TEACHER'S NAME: ______________________

STUDENT'S NAME: ______________________

BAULKHAM HILLS HIGH SCHOOL

YEAR 12

HALF YEARLY EXAMINATION

2009

MATHEMATICS
EXTENSION 2

Time allowed - Three hours
(Plus five minutes reading time)

DIRECTIONS TO CANDIDATES

- Attempt ALL questions.
- Begin each question on a fresh page.
- All necessary working should be shown.
- Write your teacher's name and your name on the cover sheet provided.
- At the end of the exam, staple your answers in order behind the cover sheet provided.

QUESTION 1 (15 Marks) **Marks**

(a) If $u = 1 - i$ and $v = -2 + 4i$ are complex numbers, evaluate: 3

(i) $\mathrm{Im}(u + \overline{v})$

(ii) $\left|\dfrac{u+v+1}{u+v+i}\right|$

(b) Find the two square roots of $-i$ 3

(c) Write $\left(\dfrac{1+\sqrt{3}i}{1-\sqrt{3}i}\right)^{10}$ in the form $a + bi$ where a and b are real 3

(d) On an Argand diagram shade the region represented by the complex number z where $\dfrac{\pi}{4} \le \arg z \le \pi, 1 \le \mathrm{Im}(z) \le 3$ and $|z| \le 4$ 3

(e) Describe algebraically and with a diagram the locus of z represented by $z.\bar{z} + 10(z + \bar{z}) = 21$. What two purely imaginary values of z satisfy this equation? 3

QUESTION 2 (15 Marks)

(a) (i) Given $|w| = 2$ and $\arg w = \dfrac{\pi}{6}$, express $\dfrac{1}{4}w^4$ in mod-arg form 1

(ii) Hence, on an Argand diagram, plot the points A, B and C which represent $w, iw, \dfrac{1}{4}w^4$ respectively. 3

(b) $OPQR$ is a rhombus. O lies at the origin, P is on the positive real axis and R corresponds to the complex number $1 + \sqrt{3}i$.

(i) Find the complex number corresponding to the point Q 1

(ii) If the figure is rotated anticlockwise by 60^0 about O to form a new rhombus $O'P'Q'R'$, show this on an Argand diagram and find the complex number corresponding to the vertex at Q' 2

(c) If $1, w, w^2$ are the 3 cube roots of unity:

(i) Show that $1 + w$ is a root of the equation $z^3 - 3z^2 + 3z - 2 = 0$ 1

(ii) Find the integer root of this equation and the third root in terms of w 2

(iii) Graph these 3 roots on an Argand diagram and find the centre and radius of the circle on which they lie 2

(d) Find the roots of the equation $z^5 + 1 = 0$ and prove that

$\cos\dfrac{\pi}{5} + \cos\dfrac{3\pi}{5} = \dfrac{1}{2}$ 3

QUESTION 3 (15 Marks) **Marks**

(a) Find the equations of the tangents to the curve $x^2 + y^2 = xy + 3$ at the point where $x = 1$ 4

(b) Let $P(x) = x^5 - x^4 - x^3 - 7x^2 - 20x - 12$. Given that $P(x) = 0$ has a double root and that $P(2i) = 0$, factorise $P(x)$ into a product of real linear and real quadratic factors 4

(c) The roots of $x^3 + 6x^2 + 5x - 8 = 0$ are α, β and χ. Derive the monic cubic polynomials whose roots are:

(i) α^2, β^2 and χ^2 2

(ii) $\frac{1}{\alpha}, \frac{1}{\beta}$ and $\frac{1}{\chi}$ 2

(d) Two of the roots of the quartic equation $x^4 + px^3 + qx + r = 0$, where $r \neq 0$, add to zero. Prove that $q^2 + p^2 r = 0$ 3

QUESTION 4 (15 Marks)

(a) When a polynomial P(x) is divided by (x-2) and (x-3) the remainders are 4 and 9 respectively. Find the remainder when P(x) is divided by (x-2)(x-3). 3

(b) P(x) is a real polynomial of degree 3 with roots α, β and χ. If $(1 - i)$ is one root and $\alpha\beta + \alpha\chi + \beta\chi = -2$, find:

(i) $\alpha + \beta + \chi$ 2

(ii) $\alpha\beta\chi$ 1

(iii) P(x) 1

(c)

(i) Given that $(a-b)^5 = a^5 - 5a^4b + 10a^3b^2 - 10a^2b^3 + 5ab^4 - b^5$, expand and simplify $(z - \frac{1}{z})^5$ 1

(ii) If $z = \cos\theta + i\sin\theta$ show that $z - \frac{1}{z} = 2i\sin\theta$ and $z^n - \frac{1}{z^n} = 2i\sin n\theta$ 2

(iii) Hence or otherwise express $\sin^5\theta$ in the form $a\sin 5\theta + b\sin 3\theta + c\sin\theta$ and then solve completely $16\sin^5\theta = \sin 5\theta$ 5

QUESTION 5 (15 Marks) **Marks**

(a) For the ellipse $x^2+4y^2=100$ find:

(i) the eccentricity 1

(ii) the coordinates of the focus 1

(iii) the equations of the directrices 1

(iv) the x and y intercepts 1

(v) Neatly sketch the curve showing all the above features 1

(b)

(i) Show that the equation of the normal to the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ at the point $P(a\sec\theta, b\tan\theta)$ is given by $ax\sin\theta+by=(a^2+b^2)\tan\theta$ 3

(ii) If this normal meets the x-axis at G and if N is the foot of the perpendicular from P to the x-axis, prove that $\frac{OG}{ON}=e^2$ 3

(c) $P(3p,\frac{3}{p})$ and $Q(3q,\frac{3}{q})$ are 2 points on the rectangular hyperbola xy=9 where $p>0$ and $q>0$

(i) Find the equation of the chord PQ 1

(ii) The tangents at P and Q intersect at M. Find the coordinates of M. 3

QUESTION 6 (15 Marks)

(a) Show that the curves $x^2-y^2=c^2$ and $xy=c^2$ cross at right angles 3

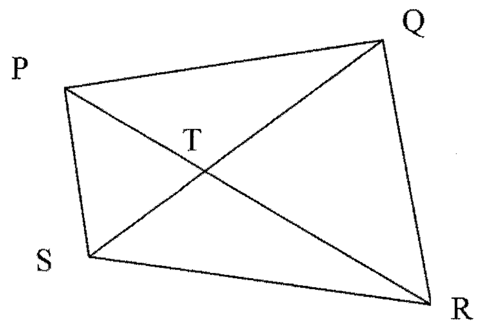
(b) The straight line $x-2y+3=0$ cuts the ellipse $x^2+2y^2=9$ at the points A and B. The tangents to the ellipse at A and B meet at the point C. Find the coordinates of C. 4

(c) P is the point $(3\sec\theta, 2\tan\theta)$ on the hyperbola $4x^2-9y^2=36$, with focus S and directrix d. The line SP is parallel to an asymptote. Show that the tangent at P and the asymptote meet at a point on the directrix. 4

(d) PQ is a chord of the rectangular hyperbola $xy=9$. If PQ has a constant length of 1, show that the locus of the midpoint of PQ has as its equation $4(xy-9)(x^2+y^2)-xy=0$. 4

QUESTION 7 (15 Marks) **Marks**

(a) PQRS is a cyclic quadrilateral whose diagonals meet at T. A circle is drawn through the vertices of triangle PQT. Prove that the tangent to this circle at T is parallel to SR. 4



(b)

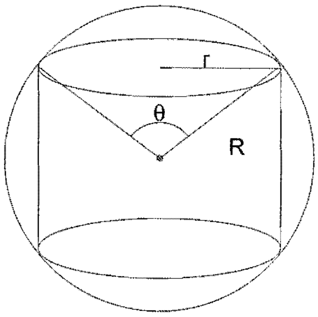
(i) By considering the expansions of cos(A+B) and cos(A-B) show that cos(A+B)+cos(A-B) = 2cosAcosB 1

(ii) Hence show that $\cos X + \cos Y = 2\cos\frac{X+Y}{2}\cos\frac{X-Y}{2}$ 1

(iii) Hence solve $\cos\theta + \cos 3\theta = 0$ for $0 \le \theta \le 2\pi$. 2

(c) Solve the inequality $\frac{x-3}{x^2-x} \ge -2$ 4

(d) A cylinder of height h and radius r is inscribed in a sphere of radius R



(i) Show that $r^2 = \frac{R^2(1-\cos\theta)}{2}$ 1

(ii) Show that $h^2 = 2R^2(1+\cos\theta)$ 1

(iii) Hence show that the maximum curved surface area of the cylinder occurs when $h = R\sqrt{2}$ 1

PTO FOR QUESTION 8

QUESTION 8 (15 Marks)

(a) Using the principle of mathematical induction prove that $a^n + b^n$ is divisible by $a+b$ for all odd positive values of *n*. 4

(b) If $y = (x+1)^x$ find $\dfrac{dy}{dx}$ 2

(c) Sketch the curve $y = 2 + \dfrac{1}{x^2 - 1}$, showing the location and nature of all stationary points and the equations of any asymptotes. 4

(d) Find the area enclosed between the curves y=cos x, y=sin x, and y=tan x in the domain $0 \le x \le \dfrac{\pi}{2}$. Draw a clear diagram to illustrate the area involved. 5

END OF EXAM

BHHS MATHS EXT 2
HALF YEARLY 2009
SOLUTIONS

Q1(a)

(i) $Im(u+\bar{v}) = Im(1-i-2-4i)$

$= Im(-1-5i)$

$= -5$

(ii) $\left|\frac{u+v+1}{u+v+i}\right| = \left|\frac{1-i+-2+4i+1}{1-i+-2+4i+i}\right|$

$= \left|\frac{3i}{-1+4i}\right|$

$= \frac{|3i|}{|-1+4i|}$

$= \frac{3}{\sqrt{1^2+4^2}}$

$= \frac{3}{\sqrt{17}}$ (2)

b) Let $x+iy = \pm\sqrt{-i}$

$\therefore (x+iy)^2 = -i$

$x^2-y^2+2ixy = -i$

Eq real & Im parts

$x^2-y^2=0$

$2xy=-1$ —1

$y = \frac{-1}{2x}$

$\therefore x^2 - \frac{1}{4x^2} = 0$

$4x^4 = 1$

$x^4 = \frac{1}{4}$

$x^2 = \pm\frac{1}{2}$ —1

But $x^2 \geq 0 \therefore x^2 = \frac{1}{2}$

$\therefore x = \pm\frac{1}{\sqrt{2}}$

$\therefore y = \frac{-1}{\pm\frac{2}{\sqrt{2}}}$

$= \mp\frac{1}{\sqrt{2}}$

$\therefore x+iy = \pm\frac{1}{\sqrt{2}} \mp \frac{i}{\sqrt{2}}$

c) $\left(\frac{1+\sqrt{3}i}{1-\sqrt{3}i}\right)^{10} = \left[\frac{2\left(\frac{1}{2}+\frac{\sqrt{3}}{2}i\right)}{2\left(\frac{1}{2}-\frac{\sqrt{3}}{2}i\right)}\right]^{10}$ —1

$= \frac{2^{10}\left(\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}\right)^{10}}{2^{10}\left(\cos\frac{-\pi}{3}+i\sin\frac{-\pi}{3}\right)^{10}}$

(by DeMoivre) $= \frac{\cos\frac{10\pi}{3}+i\sin\frac{10\pi}{3}}{\cos\frac{-10\pi}{3}+i\sin\frac{-10\pi}{3}}$ —1

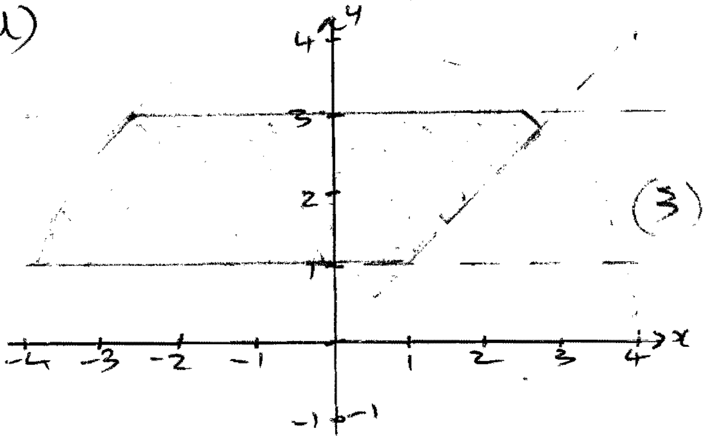
$= \frac{\cos\frac{-2\pi}{3}+i\sin\frac{-2\pi}{3}}{\cos\frac{2\pi}{3}+i\sin\frac{2\pi}{3}}$

$= \cos\left(\frac{-2\pi}{3}-\frac{2\pi}{3}\right)+i\sin\left(\frac{-2\pi}{3}-\frac{2\pi}{3}\right)$

$= \cos\frac{-4\pi}{3}+i\sin\frac{-4\pi}{3}$

$= -\frac{1}{2}+\frac{\sqrt{3}i}{2}$ —1 (3)

d)



(3)

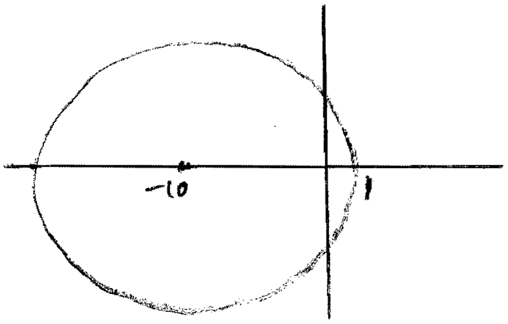
e) $z\bar{z}+10(z+\bar{z}) = 21$

$x^2+y^2+20x = 21$

$x^2+20x+100+y^2 = 121$

$(x+10)^2+y^2 = 121$ —1

$\therefore$ locus of z is a circle, centre $(-10,0)$, radius 11



—1

When $x=0$

$100+y^2 = 121$

$y^2 = 21$

$y = \pm\sqrt{21}$

$\therefore z = \pm\sqrt{21}\,i$ —1 (2)

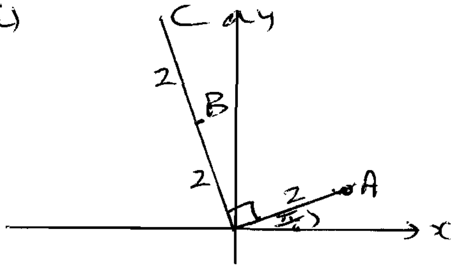
Q2(a)(i)

$\frac{1}{4}w^4 = \frac{1}{4}\left[2\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)\right]^4$

$= \frac{1}{4}\left[16\left(\cos\frac{4\pi}{6} + i\sin\frac{4\pi}{6}\right)\right]$ by DeM.

$= \underline{4\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right)}$ — 1

(ii)



(4)

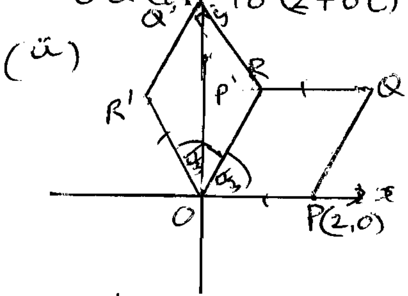
— 3

(b)(i) $|OR| = \sqrt{1+\sqrt{3}^2} = 2$

∴ OP corr. to $2+0i$ — 1

∴ OQ corr to $(2+0i)+(1+\sqrt{3}i) = \underline{3+\sqrt{3}i}$

(ii)



— 1

$Q' = \text{cis } 60^\circ(3+\sqrt{3}i)$

$= \left(\frac{1}{2} + \frac{\sqrt{3}i}{2}\right)(3+\sqrt{3}i)$

$= \frac{3}{2} + \frac{\sqrt{3}}{2}i + \frac{3\sqrt{3}i}{2} - \frac{3}{2}$ (3)

$= \underline{2\sqrt{3}i}$ — 1

(c)(i) $1+w+w^2=0 \quad \therefore 1+w = -w^2$

∴ Let $z = -w^2$

LHS $= (-w^2)^3 - 3(-w^2)^2 + 3(-w^2) - 2$

$= -w^6 - 3w^4 - 3w^2 - 2$

$= -(w^3)^2 - 3w^3.w - 3w^2 - 2$

$= -1 - 3w - 3w^2 - 2$

$= -3(1+w+w^2)$

$= -3 \times 0 = 0 =$ RHS. — 1

∴ $\underline{1+w \text{ is a root.}}$

(ii) $P(1) = 1-3+3-2 = -1 \neq 0$

$P(2) = 8-12+6-2 = 0$ — 1

∴ $\underline{z=2}$ is an integer root

$\Sigma\alpha = \alpha + 2 + (1+w) = -\frac{b}{a} = 3$

$\alpha + 3 + w = 3$ — 1

$\underline{\alpha = -w}$

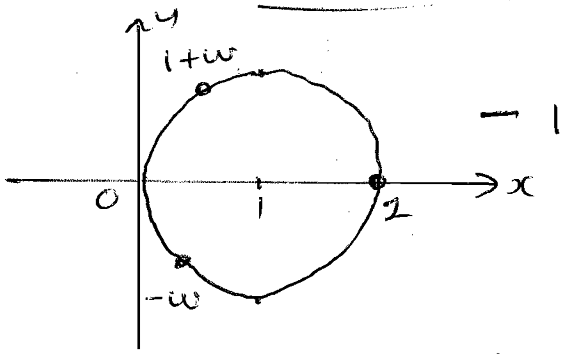
(iii) Roots are: $2 = 1+1$

$1+w = 1+w$

$-w = 1+w^2$

∴ Points translated 1 unit to the right of those for $1, w, w^2$

∴ Circle, centre (1,0) radius 1 — 1



— 1

(5)

d) $z^5 = -1$. Let $z = \text{cis }\theta$

∴ $(\text{cis }\theta)^5 = \text{cis }(2k+1)\pi$, $k = 0,1,2,3,4$

By De Moivre

$\text{cis } 5\theta = \text{cis }(2k+1)\pi$

∴ $5\theta = (2k+1)\pi$

$\theta = \left(\frac{2k+1}{5}\right)\pi$

If $k=0$, $z_1 = \cos\frac{\pi}{5} + i\sin\frac{\pi}{5}$

If $k=1$, $z_2 = \cos\frac{3\pi}{5} + i\sin\frac{3\pi}{5}$

If $k=2$, $z_3 = \cos\pi + i\sin\pi = -1$

If $k=3$, $z_4 = \cos\frac{7\pi}{5} + i\sin\frac{7\pi}{5}$

$= \cos\frac{-3\pi}{5} + i\sin\frac{-3\pi}{5}$

$= \cos\frac{3\pi}{5} - i\sin\frac{3\pi}{5} = \overline{z_2}$

If $k=4$, $z_5 = \cos\frac{9\pi}{5} + i\sin\frac{9\pi}{5}$

$= \cos\frac{-\pi}{5} + i\sin\frac{-\pi}{5}$

$= \cos\frac{\pi}{5} - i\sin\frac{\pi}{5} = \overline{z_1}$

∴ $z^5+1 = (z-z_3)(z-z_1)(z-\overline{z_1})(z-z_2)(z-\overline{z_2})$

∴ $(z+1)(z^4-z^3+z^2-z+1) = (z+1)\left(z^2 - 2\cos\frac{\pi}{5}z + 1\right)\left(z^2 - 2\cos\frac{3\pi}{5}z + 1\right)$

Equating coeffts of z in $z^4-z^3+z^2-z+1$

$-2\cos\frac{\pi}{5} - 2\cos\frac{3\pi}{5} = -1$

∴ $\underline{\cos\frac{\pi}{5} + \cos\frac{3\pi}{5} = \frac{1}{2}}$ (3)

Q3(a) $x^2+y^2=xy+3$

Diff. Imp.: $2x+2yy'=y.1+x.1.y'$

$xy'-2yy'=2x-y$

$y'(x-2y)=2x-y$

$y'=\frac{2x-y}{x-2y}$ — 1

When $x=1$, $1+y^2=y+3$

$y^2-y-2=0$

$(y-2)(y+1)=0$

$\therefore y=-1$ or 2 — 1

When $x=1$, $m_{Tang}=\frac{2--1}{1+2}=1$

$\therefore$ Tangent at $(1,-1)$ is

$y+1=1(x-1)$

$\underline{y=x-2}$ — 1

When $x=1$, $m_{Tang}=\frac{2-2}{1-4}=0$

$\therefore$ Tangent at $(1,2)$ is

$\underline{y=2}$ — 1 (4)

(b) Since coeffs are real and $P(2i)=0$ then $P(-2i)=0$ and factors are

$(x-2i)(x+2i)=x^2+4$ — 1

Since $P(x)=0$ has a double root α then $P'(\alpha)=0$

$\therefore 5\alpha^4-4\alpha^3-3\alpha^2-14\alpha-20=0$ — 1

Try factors of 20.

$P'(1)=5-4-3-14-20\neq0$

$P'(-1)=5+4-3+14-20=0$

Now $P(-1)=-1-1+1-7+20-12=0$ — 1

$\therefore x=-1$ is a double root of $P(x)=0$

$\therefore (x+1)^2$ is a factor

$\therefore P(x)=(x+1)^2(x^2+4)Q(x)$

Since $P(x)$ is monic and constant $=-12$

then $Q(x)=x-3$

$\therefore \underline{P(x)=(x+1)^2(x^2+4)(x-3)}$ — 1 (4)

c) (i) If $y=x^2$, let $x=\sqrt{y}$ in $P(x)=0$

$\therefore (\sqrt{y})^3+6(\sqrt{y})^2+5\sqrt{y}-8=0$ — 1

$\sqrt{y}(y+5)=8-6y$

square both sides

$y(y+5)^2=(8-6y)^2$

$y(y^2+10y+25)=64-96y+36y^2$

$y^3-26y^2+121y-64=0$ — 1

or $\underline{x^3-26x^2+121x-64=0}$

(ii) If $y=\frac{1}{x}$, let $x=\frac{1}{y}$ in $P(x)=0$

$\left(\frac{1}{y}\right)^3+6\left(\frac{1}{y}\right)^2+5\left(\frac{1}{y}\right)-8=0$ — 1

Mult by y^3

$1+6y+5y^2-8y^3=0$

or $\underline{8x^3-5x^2-6x-1=0}$ — 1 (4)

i.e $\underline{x^3-\frac{5}{8}x^2-\frac{3}{4}x-\frac{1}{8}=0}$

d) Let roots be $\alpha, -\alpha, \beta, \gamma$

$\Sigma\alpha=\beta+\gamma=-p$ --- (1)

$\Sigma\alpha\beta=-\alpha^2+\alpha\beta+\alpha\gamma-\alpha\beta-\alpha\gamma+\beta\gamma=0$

i.e $\beta\gamma-\alpha^2=0$ --- (2)

$\Sigma\alpha\beta\gamma=-\alpha^2\beta-\alpha^2\gamma+\alpha\beta\gamma-\alpha\beta\gamma=-q$

i.e. $-\alpha^2(\gamma+\beta)=-q$ --- (3) — 2

$\alpha\beta\gamma\delta=-\alpha^2\beta\gamma=r$ --- (4)

From (2) $\beta\gamma=\alpha^2$

$\pm$ in (4) $-\alpha^4=r$

$\alpha^4=-r$

$\alpha^2=\sqrt{-r}$, since $\alpha^2>0$ — 1

In (3) $-\sqrt{-r}\times-p=-q$

square both sides

$-r\,p^2=q^2$

$\therefore \underline{q^2+p^2r=0}$ (3)

Q4 (a) when $P(x)$ is div. by $(x-2)(x-3)$ ie x^2-5x+6, then $\deg R(x)<2$.

$\therefore$ let $R(x)=ax+b$ — 1

$\therefore P(x)=(x-2)(x-3)Q(x)+ax+b$

$P(2)=4 \therefore 4=0+2a+b$ ...(1)

$P(3)=9 \therefore 9=0+3a+b$ ...(2) — 1

(2) − (1) gives $a=5$

$\therefore$ In (1) $4=10+b$

$\therefore b=-6$

$\therefore \underline{R(x)=5x-6}$ — (3)

(b)(i) If coeffs are real and $\alpha=1-i$ then $\beta=\bar{\alpha}=1+i$. — 1

since $\alpha\beta+\alpha\gamma+\beta\gamma=-2$

then $2+(1-i)\gamma+(1+i)\gamma=-2$

$\gamma-i\gamma+\gamma+i\gamma=-4$

$\gamma=-2$

$\therefore \alpha+\beta+\gamma=(1-i)+(1+i)+-2=\underline{0}$ — 1

(ii) $\alpha\beta\gamma=(1-i)(1+i)\times-2=\underline{-4}$ — 1

(iii) $P(x)=x^3-(\Sigma\alpha)x^2-2x-\alpha\beta\gamma=0$

$\underline{=x^3-2x+4}$ — 1 (4)

(c)(i) $z^5-5z^4\frac{1}{z}+10z^3\frac{1}{z^2}-10z^2\frac{1}{z^3}+5z\cdot\frac{1}{z^4}-\frac{1}{z^5}$

$\underline{=z^5-5z^3+10z-\frac{10}{z}+\frac{5}{z^3}-\frac{1}{z^5}}$ — 1

(ii) $z-\frac{1}{z}=\cos\theta+i\sin\theta-\frac{1}{\cos\theta+i\sin\theta}$

$=\cos\theta+i\sin\theta-\frac{\cos\theta-i\sin\theta}{\cos^2\theta+\sin^2\theta}$

$=\underline{2i\sin\theta}$ — 1

$z^n-\frac{1}{z^n}=z^n-(z^n)^{-1}=z^n-z^{-n}$

$=\cos n\theta+i\sin n\theta-(\cos(-n\theta)+i\sin(-n\theta))$ by De Moivre.

$=\cos n\theta+i\sin n\theta-\cos n\theta+i\sin n\theta$

$\underline{=2i\sin n\theta}$ — 1

(iii) $(z-\frac{1}{z})^5=(z^5-\frac{1}{z^5})-5(z^3-\frac{1}{z^3})+10(z-\frac{1}{z})$ — 1

$\therefore (2i\sin\theta)^5=2i\sin5\theta-5\times2i\sin3\theta+10\times2i\sin\theta$

$32i\sin^5\theta=2i\sin5\theta-10i\sin3\theta+20i\sin\theta$

$\therefore \underline{\sin^5\theta=\frac{1}{16}\sin5\theta-\frac{5}{16}\sin3\theta+\frac{5}{8}\sin\theta}$

— 1

If $16\sin^5\theta=\sin5\theta$

then $\sin5\theta-5\sin3\theta+10\sin\theta=\sin5\theta$ — 1

$5\sin3\theta-10\sin\theta=0$

$\sin3\theta-2\sin\theta=0$

$3\sin\theta-4\sin^3\theta-2\sin\theta=0$

$\sin\theta-4\sin^3\theta=0$

$\sin\theta(1-4\sin^2\theta)=0$

$\sin\theta=0$ or $\sin^2\theta=\frac{1}{4}$ — 1

$\sin\theta=\pm\frac{1}{2}$

$\therefore \theta=0,\pm\pi,\pm2\pi,\ldots$

or $\theta=\pm\frac{\pi}{6},\pm\frac{5\pi}{6},\pm\frac{2\pi}{6},\ldots$

i.e. $\underline{\theta=n\pi, n\in J}$ — 1

$\underline{\theta=n\pi\pm\frac{\pi}{6}, n\in J}$ (8)

Q5

(a)(i) $x^2+4y^2=100$

$\frac{x^2}{100}+\frac{y^2}{25}=1$

$a=10,\ b=5$

$b^2=a^2(1-e^2)$

$1-e^2=\frac{b^2}{a^2}$

$e^2=1-\frac{b^2}{a^2}$

$=1-\frac{25}{100}$

$=\frac{75}{100}=\frac{3}{4}$

$e=\frac{\sqrt{3}}{2}$ — 1

(ii) Force $=(\pm ae,0)$

$=(\pm 5\sqrt{3},0)$ — 1

(iii) Dir. are $x=\pm\frac{a}{e}$

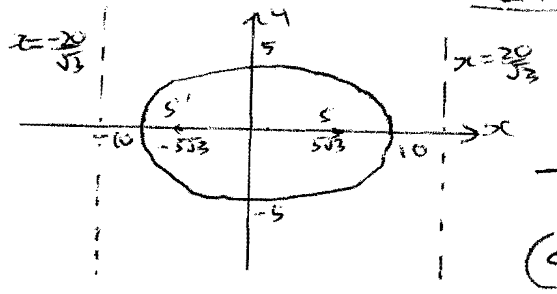
$x=\pm\frac{10}{\frac{\sqrt{3}}{2}}$

$x=\pm\frac{20}{\sqrt{3}}$ — 1

(iv) x-axis $=a=10$ ∴ $(\pm 10,0)$ — 1

y-axis $=b=5$ ∴ $(0,\pm 5)$

(v)



— 1

(5)

(b)(i) If $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$

Diff imp:

$\frac{2x}{a^2}-\frac{2y}{b^2}.y'=0$

$\frac{yy'}{b^2}=\frac{x}{a^2}$

$y'=\frac{b^2x}{a^2y}$ — 1

At P, $m_T=\frac{b^2.a\sec\theta}{a^2 b\tan\theta}$

$=\frac{b}{a\sin\theta}$

$m_N=-\frac{a\sin\theta}{b}$ — 1

∴ The normal is

$y-b\tan\theta=-\frac{a\sin\theta}{b}(x-a\sec\theta)$ — 1

$by-b^2\tan\theta=-ax\sin\theta+a^2\frac{\sin\theta}{\cos\theta}$

$ax\sin\theta+by=(a^2+b^2)\tan\theta$ (3)

(ii) At G, $y=0$ ∴ $x=\frac{(a^2+b^2)\tan\theta}{a\sin\theta}$

$x_G=\frac{(a^2+b^2)}{a\cos\theta}$ — 1

$x_N=a\sec\theta$

∴ $\frac{OG}{ON}=\frac{a^2+b^2}{a\cos\theta}\times\frac{1}{a\sec\theta}$

$=\frac{a^2+b^2}{a^2}=1+\frac{b^2}{a^2}$ — 1

since $b^2=a^2(e^2-1)$ — 1

$\frac{b^2}{a^2}=e^2-1$

$e^2=1+\frac{b^2}{a^2}$

∴ $\frac{OG}{ON}=e^2$ (3)

c) $\frac{y-\frac{3}{q}}{x-3q}=\frac{\frac{3}{p}-\frac{3}{q}}{3p-3q}=\frac{\frac{3(q-p)}{pq}}{3(p-q)}$

$=-\frac{1}{pq}$

$pqy-3p=3q-x$ — 1

$x+pqy=3(p+q)$

If $xy=9$, $y=\frac{9}{x}$ ∴ $y'=-\frac{9}{x^2}$

At $P(3p,\frac{3}{p})$, $m_{Tang}=-\frac{9}{9p^2}=-\frac{1}{p^2}$

At $Q(3q,\frac{3}{q})$, $m_{Tang}=-\frac{1}{q^2}$

∴ Tang at P is $y-\frac{3}{p}=-\frac{1}{p^2}(x-3p)$

$p^2y-3p=3p-x$

$x+p^2y=6p$ — 1

and Tang at Q is $x+q^2y=6q$

At point of int.

$6p-p^2y=6q-q^2y$

$y(p^2-q^2)=6(p-q)$

$y=\frac{6}{p+q}$ — 1

∴ $x=6p-p^2\left(\frac{6}{p+q}\right)$

$=6p-\frac{6p^2}{p+q}$

$=\frac{6p^2+6pq-6p^2}{p+q}$

$=\frac{6pq}{p+q}$ — 1

(4)

∴ $M=\left(\frac{6pq}{p+q},\frac{6}{p+q}\right)$

Q6(a) If $x^2-y^2=c^2$ then

$2x-2yy'=0$

$y'=\frac{x}{y}$

If $P(x_1,y_1)$ is the pt of intersection

$m_{Tang_1}=\frac{x_1}{y_1}$ — 1

If $xy=c^2$ then $y=\frac{c^2}{x}$

and $y'=\frac{-c^2}{x^2}$

$\therefore m_{Tang_2}=\frac{-c^2}{x_1^2}=\frac{-x_1y_1}{x_1^2}=-\frac{y_1}{x_1}$ — 1

$\therefore m_1m_2=\frac{x_1}{y_1}\times\frac{-y_1}{x_1}=-1$

$\therefore$ curves meet at right angles — 1 (3)

b) $x-2y+3=0$ --- (1)

$x^2+2y^2=9$ --- (2)

$\therefore (2y-3)^2+2y^2=9$

$4y^2-12y+9+2y^2=9$

$6y^2-12y=0$

$6y(y-2)=0$

$y=0,\ y=2$

$\therefore x=-3,\ x=1$ — 1

$\therefore A=(-3,0),\ B=(1,2)$

If $x^2+2y^2=9$

Diff Imp $2x+4yy'=0$

$y'=\frac{-x}{2y}$

At A, $m_{Tang}=\frac{3}{0}$ $\therefore$ Vertical Tangent

$\therefore$ Eqn is $x=-3$ — 1

At B, $m_{Tang}=\frac{-1}{4}$

$\therefore$ Tang is $y-2=-\frac{1}{4}(x-1)$

$4y-8=-x+1$

$x+4y-9=0$ — 1

At C, $x=-3$ $\therefore y=3$

$\therefore C=(-3,3)$ — 1 (4)

(c) $\frac{x^2}{9}-\frac{y^2}{4}=1$ $\therefore a=3, b=2$

Since $b^2=a^2(e^2-1)$ then $e^2=1+\frac{b^2}{a^2}$

$e^2=\frac{13}{9}$ $\therefore e=\frac{\sqrt{13}}{3}$

$\therefore S,S'=(\pm ae,0)=(\pm\sqrt{13},0)$

Dir are $x=\pm\frac{a}{e}$ $\therefore x=\pm\frac{9}{\sqrt{13}}$ — 1

$m_{PS}=m_{asym}$

$\therefore \frac{2\tan\theta-0}{3\sec\theta-\sqrt{13}}=-\frac{2}{3}$

$6\tan\theta=-6\sec\theta+2\sqrt{13}$

$6\tan\theta+6\sec\theta=2\sqrt{13}$

$2\tan\theta+2\sec\theta=\frac{2\sqrt{13}}{3}$ — 1

If $4x^2-9y^2=36$

Diff Imp $8x-18yy'=0$

$y'=\frac{4x}{9y}$

$\therefore m_{Tang}=\frac{12\sec\theta}{18\tan\theta}=\frac{2\sec\theta}{3\tan\theta}$

$\therefore$ Tangent is

$y-2\tan\theta=\frac{2\sec\theta}{3\tan\theta}(x-3\sec\theta)$ — 1

At asymptote $y=\frac{-2x}{3}$

$\therefore -\frac{2x}{3}-2\tan\theta=\frac{2\sec\theta}{3\tan\theta}(x-3\sec\theta)$

$-2x\tan\theta-6\tan^2\theta=2x\sec\theta-6\sec^2\theta$

$x(2\sec\theta+2\tan\theta)=6(\sec^2\theta-\tan^2\theta)$

From above

$\frac{2\sqrt{13}}{3}x=6$ — 1 (4)

$x=\frac{9}{\sqrt{13}}=\frac{a}{e}=\text{dir.}$

(d) If $PQ=1$ then $(x_2-x_1)^2+(y_2-y_1)^2=1$

$M_{PQ}=\left(\frac{x_1+x_2}{2},\frac{y_1+y_2}{2}\right)$ — 1

$=\left(\frac{x_1+x_2}{2},\frac{\frac{9}{x_1}+\frac{9}{x_2}}{2}\right)=\left(\frac{x_1+x_2}{2},\frac{9(x_2+x_1)}{2x_1x_2}\right)$

If $M=(X,Y)$ then $X=\frac{x_1+x_2}{2}$, $Y=\frac{9X}{x_1x_2}$

$\therefore x_1x_2=\frac{9X}{Y}$ — 1

Now if $(x_2-x_1)^2+(y_2-y_1)^2=1$

then $(x_2+x_1)^2-4x_1x_2+(y_2+y_1)^2-4y_1y_2=1$

$(2X)^2-4x_1x_2+(2Y)^2-4\times\frac{9}{x_1}\times\frac{9}{x_2}=1$

$4X^2-\frac{36X}{Y}+4Y^2-324\times\frac{Y}{9X}=1$ — 1

$4X^2-\frac{36X}{Y}+4Y^2-\frac{36Y}{X}=1$

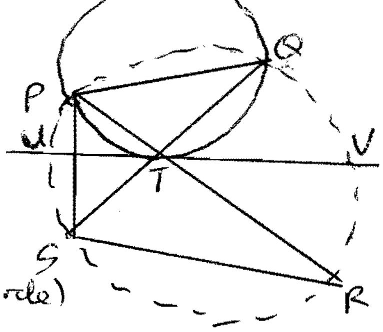
$4X^3Y-36X^2+4XY^3-36Y^2=XY$

$4(X^3Y-9X^2+XY^3-9Y^2)-XY=0$

$4[X^2(XY-9)+Y^2(XY-9)]-XY=0$

$4(XY-9)(X^2+Y^2)-XY=0$ (4)

Q7(a) Let U, V be on either side of T on the tangent



Let $\angle QPT = \alpha$

$\therefore \angle QTV = \alpha$ — 1

($\angle$ bet tang & chord = $\angle$ in alt seg of small circle)

$\angle QSR = \alpha$ — 1

($\angle$'s on circum of large circle stand on same arc)

$\angle UTS = \alpha$ (Vert. Opp. $\angle$s) — 1

$\therefore \angle QSR = \angle UTS$ (Both $= \alpha$) — 1

$\therefore UT \parallel SR$ (Alt. $\angle$'s equal) (4)

(b)(i) $\cos(A+B) = \cos A\cos B - \sin A\sin B$

$\cos(A-B) = \cos A\cos B + \sin A\sin B$ — 1

Adding:

$\cos(A+B) + \cos(A-B) = 2\cos A\cos B$

(ii) Let $X = A+B$, $Y = A-B$

Adding gives $X+Y = 2A$

$\therefore A = \frac{X+Y}{2}$

Subt. gives $X - Y = 2B$

$B = \frac{X-Y}{2}$ — 1

$\therefore \cos X + \cos Y = 2\cos\left(\frac{X+Y}{2}\right)\cos\left(\frac{X-Y}{2}\right)$

(iii) $\cos\theta + \cos 3\theta = 0$

$2\cos\frac{\theta+3\theta}{2}\cos\frac{\theta-3\theta}{2} = 0$

$2\cos 2\theta\cos(-\theta) = 0$ — 1

$\therefore \cos\theta = 0 \quad \therefore \theta = \frac{\pi}{2}, \frac{3\pi}{2}$ — 1

If $\cos 2\theta = 0$, $2\theta = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \frac{7\pi}{2}$

$\therefore \theta = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$ (4)

c) $\frac{x-3}{x^2-x} \times \frac{(x^2-x)^2}{1} \geq -2(x^2-x)^2$

$(x-3)(x^2-x) \geq -2(x^2-x)^2$ — 1

$2(x^2-x)^2 + (x-3)(x^2-x) \geq 0$

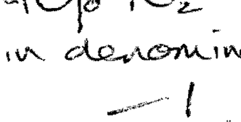
$(x^2-x)[2(x^2-x)+x-3] \geq 0$

$(x^2-x)(2x^2-x-3) \geq 0$

$x(x-1)(2x-3)(x+1) \geq 0$ — 1

$\therefore x \leq -1,\ 0 \leq x \leq 1,$ — 1

$x \geq 1\frac{1}{2}$

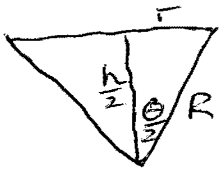


But $x(x-1) \neq 0$ in denominator

$\therefore x \neq 0, 1$ — 1

$\therefore x \leq -1,\ 0 < x < 1,\ x \geq 1\frac{1}{2}$ (4)

(d)(i) In right triangle



$\frac{r}{R} = \sin\frac{\theta}{2}$

$r = R\sin\frac{\theta}{2}$

$r^2 = R^2\sin^2\frac{\theta}{2}$

But $\cos 2\alpha = 1 - 2\sin^2\alpha$ — 1

$\therefore \sin^2\alpha = \frac{1}{2}(1-\cos 2\alpha)$

$\therefore \sin^2\frac{\theta}{2} = \frac{1}{2}(1-\cos\theta)$

$\therefore r^2 = \frac{R^2(1-\cos\theta)}{2}$

(ii) Also $\frac{\frac{h}{2}}{R} = \cos\frac{\theta}{2}$

$h = 2R\cos\frac{\theta}{2}$

$h^2 = 4R^2\cos^2\frac{\theta}{2}$

But $\cos 2\alpha = 2\cos^2\alpha - 1$ — 1

$\therefore \cos^2\alpha = \frac{1}{2}(1+\cos 2\alpha)$

$\therefore \cos^2\frac{\theta}{2} = \frac{1}{2}(1+\cos\theta)$

$\therefore h^2 = \frac{4R^2}{2}(1+\cos\theta)$

$h^2 = 2R^2(1+\cos\theta)$

(iii) Curved S.A. $= 2\pi rh$

$= 2\pi \frac{R\sqrt{1-\cos\theta}}{\sqrt{2}} \times \sqrt{2}R\sqrt{1+\cos\theta}$

$= 2\pi R^2\sqrt{1-\cos^2\theta}$

$= 2\pi R^2\sin\theta$, for $0 \leq \theta \leq \pi$

$\frac{dA}{d\theta} = 2\pi R^2\cos\theta$

$= 0$ when $\cos\theta = 0$

$\therefore \theta = \frac{\pi}{2}$ — 1

$\frac{d^2A}{d\theta^2} = -2\pi R^2\sin\theta$

$= -2\pi R^2$ when $\theta = \frac{\pi}{2}$

< 0

$\therefore$ Max when $\theta = \frac{\pi}{2}$

$\therefore h = \sqrt{2}R\left(1+\cos\frac{\pi}{2}\right)$

$= \sqrt{2}R$ (3)

Q8(a) Prove true for $n=1$

$a^1+b^1=(a+b).1$ ∴ True for $n=1$

Assume true for $n=2k-1$

i.e $a^{2k-1}+b^{2k-1}=(a+b)P$ — 1

Prove true for $n=2k+1$

i.e $a^{2k+1}+b^{2k+1}=(a+b)Q$.

$LHS=a^2.a^{2k-1}+b^{2k+1}$

$=a^2[(a+b)P-b^{2k-1}]+b^{2k+1}$ — 1

$=(a+b)a^2P-a^2b^{2k-1}+b^{2k+1}$

$=(a+b)a^2P-b^{2k-1}(a^2-b^2)$

$=(a+b)a^2P-b^{2k-1}(a-b)(a+b)$

$=(a+b)[a^2P-(a-b)b^{2k-1}]$ — 1

$=(a+b)Q$ where

$Q=a^2P-(a-b)b^{2k-1}$

∴ True for $n=2k+1$ if true for $n=2k-1$

Since true for $n=1$ and since true for $n=2k+1$ if true for $n=2k-1$ then it is true for $n=3,5,7,\ldots$ — 1

∴ True for all positive odd integer values of n. (4)

(b) $y=(x+1)^x$

∴ $\ln y=\ln(x+1)^x$

$=x\ln(x+1)$ — 1

Diff imp.

$\frac{1}{y}y'=\ln(x+1).1+x.\frac{1}{x+1}$

$=\ln(x+1)+\frac{x}{x+1}$ — 1

∴ $y'=\left(\ln(x+1)+\frac{x}{x+1}\right)(x+1)^x$ (2)

(c) $y=2+\frac{1}{x^2-1}$

Vert. Asym. when $x=\pm1$ — 1

Hor. Asym. when $y=2$ — 1

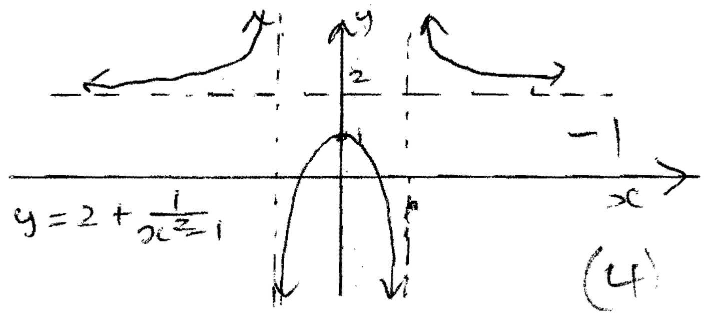
$y'=\frac{-2x}{(x^2-1)^2}$

St. pts when $y'=0$ ∴ $x=0$

If $x<0$, $y'>0$ ∴ Max at $(0,1)$ — 1

If $x>0$, $y'<0$

$f(a)=f(-a)=2+\frac{1}{a^2-1}$ ∴ even.



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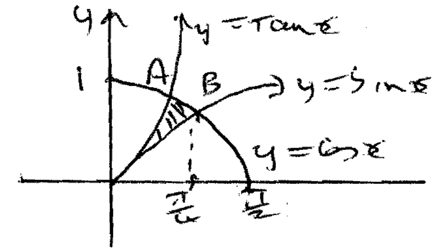
(4)

(d)



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At A, $\cos x=\tan x$

$=\frac{\sin x}{\cos x}$

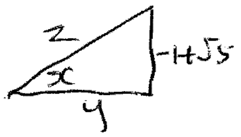
∴ $\cos^2x=\sin x$

$1-\sin^2x=\sin x$

$\sin^2x+\sin x-1=0$

$\sin x=\frac{-1\pm\sqrt{1+4}}{2}=\frac{-1\pm\sqrt5}{2}$

Since x is acute $\sin x=\frac{-1+\sqrt5}{2}$ — 1



$y=\sqrt{4-(-1+\sqrt5)^2}$

$=\sqrt{4-1+2\sqrt5-5}$

$=\sqrt{2\sqrt5-2}$

∴ $\cos x=\frac{\sqrt{2\sqrt5-2}}{2}$ — 1

At B, $\sin x=\cos x$

$x=\frac{\pi}{4}$

$\text{Area}=\int_0^A\tan x\,dx+\int_A^B\cos x\,dx-\int_0^B\sin x\,dx$ — 1

$=[-\ln(\cos x)]_0^A+[\sin x]_A^B+[\cos x]_0^B$

$=-\ln\left(\frac{\sqrt{2\sqrt5-2}}{2}\right)-\ln1+\sin\frac{\pi}{4}-\left(\frac{-1+\sqrt5}{2}\right)+\cos\frac{\pi}{4}-1$

$=-\ln\left(\frac{\sqrt{\sqrt5-1}}{\sqrt2}\right)+\frac{1}{\sqrt2}+\frac{1-\sqrt5}{2}+\frac{1}{\sqrt2}-1$

$=\sqrt2-1+\frac12-\frac{\sqrt5}{2}-\ln\left(\frac{\sqrt{\sqrt5-1}}{\sqrt2}\right)$

$=\sqrt2-\frac12-\frac{\sqrt5}{2}-\ln\frac{\sqrt{\sqrt5-1}}{\sqrt2}\ u^2$ — 1

(5)