BAULKHAM HILLS HIGH SCHOOL

2011

YEAR 12 HALF YEARLY EXAMINATION

Mathematics Extension 2

General Instructions

- Reading time – 5 minutes
- Working time – 2 hours
- Write using black or blue pen
- Board-approved calculators may be used
- A table of standard integrals is provided on the back of the cover sheet
- All necessary working should be shown in every question
- Marks may be deducted for careless or badly arranged work

Total marks – 80

- Attempt Questions 1 – 8
- All questions are of equal value
- Start a separate piece of paper for each question.
- Put your BOS number and the question number at the top of each sheet.

Total marks – 80
Attempt Questions 1 – 8
All questions are of equal value
Answer each question on a SEPARATE piece of paper clearly marked Question 1, Question 2, etc. Each piece of paper must show your BOS number.

	Marks
Question 1 ***(10 marks)*** Use a *separate* piece of paper	
a) Given that $z = -\sqrt{3} - i$, evaluate;	
(i) $\bar{z}$	1
(ii) $\lvert z \rvert$	1
(iii) $\arg z$	1
(iv) $\dfrac{1}{z}$	1
(v) z^4 in the form $x + iy$	2
b) (i) On an Argand Diagram sketch the locus of a point corresponding to the complex number z, where $\lvert z-3 \rvert = 3$	2
(ii) Use your diagram in (i) to explain why $\arg(z-3) = \arg z^2$	2
Question 2 ***(10 marks)*** Use a *separate* piece of paper	
a) For the hyperbola $\dfrac{y^2}{12} - \dfrac{x^2}{4} = 1$, find the;	
(i) eccentricity.	1
(ii) coordinates of the focii.	1
(iii) equations of the directrices.	1
(iv) equations of the asymptotes.	1
b) The polynomial $P(z)$ has the equation $P(z) = z^4 - 2z^3 - z^2 + 2z + 10$. Given that $(z-2+i)$ is a factor of $P(z)$, express $P(z)$ as a product of two real quadratic factors.	3
c) Solve the inequality $\dfrac{4x}{3+2x-x^2} \le 1$	3

Marks

<u>Question 3</u> ***(10 marks)*** Use a *separate* piece of paper

a) (i) Find a monic cubic equation with roots α, β and γ such that $\alpha\beta\gamma = 5$, $\alpha+\beta+\gamma = 7$ and $\alpha^2+\beta^2+\gamma^2 = 29$. *2*

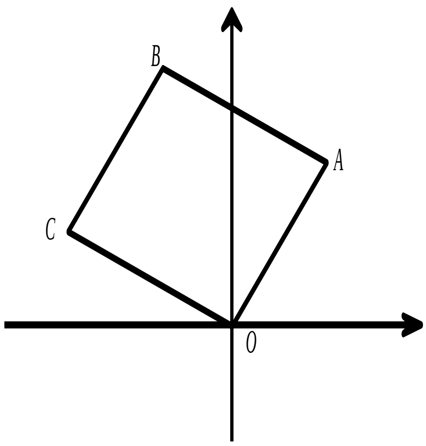
(ii) Find $\alpha^3+\beta^3+\gamma^3$ *2*

b) (i) Given that $z = \cos\theta + i\sin\theta$, prove that $z^n + \dfrac{1}{z^n} = 2\cos n\theta$ *2*

(ii) Hence, or otherwise, solve $2z^4+3z^3+5z^2+3z+2=0$ *4*

<u>Question 4</u> ***(10 marks)*** Use a *separate* piece of paper

a) In the square *OABC*, shown below, the point *A* represents the complex number *z* and the point *C* represents the complex number ω.



(i) Find $\arg\left(\dfrac{z-\omega}{z+\omega}\right)$, *2*

(ii) If $z = 2+i$, what complex numbers do *B* and *C* represent? *3*

b) How many different ways are there of arranging four married couples at a circular table with men and women in alternate positions, and no wife next to her husband? *2*

c) Prove that $1 + x + \dfrac{x^2}{2!} + \dfrac{x^3}{3!} + \ldots + \dfrac{x^n}{n!} = 0$ *3*

has no multiple roots for $n \geq 1$

Marks

<u>Question 5</u> ***(10 marks)*** Use a *separate* piece of paper

a) The equation $x^3 - 6x^2 + 11x - 6 = 0$, has roots α, β and γ.

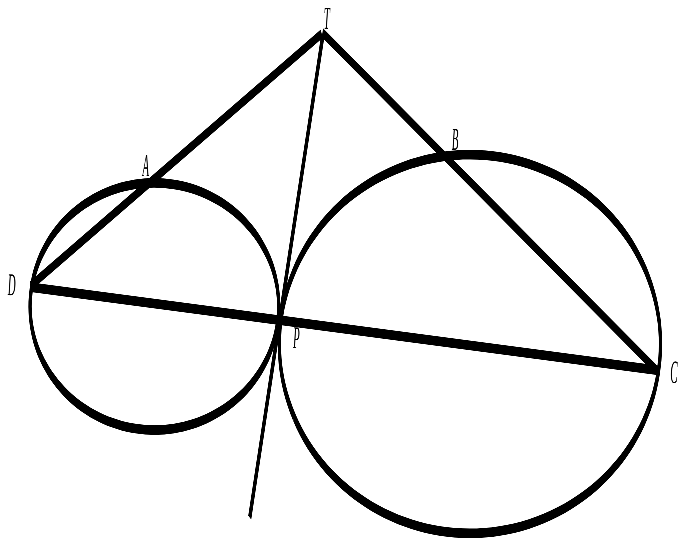
(i) Without finding the roots α, β and γ, find the equation with roots $\alpha - 2$, $\beta - 2$ and $\gamma - 2$. *2*

(ii) Hence, or otherwise, solve $x^3 - 6x^2 + 11x - 6 = 0$. *3*

b) In the diagram, two circles touch at *P* and share a common tangent *TP*.

C lies on the circumference of one circle and *D* lies on the circumference of the other, such that *D*, *P* and *C* are collinear.

TC and *TD* intersect the circles at *B* and *A* respectively.



(Note: on page 7 there is a copy of the above diagram that you can use, please detach and include with your solutions)

(i) Show, giving reasons, that *ATBP* is a cyclic quadrilateral. *3*

(ii) Show, giving reasons, that *ABCD* is a cyclic quadrilateral. *2*

Marks

Question 6 ***(10 marks)*** Use a *separate* piece of paper

a) Let ω be one of the cube roots of unity.

(i) Find the two possible values of $1+\omega+\omega^2$. *2*

(ii) If ω is one of the non-real cube roots of unity, find the value of $(2-\omega)(2-\omega^2)(2-\omega^4)(2-\omega^5)$ *2*

b) The tangent to the rectangular hyperbola $xy=4$ at the point $T\left(2t,\frac{2}{t}\right)$ has the equation $x+t^2y=4t$. (Do not prove this)

(i) The tangent cuts the x-axis at the point Q. Show that the line through Q which is perpendicular to the tangent at T has equation $t^2x-y=4t^3$. *2*

(ii) This line through Q cuts the rectangular hyperbola at the points R and S. Show that the midpoint M of RS has coordinates $M\left(2t,-2t^3\right)$. *2*

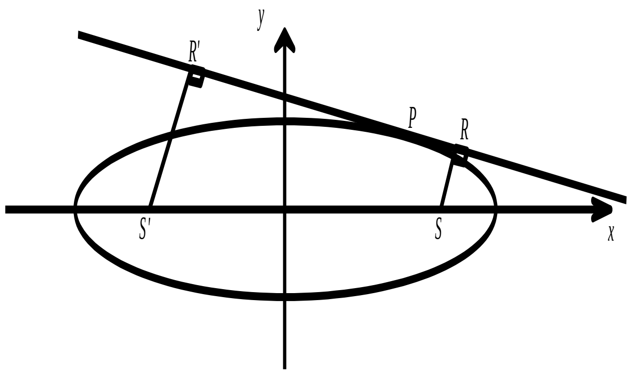
(iii) Find the equation of the locus of M as T moves on the rectangular hyperbola, stating any restrictions that may apply. *2*

Question 7 ***(10 marks)*** Use a *separate* piece of paper

a) The complex number $\frac{\sqrt{3}}{2}+\frac{i}{2}$ is one of the nth roots of – 1. *2*
Find the least value of n for this to be so.

b) Solve the equation $x^4+x^3+x^2+x+1=0$ *3*

c) Consider the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, where $a > b > 0$.



(i) Show that the tangent to the ellipse at the point $(a\cos\theta, b\sin\theta)$ has the equation $bx\cos\theta+ay\sin\theta-ab=0$. *2*

(ii) R and R' are the feet of the perpendiculars from the foci S and S' to the tangent at P. Show that $SR\times S'R'=b^2$. *3*

Marks

<u>Question 8</u> ***(10 marks)*** Use a *separate* piece of paper

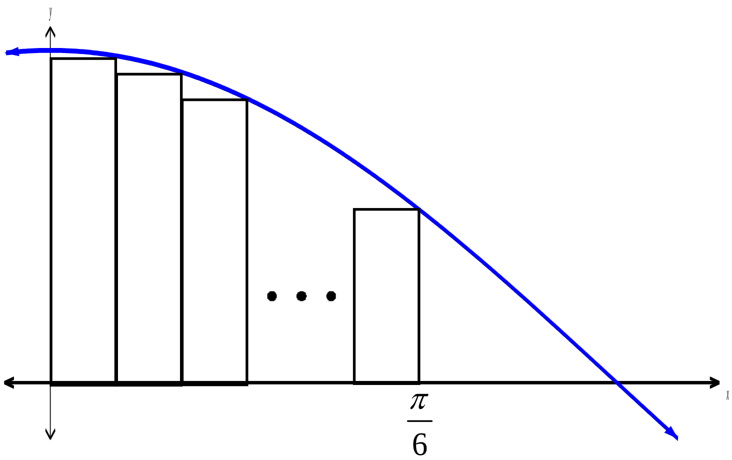
a) 47231 is a five digit number whose digits sum to 17 (i.e. $4+7+2+3+1=17$) *2*
How many five digit numbers sum to 43?

b) (i) Prove by mathematical induction, that for all positive integers n; *4*

$$\frac{1}{2}+\cos\theta+\cos 2\theta+\ldots+\cos(n-1)\theta=\frac{\sin\frac{1}{2}(2n-1)\theta}{2\sin\frac{1}{2}\theta}$$

(Hint: you may use the identity $2\cos A\sin B=\sin(A+B)-\sin(A-B)$)

(ii) In the diagram, n rectangles of equal width are constructed under the curve *2*
$y=\cos x$ from $x=0$ to $x=\frac{\pi}{6}$.



Use the result of part (i) to show that the sum S_n of the areas of the n rectangles is given by

$$S_n=\frac{\pi}{12n}\left\{\left(\sqrt{3}-1\right)+\frac{\sin(2n-1)\frac{\pi}{12n}}{\sin\frac{\pi}{12n}}\right\}$$

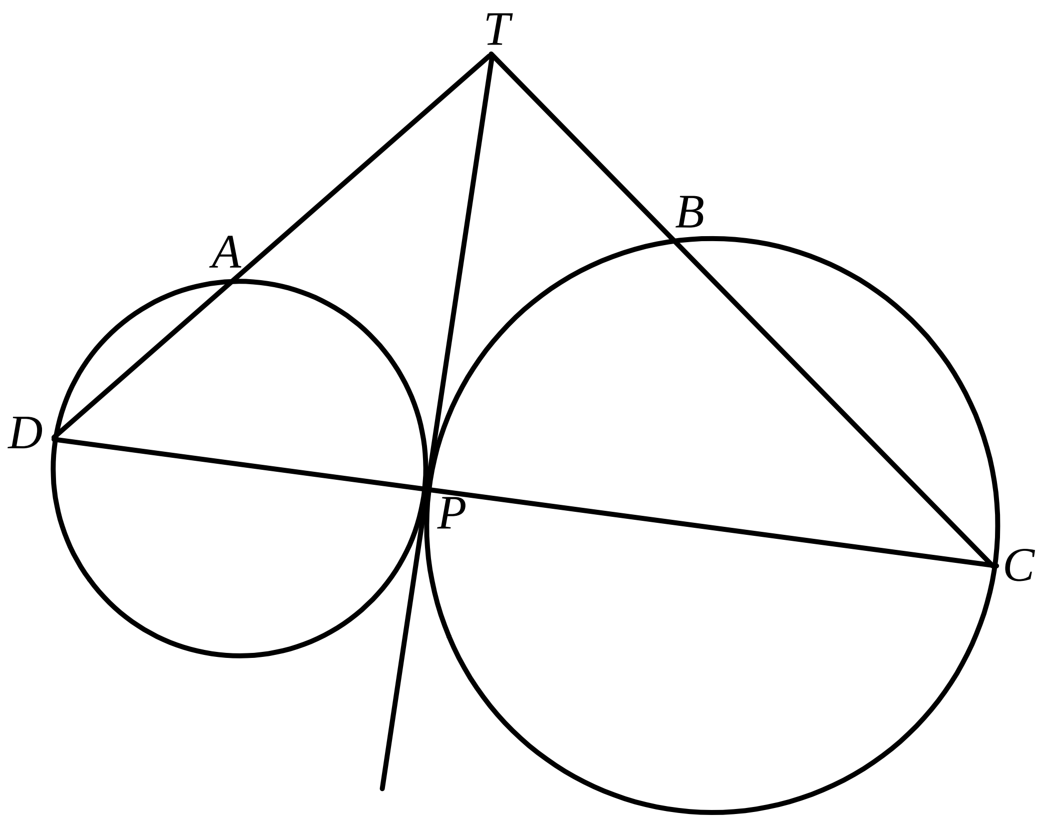
(iii) Hence find $\lim_{n\to\infty} S_n$ *2*

(Hint: Let $n=\frac{1}{h}$)

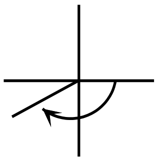
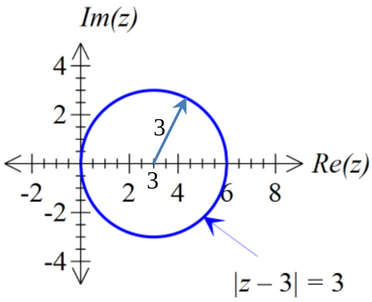
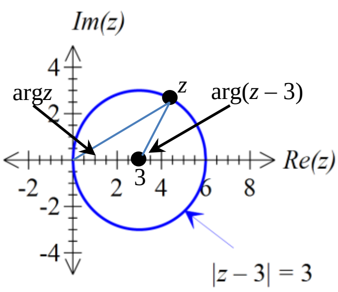
End of paper

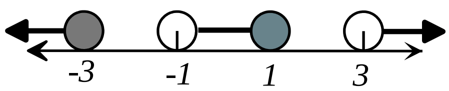
Question 5 b)

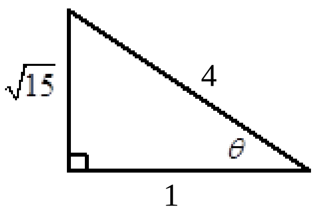
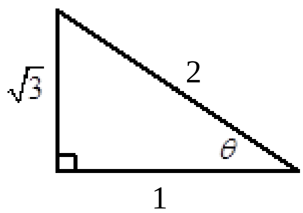
Please detach and include with your solutions

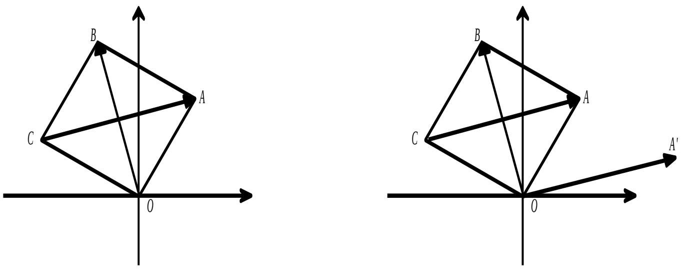
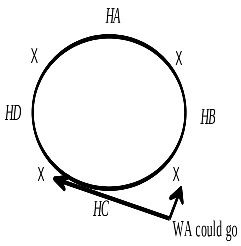


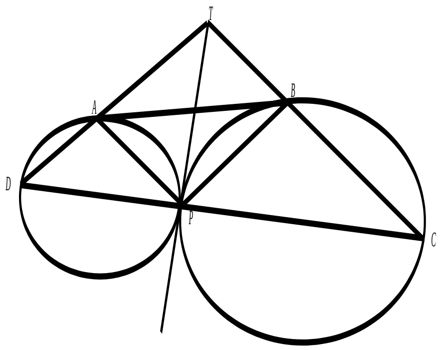
BAULKHAM HILLS HIGH SCHOOL
EXTENSION 2 HALF YEARLY 2011 SOLUTIONS

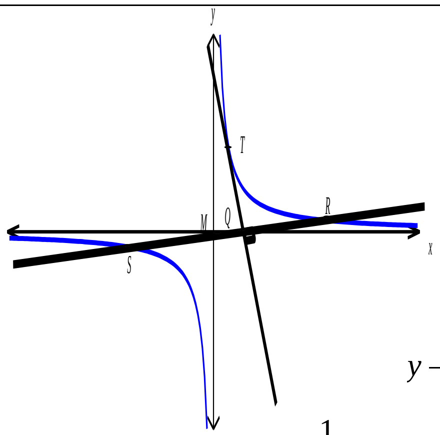
Solution	Marks	Comments
QUESTION 1		
1a) (i) $\bar{z} = -\sqrt{3} + i$	**1**	
1a) (ii) $\lvert z\rvert = \sqrt{(-\sqrt{3})^2 + (-1)^2}$ $= 2$	**1**	
1a) (iii) $\arg z = \tan^{-1}\left(\frac{-1}{-\sqrt{3}}\right)$ $= -\frac{5\pi}{6}$ 	**1**	*Note: argument should be quoted* $-\pi < \arg z \leq \pi$
1a) (iv) $\frac{1}{z} = \frac{\bar{z}}{\lvert z\rvert^2}$ $= \frac{-\sqrt{3} + i}{4}$ $= -\frac{\sqrt{3}}{4} + \frac{i}{4}$	**1**	
1a) (v) $z^4 = \left(2\text{cis}\left(-\frac{5\pi}{6}\right)\right)^4$ $= 2^4\text{cis}\left(-\frac{20\pi}{6}\right)$ $= 16\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right)$ $= -8 + 8\sqrt{3}\,i$	**2**	**1 mark** • Correct use of De Moivre's theorem or equivalent merit • Correctly changing from mod-arg to $x + iy$ *Note: correct use of student's answers to (ii) & (iii) should gain full marks*
1b) (i) 	**2**	**1 mark** • Recognises the locus is a circle
1b) (ii)  $\arg(z - 3) = 2\arg z$ $(\angle$ *at centre twice* $\angle$ *at circumference*$)$ $\therefore \arg(z - 3) = \arg z^2$	**2**	**1 mark** • Uses diagram to make progress towards a solution

Solution	Marks	Comments
QUESTION 2		
2a) (i) $a^2 = b^2(e^2 - 1)$ $4 = 12(e^2 - 1)$ $e^2 - 1 = \frac{1}{3}$ $e^2 = \frac{4}{3}$ $e = \frac{2}{\sqrt{3}}$	**1**	
2a) (ii) $\left(0\ ,\ \pm\sqrt{12}\ \times \frac{2}{\sqrt{3}}\right)$ $= (0\ , \pm 4)$	**1**	**1 mark** • Correct use of formulae and foci located on x axis and directrices parallel to y axis
2a) (iii) $y = \pm\sqrt{12}\ \times \frac{\sqrt{3}}{2}$ $y = \pm 3$	**1**	
2a) (iv) $y = \pm\frac{\sqrt{12}}{2}x$ $y = \pm\sqrt{3}\,x$	**1**	
2b) If $(z - 2 + i)$ is a factor, so is $(z - 2 - i)$ $\therefore (z^2 - 4z + 5)$ is a factor $P(z) = z^4 - 2z^3 - z^2 + 2z + 10$ $= (z^2 - 4z + 5)(z^2 + 2z + 2)$	**3**	**2 marks** • Finds one quadratic factor **1 mark** • Identifies conjugate as another root
2c) $\frac{4x}{3 + 2x - x^2} \le 1$ $3 + 2x - x^2 \ne 0$ $x^2 - 2x - 3 \ne 0$ $(x - 3)(x + 1) \ne 0$ $x \ne 3 \ or\ x \ne -1$ $\frac{4x}{3 + 2x - x^2} = 1$ $4x = 3 + 2x - x^2$ $x^2 + 2x - 3 = 0$ $(x + 3)(x - 1) = 0$ $x = -3 \ or\ x = 1$  $x \le -3 \ or\ -1 < x \le 1 \ or\ x > 3$	**3**	**3 marks** • Correct graphical solution on number line or algebraic solution, with correct working **2 marks** • Bald answer • Identifies the four correct critical points via a correct method • Correct conclusion to their critical points obtained using a correct method **1 mark** • Uses a correct method • Acknowledges a problem with the denominator. **0 marks** • Solves like a normal equation , with no consideration of the denominator.

Solution	Marks	Comments
QUESTION 3		
3a) (i) $\alpha^2 + \beta^2 + \gamma^2 = (\alpha + \beta + \gamma)^2 - 2(\alpha\beta + \alpha\gamma + \beta\gamma)$ $29 = 7^2 - 2(\alpha\beta + \alpha\gamma + \beta\gamma)$ $2(\alpha\beta + \alpha\gamma + \beta\gamma) = 20$ $\alpha\beta + \alpha\gamma + \beta\gamma = 10$ $\therefore$ equation is $x^3 - 7x^2 + 10x - 5 = 0$	**2**	**1 mark** • Correctly evaluates $\Sigma\alpha\beta$ • Correct equation for their $\Sigma\alpha\beta$
3a) (ii) $\sum\alpha^3 - 7\sum\alpha^2 + 10\sum\alpha - 15 = 0$ $\sum\alpha^3 = 7\sum\alpha^2 - 10\sum\alpha + 15$ $= 7(29) - 10(7) + 15$ $= 148$	**2**	**2 marks** • Correctly solution for incorrect equation found in part (i) **1 mark** • Correctly deriving a formula for $\Sigma\alpha^3$ using their equation. • Substitutes correctly into an incorrect $\Sigma\alpha^3$ obtained using a correct method.
3b) (i) $z^n + \frac{1}{z^n} = (\text{cis}\theta)^n + (\text{cis}\theta)^{-n}$ $= \text{cis}(n\theta) + \text{cis}(-n\theta)$ $= \cos n\theta + i\sin n\theta + \cos n\theta + i\sin(-n\theta)$ $= \cos n\theta + i\sin n\theta + \cos n\theta - i\sin n\theta$ $= 2\cos n\theta$	**2**	**1 mark** • Uses De Moivre's Theorem in an attempt to show desired result.
3b) (ii) $2z^4 + 3z^3 + 5z^2 + 3z + 2 = 0$ $2z^2 + 3z + 5 + \frac{3}{z} + \frac{2}{z^2} = 0$ $2\left(z^2 + \frac{1}{z^2}\right) + 3\left(z + \frac{1}{z}\right) + 5 = 0$ $4\cos 2\theta + 6\cos\theta + 5 = 0$ $8\cos^2\theta - 4 + 6\cos\theta + 5 = 0$ $8\cos^2\theta + 6\cos\theta + 1 = 0$ $(4\cos\theta + 1)(2\cos\theta + 1) = 0$ $\cos\theta = -\frac{1}{4}$ *or* $\cos\theta = -\frac{1}{2}$   $\therefore z = -\frac{1}{4} \pm \frac{\sqrt{15}}{4}i$ *or* $z = -\frac{1}{2} \pm \frac{\sqrt{3}}{2}i$	**4**	**3 marks** • Correctly finding values for $\cos\theta$ • Correct solution for incorrect values of $\cos\theta$ **2 marks** • Correctly finding the equation $4\cos 2\theta + 6\cos\theta + 5 = 0$ or equivalent **1 mark** • Dividing equation by z^2

Solution	Marks	Comments
QUESTION 4		
4a) (i)  $\overrightarrow{OB} = z + \omega$ *translate* $\overrightarrow{CA}$ *to origin* $\overrightarrow{CA} = z - \omega$ $\angle BOA' = \frac{\pi}{2}$ (*diagonals* $\perp$ *in square*) $\arg\left(\frac{z-\omega}{z+\omega}\right) = \arg(z-\omega) - \arg(z+\omega)$ $= -\left[\arg(z+\omega) - \arg(z-\omega)\right]$ $= -\frac{\pi}{2}$	2	**1 mark** • Linking the problem to the diagonals of the square • Answer of $\frac{\pi}{2}$
4a) (ii) $C = iz$ $= i(2+i)$ $= -1 + 2i$ $B = z + \omega$ $= 2 + i + (-1 + 2i)$ $= 1 + 3i$	3	**2 mrks** • Correctly finds one of B or C and makes progress towards finding other. **1 mark** • Correctly finds B or C • States $B = z + \omega$ and $C = iz$, but does not evaluate *Note: statements may be made in part (i)*
4b) ① Place husbands around table = 3! ② Wife A has two spots to choose ③ Wife B (or D) then only has one spot ④ Leaving only one spot for each of the other two wives. *Ways* $= 3! \times 2 \times 1 \times 1 \times 1$ $= 12$  	2	**1 mark** • Partially correct logic that leads to an incorrect answer.
4c) $P(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots + \frac{x^n}{n!}$ $P'(x) = 1 + \frac{2x}{2!} + \frac{3x^2}{3!} + \ldots + \frac{nx^{n-1}}{n!}$ $P'(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots + \frac{x^{n-1}}{(n-1)!}$ In order for multiple roots to exist $P(x) = P'(x) = 0$ $P(x) = P'(x)$ $1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots + \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots + \frac{x^{n-1}}{(n-1)!}$ $\frac{x^n}{n!} = 0$ $x = 0$ But $P(0) = 1 \neq 0$, $\therefore$ $x = 0$ is not a root, thus no multiple roots exist.	3	**2 marks** • Finds $P'(x)$ and equates with $P(x)$ to find possible multiple root. **1 mark** • Uses $P'(x) = 0$ to find multiple roots.

Solution	Marks	Comments
QUESTION 5		
5a) (i) *Let* $y = x - 2$ $x = y + 2$ $x^3 - 6x^2 + 11x - 6 = 0$ $(y + 2)^3 - 6(y + 2)^2 + 11(y + 2) - 6 = 0$ $y^3 + 6y^2 + 12y + 8 - 6y^2 - 24y - 24 + 11y + 22 - 6 = 0$ $y^3 - y = 0$	**2**	**1 mark** • Substitutes correctly for *x*.
5a) (ii) $y^3 - y = 0$ $y\left(y^2 - 1\right) = 0$ $y = 0$ *or* $y = \pm 1$ $\therefore x - 2 = 0$ *or* $x - 2 = \pm 1$ $x = 2$ *or* $x = 2 \pm 1$ $x = 2$ *or* $x = 1$ *or* $x = 3$	**3**	**2 marks** • Finds some of the correct roots of original equation • Finds all three roots using incorrect roots of equation found in (i) **1 mark** • Finds roots of equation found in (i)
5b) (i)  $\angle TPB = \angle BCP$ (alternate segment theorem) $\angle TPA = \angle ADP$ (“) $\angle TCD + \angle TDC + \angle CTD = 180^\circ$ ($\angle$ sum of ΔTCD) $\therefore \angle TPB + \angle TPA + \angle CTD = 180^\circ$ $\angle APB = \angle TPB + \angle TPA$ (common $\angle$) $\angle APB + \angle CTD = 180^\circ$ $\therefore ATBP$ is a cyclic quadrilateral (opposite $\angle$'s supplementary)	**3**	**2 marks** • Correct solution with poor reasoning • Significant progress towards solution with good reasoning. **1 mark** • Significant progress towards solution with poor reasoning. • Progress towards solution with good reasoning.
5b) (ii) $\angle TPA = \angle TBA$ ($\angle$'s in same segment are =) $\therefore \angle PDA = \angle TBA$ ($\angle TPA = \angle PDA$, proven in (i)) $\therefore ABCD$ is a cyclic quadrilateral (exterior $\angle$ = opposite interior $\angle$)	**2**	**1 mark** • Correct solution with poor reasoning • Progress towards solution with good reasoning.

Solution	Marks	Comments
QUESTION 6		
6a) (i) If ω is a complex cube root then other roots are 1 and ω^2 $\therefore 1+\omega+\omega^2=0$ (sum of roots of $x^3-1=0$) If ω is a real cube root then $\omega=1$ $\therefore 1+\omega+\omega^2=1+1+1^2$ $=3$ $\therefore 1+\omega+\omega^2=0 \text{ or } 3$	**2**	**1 mark** • Correctly find one possible value.
6a) (ii) $(2-\omega)\left(2-w^2\right)\left(2-\omega^4\right)\left(2-w^5\right)$ ($\omega^3=1$) $=(2-\omega)\left(2-\omega^2\right)(2-w)\left(2-\omega^2\right)$ $=(2-\omega)^2(3+\omega)^2$ ($\omega^2=-1-\omega$) $=\left(6-\omega-\omega^2\right)^2$ $=(7)^2$ $=49$ ($1=-\omega-\omega^2$)	**2**	**1 mark** • Uses $\omega^3=1$. • Uses $1+\omega+\omega^2=0$
6b) (i)  x intercepts occur when $y=0$ $t^2x-0=4t^3$ $t^2x=4t^3$ $x=4t$ Slope of tangent $=-\dfrac{1}{t^2}$ $\therefore$ required slope $=t^2$ $y-0=t^2(x-4t)$ $y=t^2x-4t^3$ $t^2x-y-4t^3=0$	**2**	**1 mark** • Finds x intercept. • Finds slope
6b) (ii) Solving line and hyperbola simultaneously $xy=4$ $x\left(t^2x-4t^3\right)=4$ $t^2x^2-4t^3x-4=0$ The roots of this equation are the x coordinates of R and S. $x_M=\dfrac{\Sigma\alpha\beta}{2}$ $=\dfrac{4t^3}{2t^2}$ $=2t$ $y_M=t^2(2t)-4t^3$ $=2t^3-4t^3$ $=-2t^3$ $\therefore M\left(2t,-2t^3\right)$	**2**	**1 mark** • Successfully finds x or y coordinate • Uses $\Sigma\alpha\beta$ in attempt to find midpoint
6b) (iii) $x=2t$ $t=\dfrac{x}{2}$ $y=-2t^3$ $=-2\left(\dfrac{x}{2}\right)^3$ $=-\dfrac{x^3}{4}$ However $t\neq 0$, $\therefore x\neq 0$ Locus is $y=-\dfrac{x^3}{4}$, excluding the point (0,0)	**2**	**1 mark** • Identifies locus without the point of exclusion

<table>
<tr><th>Solution</th><th>Marks</th><th>Comments</th></tr>
<tr><td colspan="3">QUESTION 7</td></tr>
<tr><td>7a) $\frac{\sqrt{3}}{2}+\frac{i}{2}=\text{cis}\frac{\pi}{6}$
$\left(\text{cis}\frac{\pi}{6}\right)^n = -1$
$\text{cis}\frac{n\pi}{6}=\text{cis}\pi$
$\frac{n\pi}{6}=\pi$
$\therefore \quad n=6$</td><td>2</td><td>1 mark
• Another multiple of 6.
• $\text{cis}\frac{n\pi}{6}=\text{cis}\pi$</td></tr>
<tr><td>7b) $x^4+x^3+x^2+x+1=0$
$\frac{(x-1)\left(x^4+x^3+x^2+x+1\right)}{x-1}=0$
$\frac{x^5-1}{x-1}=0$
$x^5-1=0 \quad , \quad x\neq 1$
$x=\text{cis}\frac{2\pi}{5}\ ,\ \text{cis}\ -\frac{2\pi}{5},\ \text{cis}\frac{4\pi}{5},\ \text{cis}\ -\frac{4\pi}{5}$</td><td>3</td><td>2 marks
• Finds the solutions, but includes $x=1$
• Incorrectly solves $x^5-1=0$ but acknowledges $x\neq 1$
1 mark
• Attempts to link the question with roots of unity</td></tr>
<tr><td>7c) (i) $x=a\cos\theta$ $y=b\sin\theta$ $\frac{dy}{dx}=\frac{dy}{d\theta}\times\frac{d\theta}{dx}$
$\frac{dx}{d\theta}=-a\sin\theta$ $\frac{dy}{d\theta}=b\cos\theta$
$=-\frac{b\cos\theta}{a\sin\theta}$
$y-b\sin\theta=-\frac{b\cos\theta}{a\sin\theta}\left(x-a\cos\theta\right)$
$ay\sin\theta-ab\sin^2\theta=-bx\cos\theta+ab\cos^2\theta$
$bx\cos\theta+ay\sin\theta-ab\left(\sin^2\theta+\cos^2\theta\right)=0$
$bx\cos\theta+ay\sin\theta-ab=0$</td><td>2</td><td>1 mark
• Correctly derives slope of the tangent</td></tr>
<tr><td>7c) (ii) $SR=\left|\frac{abe\cos\theta-0-ab}{\sqrt{b^2\cos^2\theta+a^2\sin^2\theta}}\right|$ $S'R'=\left|\frac{-abe\cos\theta-0-ab}{\sqrt{b^2\cos^2\theta+a^2\sin^2\theta}}\right|$
$SR\times S'R'=\left|\frac{abe\cos\theta-0-ab}{\sqrt{b^2\cos^2\theta+a^2\sin^2\theta}}\right|\times\left|\frac{-abe\cos\theta-0-ab}{\sqrt{b^2\cos^2\theta+a^2\sin^2\theta}}\right|$
$=\frac{\left|a^2b^2e^2\cos^2\theta-a^2b^2\right|}{b^2\cos^2\theta+a^2\sin^2\theta}$
$=b^2\left(\frac{\left|a^2e^2\cos^2\theta-a^2\right|}{b^2\cos^2\theta+a^2\sin^2\theta}\right)$
$=b^2\left\{\frac{\left|a^2e^2\cos^2\theta-a^2\right|}{a^2\left(1-e^2\right)\cos^2\theta+a^2\sin^2\theta}\right\}$
$=b^2\left\{\frac{\left|-1\right|\left|a^2-a^2e^2\cos^2\theta\right|}{a^2\left(\sin^2\theta+\cos^2\theta\right)-a^2e^2\cos^2\theta}\right\}$ $=b^2\left(\frac{a^2-a^2e^2\cos^2\theta}{a^2-a^2e^2\cos^2\theta}\right)=b^2$</td><td>3</td><td>2 marks
• Makes substantial progress towards solution.
1 mark
• Finds a correct expression for $SR\times S'R'$</td></tr>
</table>

Solution	Marks	Comments
QUESTION 8		
8a) Possible sets of digits $9 + 9 + 9 + 9 + 7 = 43$ $9 + 9 + 9 + 8 + 8 = 43$ # of numbers $= \frac{5!}{4!} + \frac{5!}{3! \times 2!}$ $= 5 + 10$ $= 15$	**2**	**1 mark** • Identifies the two sets of digits • Evaluates the number of ways for one set of digits
8b) (i) Prove for $n = 1$ $LHS = T_1$ $= \frac{1}{2}$ $RHS = \frac{\sin\frac{1}{2}\{2(1) - 1\}\theta}{2\sin\frac{1}{2}\theta}$ $= \frac{\sin\frac{1}{2}\theta}{2\sin\frac{1}{2}\theta}$ $= \frac{1}{2}$ $\therefore LHS = RHS$ Hence the result is true for $n = 1$ Assume the result is true for $n = k$, where k is a positive integer i.e. $\frac{1}{2} + \cos\theta + \cos(2\theta) + \ldots + \cos(k - 1)\theta = \frac{\sin\frac{1}{2}(2k - 1)\theta}{2\sin\frac{1}{2}\theta}$ Prove the result true for $n = k + 1$ i.e. Prove $\frac{1}{2} + \cos\theta + \cos(2\theta) + \ldots + \cos k\theta = \frac{\sin\frac{1}{2}(2k + 1)\theta}{2\sin\frac{1}{2}\theta}$ ***PROOF:*** $\frac{1}{2} + \cos\theta + \cos(2\theta) + \ldots + \cos(k - 1)\theta + \cos k\theta$ $= \frac{\sin\frac{1}{2}(2k - 1)\theta}{2\sin\frac{1}{2}\theta} + \cos k\theta$ $= \frac{\sin\frac{1}{2}(2k - 1)\theta + 2\sin\frac{1}{2}\theta\cos k\theta}{2\sin\frac{1}{2}\theta}$ $= \frac{\sin\frac{1}{2}(2k - 1)\theta + \sin\left(k\theta + \frac{1}{2}\theta\right) - \sin\left(k\theta - \frac{1}{2}\theta\right)}{2\sin\frac{1}{2}\theta}$ $= \frac{\sin\frac{1}{2}(2k - 1)\theta + \sin\frac{1}{2}(2k + 1)\theta - \sin\frac{1}{2}(2k - 1)\theta}{2\sin\frac{1}{2}\theta}$ $= \frac{\sin\frac{1}{2}(2k + 1)\theta}{2\sin\frac{1}{2}\theta}$	**4**	There are 4 key parts of the induction; 1. Proving the result true for $n = 1$ 2. Clearly stating the assumption and what is to be proven 3. Using the assumption in the proof 4. Correctly proving the required statement **4 marks** • Successfully does all of the 4 key parts **3 marks** • Successfully does 3 of the 4 key parts **2 marks** • Successfully does 2 of the 4 key parts **1 mark** • Successfully does 1 of the 4 key parts

Solution	Marks	Comments
QUESTION 8…continued		
Hence the statement is true for $n = k + 1$, if it is true for $n = k$ Since the result is true for $n = 1$, then it is true for all positive integers, by induction.		
8b) (ii) The width of each rectangle is $\frac{\pi}{6n}$ $S_n = \frac{\pi}{6n}\left[\cos\frac{\pi}{6n} + \cos\frac{2\pi}{6n} + \cos\frac{3\pi}{6n} + \ldots + \cos\frac{(n-1)\pi}{6n} + \cos\frac{n\pi}{6n}\right]$ $= \frac{\pi}{6n}\left[\cos\frac{\pi}{6n} + \cos\frac{2\pi}{6n} + \cos\frac{3\pi}{6n} + \ldots + \cos\frac{(n-1)\pi}{6n} + \cos\frac{\pi}{6}\right]$ $= \frac{\pi}{6n}\left[\cos\frac{\pi}{6n} + \cos\frac{2\pi}{6n} + \cos\frac{3\pi}{6n} + \ldots + \cos\frac{(n-1)\pi}{6n} + \frac{\sqrt{3}}{2}\right]$ $= \frac{\pi}{6n}\left[\frac{\sqrt{3}}{2} - \frac{1}{2} + \left\{\frac{1}{2} + \cos\frac{\pi}{6n} + \cos\frac{2\pi}{6n} + \cos\frac{3\pi}{6n} + \ldots + \cos\frac{(n-1)\pi}{6n}\right\}\right]$ $= \frac{\pi}{6n}\left\{\frac{\sqrt{3}}{2} - \frac{1}{2} + \frac{\sin\frac{1}{2}(2n-1)\frac{\pi}{6n}}{2\sin\frac{1}{2} \times \frac{\pi}{6n}}\right\}$ $= \frac{\pi}{12n}\left\{\sqrt{3} - 1 + \frac{\sin(2n-1)\frac{\pi}{12n}}{\sin\frac{\pi}{12n}}\right\}$	**2**	**1 mark** • Successfully uses the result of part(i) in a method that would lead to a correct solution. • Writes S_n as a sum of the areas of the rectangles.
8b) (iii) $\lim\limits_{n \to \infty} S_n = \lim\limits_{h \to 0} S_{\frac{1}{h}}$ $= \lim\limits_{h \to 0} \frac{h\pi}{12}\left[\sqrt{3} - 1 + \frac{\sin\left(\frac{2}{h} - 1\right)\frac{h\pi}{12}}{\sin\frac{h\pi}{12}}\right]$ $= \lim\limits_{h \to 0}\left[\frac{\left(\sqrt{3} - 1\right)h\pi}{12} + \frac{\frac{h\pi}{12}}{\sin\frac{h\pi}{12}} \times \sin\left(\frac{2}{h} - 1\right)\frac{h\pi}{12}\right]$ $= 0 + 1 \times \sin\left(\frac{\pi}{6} - 0\right)$ $= \frac{1}{2}$	**2**	**1 mark** • Bald answer • Uses $\lim\limits_{x \to 0}\frac{x}{\sin x}$ • Correctly substitutes $n = \frac{1}{h}$