BAULKHAM HILLS HIGH SCHOOL

HALF YEARLY EXAMINATION

2010

YEAR 12

MATHEMATICS
EXTENSION 2

Time Allowed: 2 hours
plus 5 minutes reading time

Instructions:

- Attempt all questions.
- Do not write on this question paper. Use the answer booklets provided
- Start a new page for **each** question.
- Write your student number at the top of each page
- Use black or blue pen only
- Board approved calculators may be used
- Staple your answer pages together in ONE bundle with the QUESTIONS stapled at the back of your solutions.

Marks

Question 1 Start on a new page (10 marks)

a) Given $z = 1 - 3i$ and $w = 2 + i$

i) Express zw in the form $a + ib$ **1**

ii) Find $|zw|$ and Arg (zw) **3**

iii) Hence, find x if $\dfrac{\sqrt{2}\,(\cos x - i\sin x)}{2+i} = \dfrac{1-3i}{5}$ and $0 \le x \le \dfrac{\pi}{2}$. **3**

b) On an Argand diagram, shade the region containing all points representing complex numbers z such that Re $(z) \le 1$ and $0 \le \text{Arg}\,(z + i) \le \dfrac{\pi}{4}$. **3**

Question 2 Start on a new page (10 marks)

a) i) Express $\sqrt{8-6i}$ in the form $a + ib$ where a and b are real. **3**

ii) Hence solve $2z^2 + (1 - 3i)\,z - 2 = 0$ expressing the answers in the form $c + id$ where c and d are real. **2**

b) The roots of $x^3 + 6x^2 + 5x - 8 = 0$ are α, β and γ.

Find

i) the value of $\alpha^3 + \beta^3 + \gamma^3$ **3**

ii) the monic polynomial with roots α^2, β^2 and γ^2. **2**

Marks

Question 3 Start on a new page (10 marks)

a) i) Show that $z = i$ is a root of $(2 - i)z^2 - (1+i)z + 1 = 0$. **1**

ii) Find the other root in the form $z = a + ib$ where a and b are real numbers. **2**

b) i) If $P(x) = 0$ where $P(x)$ is a polynomial of degree n (where ≥ 2) has a double root at $x = \alpha$, prove that $x = \alpha$ is a single root of $P'(x)$. **2**

ii) Hence find the double root of $x^3 + x^2 - 5x + 3 = 0$. **2**

c) Given $2 + i$ and $1 - 3i$ are two roots of the equation $x^4 + bx^3 + cx^2 + dx + e = 0$ where b, c, d and e are real numbers

i) write down the other two roots, giving a reason for your answers. **1**

ii) hence find the values of b and e only. **2**

Question 4 Start on a new page (10 marks)

a) i) If w is a complex cube root of unity, prove that $1 + w + w^2 = 0$ **2**

ii) Form an equation whose roots are 4, $1 + w$, $1 + w^2$ **3**

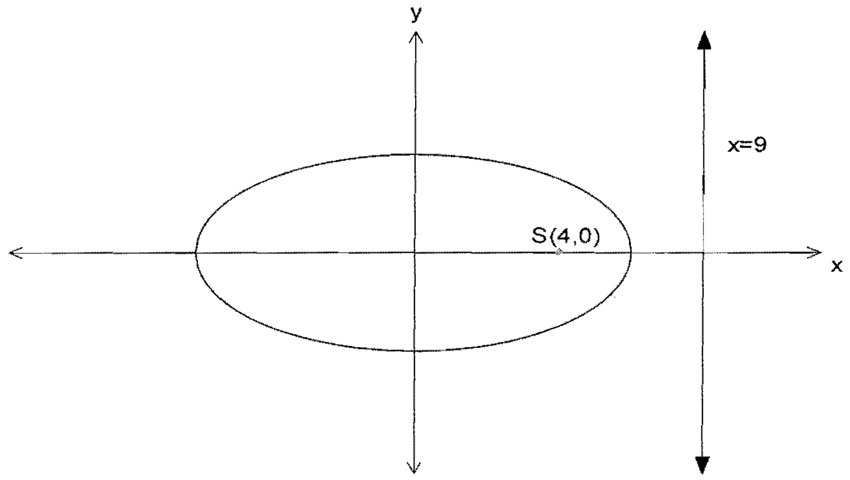
b) An ellipse has equation $\frac{x^2}{9} + y^2 = 1$.

i) Find the condition for the line $y = mx + c$ to be a tangent to the ellipse. **4**

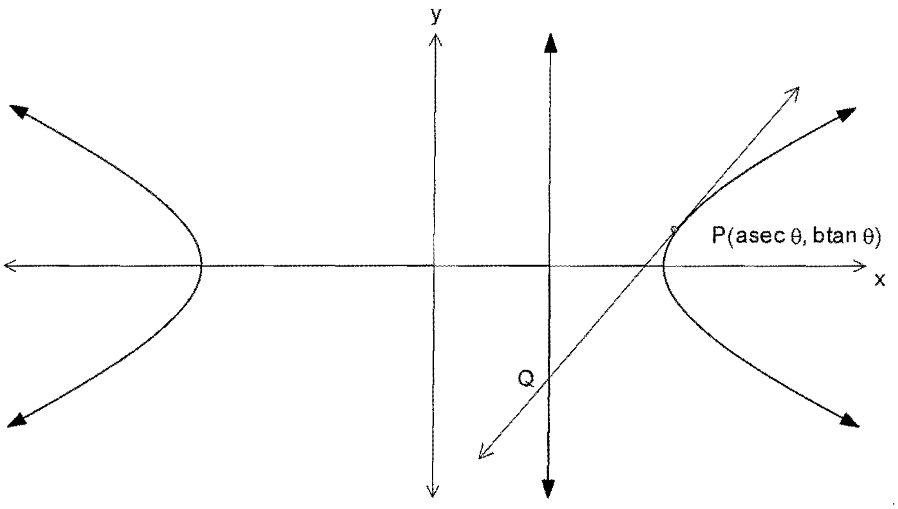
ii) Hence what are the equations of the tangents with a gradient of $\frac{1}{3}$. **1**

Question 5 Start on a new page (10 marks)

a) Find the equation of the ellipse shown in the diagram below. **3**
A focus and directrix are shown.



b)



i) Derive the equation of the tangent to the hyperbola $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$ at the point $P\ (a \sec\theta,\ b \tan\theta)$ **3**

ii) This tangent meets the directrix at Q. Find the coordinates of Q. **1**

iii) If S is the focus, find the gradient of QS **1**

(iv) Prove that PQ subtends a right angle at S **2**

Question 6 Start on a new page (10 marks)

a) Consider the ellipse $\frac{x^2}{9} + \frac{y^2}{16} = 1$

i) Find its eccentricity **1**

ii) Find the coordinates of its foci S and S' **1**

iii) Find the equations of its directrices **1**

iv) Sketch the ellipse showing the foci, intercepts and directrices **1**

b) If β is the acute angle between the asymptotes of the hyperbola $\frac{x^2}{r^2} - \frac{y^2}{s^2} = 1$ with eccentricity e (where $r > s$)

i) find β and e in terms of r and s **3**

ii) hence show that $e = \sec\frac{\beta}{2}$ **3**

Question 7 Start on a new page (10 marks)

a) It is given that $|z| = z + \bar{z}$.
Sketch on an Argand diagram the locus of the point P representing z. **2**

b) i) Express the roots of the equation $z^5 + 32 = 0$ in modulus/argument form. **2**

ii) Hence show that

$$z^4 - 2z^3 + 4z^2 - 8z + 16 = (z^2 - 4\cos\frac{\pi}{5}z + 4)(z^2 - 4\cos\frac{3\pi}{5}z + 4)$$ **3**

iii) Hence find the exact value of $\cos\frac{\pi}{5}$ in simplest surd form. **3**

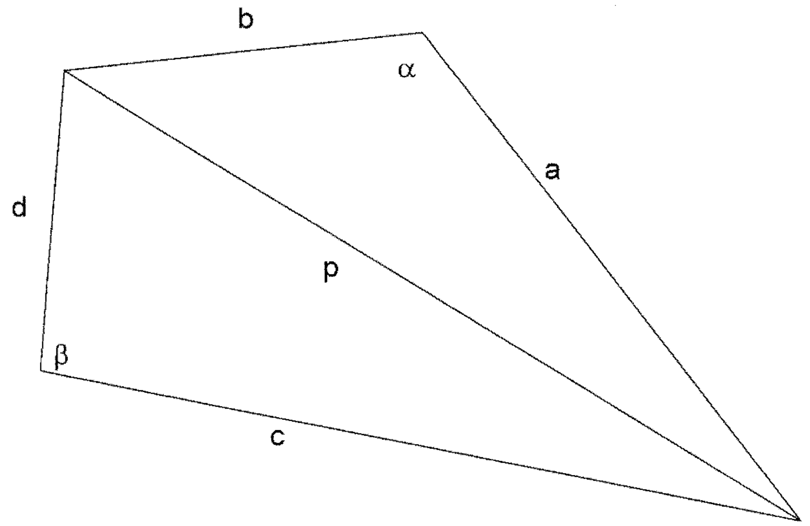
Question 8 Start a new page

a) Two of the roots of the quartic $6x^4 - 13x^3 - 90x^2 + 208x - 96 = 0$ are equal in magnitude but opposite in sign.

The other two roots are reciprocals of each other.

Find the four roots. **3**

b) Four lengths, a, b, c and d are joined together to form a quadrilateral.



i) Find an expression for the area, A, of the quadrilateral. **1**

ii) Prove that $a^2 + b^2 - c^2 - d^2 = 2ab\cos\alpha - 2cd\cos\beta$ **1**

iii) Show that $\dfrac{dA}{d\alpha} = \dfrac{1}{2}\left(ab\cos\alpha + cd\cos\beta\dfrac{d\beta}{d\alpha}\right)$ and that **2**

$$\frac{d\beta}{d\alpha} = \frac{ab\sin\alpha}{cd\sin\beta}$$

iv) Prove that the greatest area occurs when the quadrilateral is cyclic **3**

End of Paper

2010 YR 12 HALF YEARLY EXT 2 SOLNS

1 a i) $(1-3i)(2+i) = 2+i-6i-3i^2$

$= 5-5i$ ✓

ii) $|5-5i| = \sqrt{25+25}$

$= 5\sqrt{2}$ ✓

$\arg(5-5i) = \tan^{-1}(-1)$ ✓

$= -\frac{\pi}{4}$ ✓

⅔ for $\frac{\pi}{4}$

(iii) $\frac{\sqrt{2}(\cos x - i\sin x)}{2+i} = \frac{1-3i}{5}$

$\sqrt{2}(\cos x - i\sin x) = \frac{5-5i}{5}$ ✓

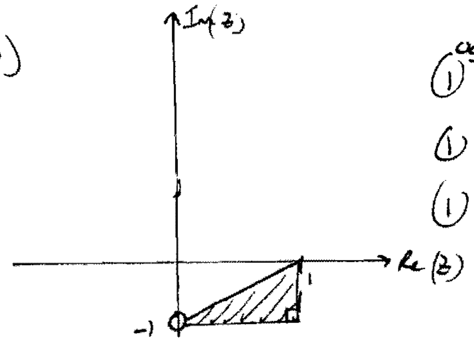
$\sqrt{2}(\cos x - i\sin x) = 1-i$ ✓

$\therefore \sqrt{2}\cos x = 1 \quad -\sqrt{2}\sin x = -1$ equating real and imaginary parts

$\therefore \tan x = 1$

$x = \frac{\pi}{4}$ ✓

1(b)



(1) $\arg(z+i) = \frac{\pi}{4}$

(1) $\mathrm{Re}(z) \leq 1$

(1) excluding $z=-i$

2a(i) $\sqrt{8-6i} = x+iy$

$8-6i = x^2-y^2+2xyi$

$\therefore \begin{rcases} 8 = x^2-y^2 \\ -6 = 2xy \end{rcases}$ ✓

$y = \frac{-3}{x}$

$\therefore 8 = x^2 - \frac{9}{x^2}$

$8x^2 = x^4-9$

$x^4-8x^2-9=0$

$(x^2-9)(x^2+1)=0$ ✓

$x^2=9 \qquad x^2+1=0$

$x=\pm 3 \qquad$ no real sol for $x \quad \therefore y = \mp 1$ ✓

$\therefore \pm(3-i) = \sqrt{8-6i}$

(ii) $2z^2+(1-3i)z-2=0$

$z = \frac{-(1-3i) \pm \sqrt{(1-3i)^2+4\times 2\times 2}}{4}$ ✓

$z = \frac{-(1-3i) \pm \sqrt{1-6i-9+16}}{4}$

$z = \frac{-(1-3i) \pm \sqrt{8-6i}}{4}$

$z = \frac{-1+3i \pm (3-i)}{4}$

$z = \frac{2+2i}{4}, \frac{-4+4i}{4}$

$z = \frac{1}{2}+\frac{i}{2}, \; -1+i$ ✓

2(b) i) $\alpha^3 = -6\alpha^2-5\alpha+8$

$\beta^3 = -6\beta^2-5\beta+8$

$\gamma^3 = -6\gamma^2-5\gamma+8$

$\alpha^3+\beta^3+\gamma^3 = -6(\gamma^2+\alpha^2+\beta^2)-5(\alpha+\beta+\gamma)+24$ ✓

Now $\alpha^2+\beta^2+\gamma^2 = (\alpha+\beta+\gamma)^2-2(\alpha\beta+\alpha\gamma+\beta\gamma)$

$= (-6)^2-2\times 5$

$= 26$ ✓

$\therefore \alpha^3+\beta^3+\gamma^3 = -6\times 26-5\times -6+24$

$= -102$ ✓

2b(ii) let $y=x^2 \quad \therefore$ roots are $\alpha^2, \beta^2, \gamma^2$

ie $x=\sqrt{y}$ ✓

$y\sqrt{y}+6y+5\sqrt{y}-8=0$

$[\sqrt{y}(y+5)]^2 = [-6y+8]^2$

$y(y^2+10y+25) = 36y^2-96y+64$

$y^3-26y^2+121y-64=0$

$\therefore$ Eqn is $x^3-26x^2+121x-64=0$ ✓

3 a(i) $P(z) = (2-i)z^2 - (1+i)z + 1$

$P(i) = (2-i)(-1) - i(1+i) + 1$

$= -2 + i - i + 1 + 1$

$= 0$

$\therefore z = i$ is a root ✓

(ii) $\therefore z - i$ is a factor. By inspection,

$(2-i)z^2 - (1+i)z + 1 = (z-i)((2-i)z + i)$ ✓

$\therefore$ other root is $(2-i)z = i$,

$z = \frac{-i}{2-i} \cdot \frac{2+i}{2+i}$

$z = \frac{1-2i}{5}$

$= \frac{1}{5} - \frac{2i}{5}$ ✓

3(b) let $P(x) = (x-\alpha)^2 Q(x)$ ✓

$P'(x) = Q(x) . 2(x-\alpha) + (x-\alpha)^2 Q'(x)$

$P'(x) = (x-\alpha)(2Q(x) + (x-\alpha) Q'(x))$ ✓

$P'(\alpha) = 0 . (T(x))$

$= 0$

$\therefore x = \alpha$ is also a root of $P'(x)$

(ii) let $P(x) = x^3 + x^2 - 5x + 3$

$P'(x) = 3x^2 + 2x - 5$

$P'(x) = (3x+5)(x-1)$

$\therefore P'(x) = 0 \rightarrow x = 1, -\frac{5}{3}$ ✓

Test in $P(x)$ $P(1) = 1 + 1 - 5 + 3$

$= 0$

$\therefore x = 1$ is the double root ✓

3 c(i) $2-i$ and $1+3i$ since coeffecients are real, roots are in conjugate pairs

(ii) sum of roots

$(2-i) + (2+i) + (1+3i) + (1-3i) = -b$

$6 = -b$

$b = -6$ ✓

Product $(2+i)(2-i)(1-3i)(1+3i) = e$

$5 \times 10 = e$

$e = 50$ ✓

$\therefore b = -6$ and $e = 50$

4 a(i) For $z^3 - 1 = 0$

$z^3 - 1 = (z-1)(z^2 + z + 1)$

If w is a complex root

$w^3 - 1 = (w-1)(w^2 + w + 1)$ ✓

But $w \neq 1$ (as it is complex)

$\therefore 0 = w^2 + w + 1$ ✓

(ii) sum of roots 1 at a time $= 4 + 1 + w + (-w)$

$= 5$ ✓ for 1 correct expression

sum two at a time:

$4(1+w) + 4(1+w^2) + (1+w)(1+w^2)$

$= 4 + 4w + (4 - 4 - w) + (1+w)(-w)$

$= 4 - w - w^2$ ✓ for 2 correct expressions

$= 5$

three at a time $4(1+w)(-w)$

$= 4(-w - w^2)$

$= 4$

$\therefore$ Eqn is $x^3 - 5x^2 + 5x - 4 = 0$ (or equivalent) ✓

4(b)(i) $\frac{x^2}{9} + (mx+c)^2 = 1$ ✓

$\frac{x^2}{9} + m^2x^2 + 2mcx + c^2 = 1$

$x^2 + 9m^2x^2 + 18mcx + 9c^2 - 9 = 0$

$(9m^2+1)x^2 + 18mcx + 9c^2 - 9 = 0$ ✓

To be a tangent $\Delta = 0$

$(18mc)^2 - 4(9m^2+1)9(c^2-1) = 0$ ✓

$36(9m^2c^2 - (9m^2c^2 - 9m^2 + c^2 - 1)) = 0$

$9m^2 = c^2 - 1$ ✓

(ii) when $m = \frac{1}{3}$

$1 = c^2 - 1$

$c = \pm\sqrt{2}$

$\therefore y = \frac{x}{3} \pm \sqrt{2}$ ✓

5 a) Focus $(ae, 0)$ Directrix $x = \frac{a}{e}$

$ae = 4$ $\qquad$ $\frac{a}{e} = 9 \rightarrow e = \frac{a}{9}$

$\frac{a^2}{9} = 4$

$a^2 = 36 \rightarrow a = 6$ ✓

$\therefore e = \frac{2}{3}$ ✓

$b^2 = a^2(1 - e^2)$

$b^2 = 36\left(1 - \frac{4}{9}\right)$

$b^2 = 20$

$\therefore$ Eqn is $\frac{x^2}{36} + \frac{y^2}{20} = 1$ ✓

5 b) i) Different. w.r.t. x.

$\frac{2x}{a^2} - \frac{2y}{b^2} \cdot \frac{dy}{dx} = 0$

$\frac{dy}{dx} = \frac{xb^2}{a^2y}$ ✓

At P, $m = \frac{a\sec\theta b^2}{a^2 b\tan\theta}$

Eqn of tangent ✓

$y - b\tan\theta = \frac{a\sec\theta b}{a\tan\theta}(x - a\sec\theta)$

$\frac{y\tan\theta}{b} - \tan^2\theta = \frac{x\sec\theta}{a} - \sec^2\theta$

$\therefore \frac{x\sec\theta}{a} - \frac{y\tan\theta}{b} = 1$ is eqn. ✓

5 b(ii) At directrix $x = \frac{a}{e}$

$\frac{a}{e}\frac{\sec\theta}{a} - \frac{y\tan\theta}{b} = 1$

$\frac{\sec\theta}{e} - 1 = \frac{y\tan\theta}{b}$

$y = \frac{b}{\tan\theta}\left(\frac{\sec\theta - e}{e}\right)$

$\therefore Q$ is $\left(\frac{a}{e}, \frac{b(\sec\theta - e)}{e\tan\theta}\right)$ or equivalent ✓

b(iii) $m_{QS} = \dfrac{\frac{b(\sec\theta - e)}{e\tan\theta} - 0}{\frac{a}{e} - ae}$

$= \frac{b(\sec\theta - e)}{\tan\theta(a - ae^2)}$ ✓

iv) $m_{PS} = \frac{b\tan\theta - 0}{a\sec\theta - ae}$

$= \frac{b\tan\theta}{a(\sec\theta - e)}$ ✓

$m_{PS} \cdot m_{QS} = \frac{b\tan\theta}{a(\sec\theta - e)} \cdot \frac{b(\sec\theta - e)}{\tan\theta\, a(1 - e^2)}$

$= \frac{b^2}{a^2(1 - e^2)}$

$= \frac{b^2}{-a^2(e^2 - 1)}$ ✓

$= -1$ as $b^2 = a^2(e^2 - 1)$

$\therefore PS \perp QS$

6 (a)(i) $9 = 16(1 - e^2)$

$\frac{9}{16} = 1 - e^2$

$e^2 = \frac{7}{16}$

$e = \frac{\sqrt{7}}{4}$ ✓

(ii) Foci are: $(0, \pm be)$

$= \left(0, \pm 4 \times \frac{\sqrt{7}}{4}\right)$

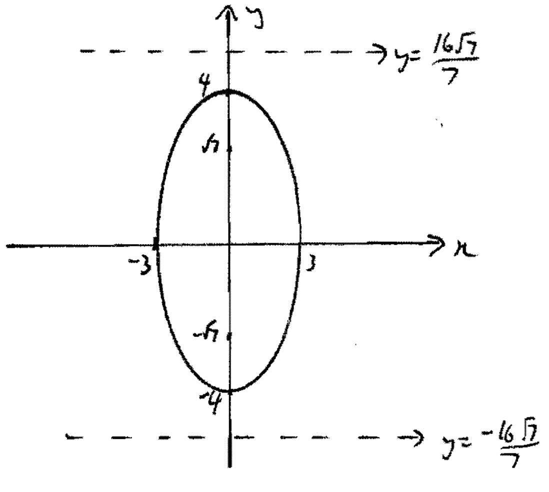
$= (0, \pm\sqrt{7})$ ✓

(iii) $y = \pm\frac{b}{e}$

$= \pm\frac{4}{\frac{\sqrt{7}}{4}}$

$= \pm\frac{16}{\sqrt{7}}$

$y = \pm\frac{16\sqrt{7}}{7}$ ✓



✓

(b) $s^2 = r^2(e^2-1)$

$\frac{s^2}{r^2} = e^2 - 1$

$e^2 = \frac{s^2+r^2}{r^2}$

$e = \frac{\sqrt{s^2+r^2}}{r}$ (since $e>0$) ✓

Asymptotes are $y = \pm\frac{s}{r}x$

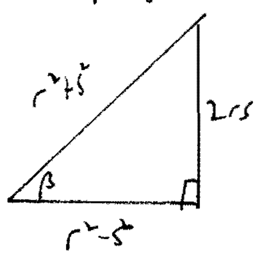
$\tan\beta = \left|\frac{m_1-m_2}{1+m_1m_2}\right|$

$= \left|\frac{\frac{s}{r} - -\frac{s}{r}}{1+\left(-\frac{s^2}{r^2}\right)}\right|$ ✓

$= \left|\frac{\frac{2s}{r}}{\frac{r^2-s^2}{r^2}}\right|$

$\tan\beta = \left|\frac{2sr}{r^2-s^2}\right|$

$\tan\beta = \frac{2rs}{r^2-s^2}$ as $r>s$ ∴ $\beta = \tan^{-1}\left(\frac{2rs}{r^2-s^2}\right)$ ✓



$\cos\beta = \frac{r^2-s^2}{r^2+s^2}$ ✓

$\cos\left(2\frac{\beta}{2}\right) = 2\cos^2\frac{\beta}{2} - 1$ ✓

$\frac{r^2-s^2}{r^2+s^2} = 2\cos^2\frac{\beta}{2} - 1$

$\frac{r^2-s^2+r^2+s^2}{r^2+s^2} = 2\cos^2\frac{\beta}{2}$

$\frac{2r^2}{2(r^2+s^2)} = \cos^2\frac{\beta}{2}$

$\sec\frac{\beta}{2} = \frac{\sqrt{r^2+s^2}}{r}$ ✓

$\sec\frac{\beta}{2} = e$

7 (a) $|z| = z+\bar{z}$ let $z = x+iy$

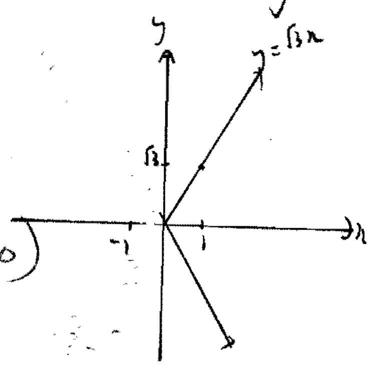
$\sqrt{x^2+y^2} = 2x$ ($x \geq 0$)

$x^2+y^2 = 4x^2$

$3x^2 - y^2 = 0$

$y^2 = 3x^2$

$y = \pm\sqrt{3}x$ ✓ (for $x>0$)



7b(i) $z^5 = +32(\cos(\pi+2k\pi) + i\sin(\pi+2k\pi))$ ✓

$z = 2\left(\cos\left(\frac{\pi+2k\pi}{5}\right) + i\sin\left(\frac{2k\pi+\pi}{5}\right)\right)$ $k=0,1,2,3,4$ ✓

$z_1 = 2\left(\cos\frac{\pi}{5} + i\sin\frac{\pi}{5}\right)$

$z_2 = 2\left(\cos\left(\frac{3\pi}{5}\right) + i\sin\left(\frac{3\pi}{5}\right)\right)$

$z_3 = 2(\cos\pi + i\sin\pi) = -2$

$z_4 = 2\left(\cos\frac{7\pi}{5} + i\sin\frac{7\pi}{5}\right) = 2\,\text{cis}\left(-\frac{3\pi}{5}\right)$

$z_5 = 2\left(\cos\frac{9\pi}{5} + i\sin\frac{9\pi}{5}\right) = 2\,\text{cis}\left(-\frac{\pi}{5}\right)$

(ii) $z^5+32 = (z+2)(z^4-2z^3+4z^2-8z+16)$

Roots of $z^5+32=0$ are z_1, z_2, z_3, z_4, z_5.

∴ Roots of $z^4-2z^3+4z^2-8z+16=0$ are complex roots of $z^5+32=0$ ie $z_1, z_2, z_4=\bar{z}_2, z_5=\bar{z}_1$ ✓

$z^4-2z^3+4z^2-8z+16 = (z-z_1)(z-\bar{z}_1)(z-z_2)(z-\bar{z}_2)$ ✓

$z_1+\bar{z}_1 = 4\cos\frac{\pi}{5}$ $z_1\bar{z}_1 = 4$ $z_2+\bar{z}_2 = 4\cos\frac{3\pi}{5}$ $z_2\bar{z}_2 = 4$

$z^4-2z^3+4z^2-8z+16 = (z^2-(z_1+\bar{z}_1)z+z_1\bar{z}_1)(z^2-(z_2+\bar{z}_2)z+z_2\bar{z}_2)$

$= \left(z^2-4\cos\frac{\pi}{5}z+4\right)\left(z^2-4\cos\frac{3\pi}{5}z+4\right)$

(iii) equating coefficients of z:

$-8 = -16\cos\frac{\pi}{5} - 16\cos\frac{3\pi}{5}$ ✓

$1 = 2\left(\cos\frac{\pi}{5} + \cos\frac{3\pi}{5}\right)$

equating coeff of z^2

$4 = 4 + 16\cos\frac{\pi}{5}\cos\frac{3\pi}{5} + 4$

$\cos\frac{\pi}{5}\cos\frac{3\pi}{5} = -\frac{1}{4}$

$\cos\frac{3\pi}{5} = \frac{-1}{4\cos\pi/5}$

$\therefore 1 = 2\cos\frac{\pi}{5} - \frac{1}{2\cos\frac{\pi}{5}}$

$2\cos\frac{\pi}{5} = 4\cos^2\frac{\pi}{5} - 1$

$4\cos^2\frac{\pi}{5} - 2\cos\frac{\pi}{5} - 1 = 0$ ✓

$\cos\frac{\pi}{5} = \frac{2 \pm \sqrt{4+16}}{8}$

$= \frac{1\pm\sqrt{5}}{4}$

since $\cos\frac{\pi}{5} > 0$, $\cos\frac{\pi}{5} = \frac{1+\sqrt{5}}{4}$ ✓

8(a) let roots be $\alpha, -\alpha, \beta, \frac{1}{\beta}$

sum of roots $= \beta + \frac{1}{\beta} = \frac{13}{6}$ ✓

$6\beta^2 + 6 = 13\beta$

$6\beta^2 - 13\beta + 6 = 0$

$(2\beta - 3)(3\beta - 2) = 0$

$\beta = \frac{3}{2}, \beta = \frac{2}{3}$ ✓

product of roots

$-\alpha^2 = \frac{-96}{6}$

$\alpha = \pm 4$ ✓

$\therefore$ Roots are $4, -4, \frac{2}{3}, \frac{3}{2}$

b(i) $A = \frac{1}{2}ab\sin\alpha + \frac{1}{2}cd\sin\beta$ ✓

(ii) using cosine rule

$p^2 = a^2+b^2 - 2ab\cos\alpha$

and $p^2 = c^2+d^2 - 2cd\cos\beta$

$\therefore a^2+b^2 - 2ab\cos\alpha = c^2+d^2 - 2cd\cos\beta$

$a^2+b^2-c^2-d^2 = 2ab\cos\alpha - 2cd\cos\beta$ ✓

(iii) $\frac{dA}{d\alpha} = \frac{1}{2}ab\cos\alpha + \frac{1}{2}cd\cos\beta\cdot\frac{d\beta}{d\alpha}$

$= \frac{1}{2}\left(ab\cos\alpha + cd\cos\beta\frac{d\beta}{d\alpha}\right)$ ✓

$\frac{d(a^2+b^2-c^2-d^2)}{d\alpha} = -2ab\sin\alpha + 2cd\sin\beta\cdot\frac{d\beta}{d\alpha}$

$0 = -2ab\sin\alpha + 2cd\sin\beta\frac{d\beta}{d\alpha}$

$2ab\sin\alpha = 2cd\sin\beta\frac{d\beta}{d\alpha}$ ✓

$\frac{d\beta}{d\alpha} = \frac{ab\sin\alpha}{cd\sin\beta}$

(iv) $\frac{dA}{d\alpha} = \frac{1}{2}\left(ab\cos\alpha + cd\cos\beta\,\frac{ab\sin\alpha}{cd\sin\beta}\right)$ ✓

$= \frac{1}{2}ab\left(\frac{\cos\alpha\sin\beta + \sin\alpha\cos\beta}{\sin\beta}\right)$

For max or min, $\frac{dA}{d\alpha} = 0$

$\frac{1}{2}ab\left(\frac{\sin(\alpha+\beta)}{\sin\beta}\right) = 0$ ✓

ie $\sin(\alpha+\beta) = 0$

$\alpha+\beta = 0 \rightarrow$ min area

$\alpha+\beta = \pi \rightarrow$ max area

$\therefore \alpha$ and β are supplementary

$\therefore$ Quadrilateral is cyclic ✓