub in $(5,0)$

$\therefore 0=5m\pm\sqrt{16m^2+9}$

$16m^2+9=25m^2$

$9m^2=9$

$m^2=1$

$m=\pm 1$

Tangents with gradients 1 and -1

$\therefore$ Perpendicular.

(c)(i) $xy=c^2 \therefore y=c^2x^{-1}$

$\therefore y'=-c^2x^{-2}=\frac{-c^2}{x^2}$

At $P(cp,\frac{c}{p})$ $m_T=-\frac{c^2}{c^2p^2}=\frac{-1}{p^2}$

$\therefore$ Eqn of tangent is

$y-\frac{c}{p}=-\frac{1}{p^2}(x-cp)$

$p^2y-cp=-x+cp$

$p^2y+x=2cp$

(ii) Tangent passes thru $(cq,0)$

$\therefore 0+cq=2cp \therefore q=2p$

(iii) $M_{PQ}=\left(\frac{c(p+q)}{2},\frac{\frac{c}{p}+\frac{c}{q}}{2}\right)$

Let $q=2p$

$\therefore M_{PQ}=\left(\frac{3pc}{2},\frac{c}{2}\cdot\frac{3}{2p}\right)$

$\therefore$ If $x=\frac{3cp}{2}$ and $y=\frac{3c}{4p}$

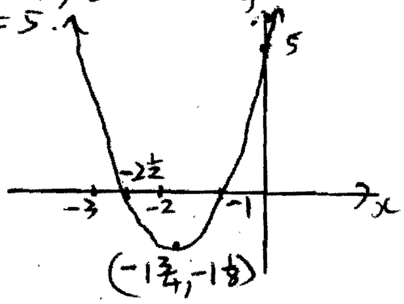
then $xy=\frac{9c^2}{8}$

6(a) $y=(2x+5)(x+1)$

$=0$ at $x=-2\frac{1}{2}, x=-1$

Vertex at $x=-1\frac{3}{4}, y=-\frac{9}{8}$

When $x=0$, $y=5$.

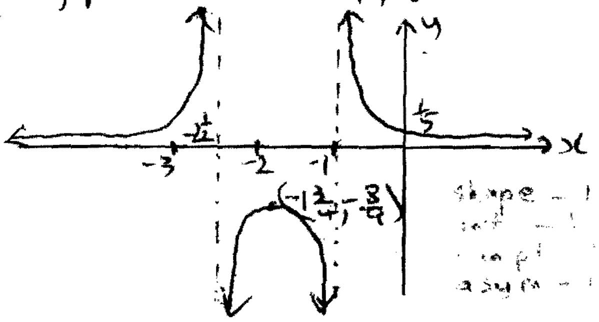


(b) $y=\frac{1}{(2x+5)(x+1)}$

Asym. at $x=-2\frac{1}{2}$, $x=-1$

y-int at $y=\frac{1}{5}$

Turning pt at $x=-1\frac{3}{4}, y=-\frac{8}{9}$

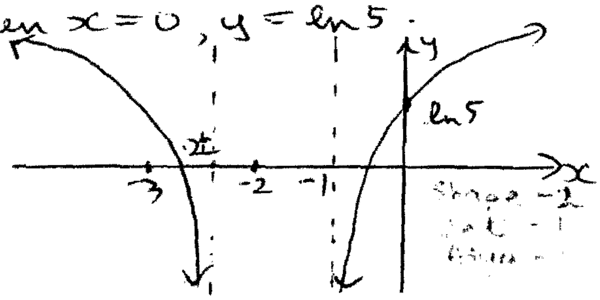


(c) $y=\ln[(2x+5)(x+1)]$

Exists when $y=(2x+5)(x+1)$ is pos.

$\therefore$ For $x<-2\frac{1}{2}$ and $x>-1$

When $x=0$, $y=\ln 5$.



(d) $y'=\frac{(x+1)2+(2x+5)1}{(2x+5)(x+1)}=\frac{4x+7}{(2x+5)(x+1)}$

When $x=0$, $m_T=\frac{7}{5}$ and $y=\ln 5$

$\therefore$ Eqn. is $y=\frac{7}{5}x+\ln 5$

(e) If $k=\frac{7}{5}$, tangent touches at $x=0$ and crosses for $x<-2\frac{1}{2}$

If $0<k<\frac{7}{5}$, line crosses for $x<-2\frac{1}{2}$, for $x=0$ and for $x>0$

$\therefore$ One positive soln if $0<k<\frac{7}{5}$

Q7.(a) Asymptotes at $x = -3$, $x = 2$ — 1

St. pts when $f'(x) = 0$

$\therefore x = \frac{1}{3}$, $y = 1\frac{1}{5}$ and $x = 7$, $y = 2\frac{4}{5}$ — 1

When $-3 < x < \frac{1}{3}$, $f'(x) > 0$; when $\frac{1}{3} < x < 2$, $f'(x) < 0$ } $\therefore$ Max at $x = \frac{1}{3}$

When $2 < x < 7$, $f'(x) < 0$; when $x > 7$, $f'(x) > 0$ } $\therefore$ Min at $x = 7$

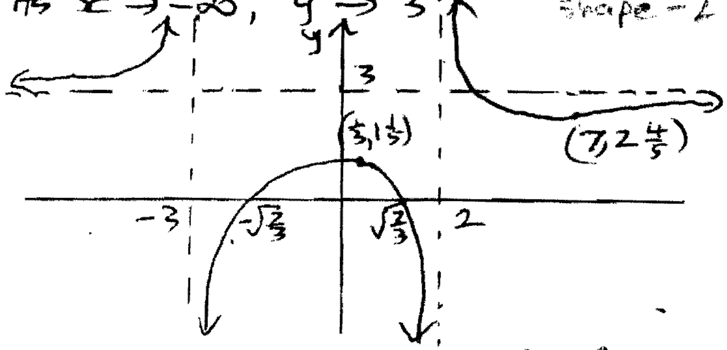
For $f(x) = \dfrac{3x^2 - 7}{x^2 + x - 6} = 3 + \dfrac{11 - 3x}{x^2 + x - 6}$ by division

$\therefore$ Hor. asym at $y = 3$ — 1

As $x \to \infty$, $y \to 3^-$

As $x \to -\infty$, $y \to 3^+$

Shape — 2



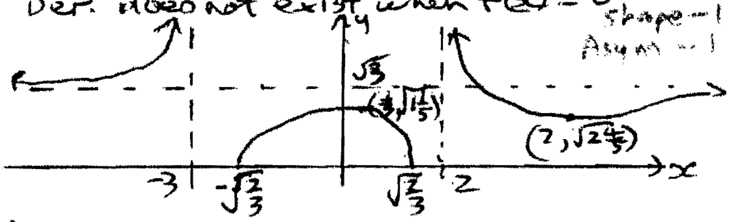
(b)(i) $y = \sqrt{f(x)}$ Only exists for $f(x) \geq 0$

Asym at $x = -3$, $x = 2$. Hor As, $y = \sqrt{3}$

$y' = \dfrac{f'(x)}{2\sqrt{f(x)}}$ $\therefore$ St pts when $f'(x) = 0$

Der. does not exist when $f(x) = 0$

Shape — 1, Asym — 1

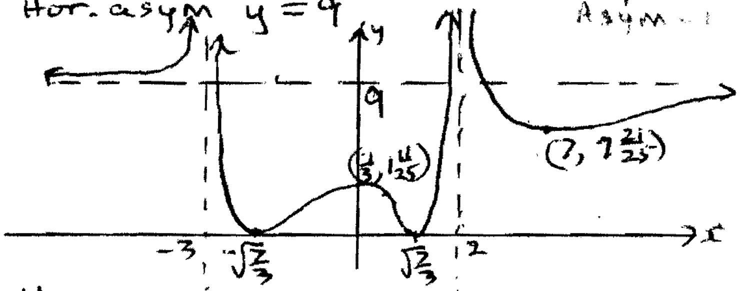


(ii) $y = (f(x))^2$ $\therefore y' = 2(f(x))^1 f'(x) = 0$ when $f(x) = 0$ and $f'(x) = 0$

i.e when $x = \frac{1}{3}, 7, \pm\sqrt{\frac{2}{3}}$

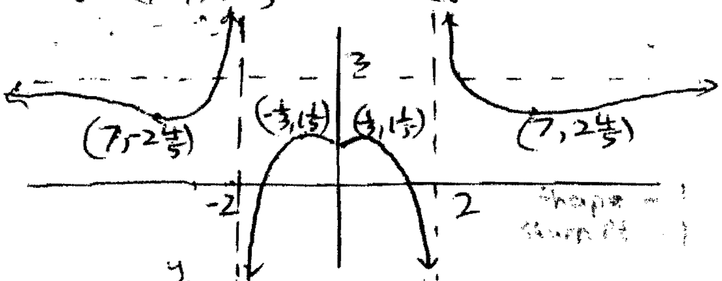
Hor. asym $y = 9$

Shape — 1, Asym — 1

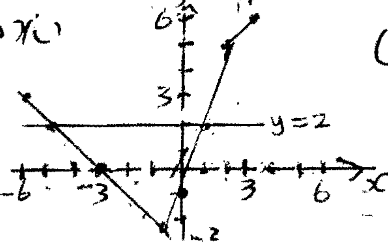


(iii) $y = f(|x|)$. y' is undef at $x = 0$

Shape — 1, Sharp Pt — 1



(c)(i)



(ii) $-5 \leq x \leq 1$

Q8(a) $\int_1^a \left(x + \frac{1}{2}\right) dx = 2\int_0^{\frac{\pi}{4}} \sec^2 x \, dx$

$\left[\frac{x^2}{2} + \frac{x}{2}\right]_1^a = 2\left[\tan x\right]_0^{\frac{\pi}{4}}$ Int$_1$ — 1, Int$_2$ — 1

$\left(\frac{a^2}{2} + \frac{a}{2}\right) - \left(\frac{1}{2} + \frac{1}{2}\right) = 2 - 0$

$\frac{a^2}{2} + \frac{a}{2} - 3 = 0$

$a^2 + a - 6 = 0$

$(a + 3)(a - 2) = 0$

$\therefore a = -3$ or 2 — 1

(b)(i) $\cos(A+B) = \cos A \cos B - \sin A \sin B$ ---(1)

$\cos(A-B) = \cos A \cos A + \sin A \sin B$ ---(2)

$\therefore$ (2) − (1) gives

$\cos(A-B) - \cos(A+B) = 2\sin A \sin B$ — 1

(ii) $\sin x \sin 3x + \sin x \sin 5x + \sin x \sin 7x$

$= \frac{1}{2}(\cos 2x - \cos 4x + \cos 4x - \cos 6x + \cos 6x - \cos 8x)$

$= \frac{1}{2}(\cos 2x - \cos 8x)$ Since $A - B = 2x$ and $A + B = 8x$

$= \frac{1}{2} \times 2\sin 5x \sin 3x$ Then $A = 5x$ and $B = 3x$

$= \sin 5x \sin 3x$

$\therefore a = 5$ and $b = 3$ — 1

or $a = 3$ and $b = 5$

(c) $p + q = -\frac{b}{a}$ $\therefore ap + aq = -b$ --(1)

$pq = \frac{c}{a}$ $\therefore apq = c$

$\therefore \dfrac{q}{ap+b} + \dfrac{p}{aq+b} = \dfrac{q}{-aq} + \dfrac{p}{-ap}$ using (1) — 1

$= -\frac{1}{a} + \frac{1}{-a} = -\frac{2}{a}$ — 1

(d)(i) $\int_0^4 f(x)dx = \frac{1}{2} \times 1 \times 1 - \frac{1}{2}\left(2\frac{1}{2} + 1\right) \times 1 + \frac{1}{2} \times \frac{1}{2} \times 1$

$= \frac{1}{2} - 1\frac{3}{4} + \frac{1}{4}$

$= -1$ — 1

(ii) $\int_0^a f(x)dx$ has the smallest value when it is most negative

$\therefore a = 3\frac{1}{2}$ — 1