BAULKHAM HILLS HIGH SCHOOL

2012

YEAR 12 HALF-YEARLY

Mathematics Extension 2

General Instructions

- Reading time – 5 minutes
- Working time – 120 minutes
- Write using black or blue pen
- Board-approved calculators may be used
- All necessary working should be shown in every question
- Marks may be deducted for careless or badly arranged work
- Attempt all questions
- Start a new page for each question

Total marks – 70
Exam consists of 8 pages.

This paper consists of TWO sections.

Section 1 – Pages 3-5
Multiple Choice
Question 1-10 (10 marks)

Section 2 – Pages 6-9
Extended Response
Question 11- 14 (60 marks)

Standard integrals provided on page 10

Section I
10 marks
Attempt questions 1-10
Use the multiple choice answer sheet for question 1-10

1. Simplify $\frac{1}{i^9}$

a) i b) $-i$ c) 1 d) -1

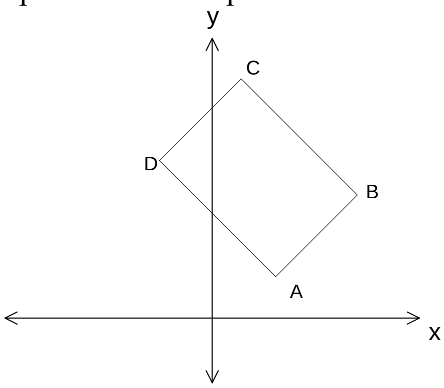
2. z^{-1} is equivalent to

a) $\bar{z}$ b) $|z|^2$ c) $\frac{\bar{z}}{|z|}$ d) $\frac{\bar{z}}{|z|^2}$

3. $P(x) = x^3 + 8x^2 - ax + b$ where a and b are integers. If $1+3i$ is a root, which of the following is also a root?

a) -10 b) -8 c) -3 d) 3

4. In the Argand diagram, ABCD is a rectangle and AD = 2AB.
The vertices A and B correspond to the complex numbers u and v.



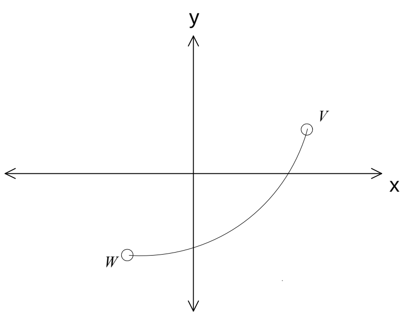
The point C is given by

a) $u + 2iu$ b) $v + 2i(v - u)$ c) $v + 2iu$ d) $v + u + 2iu$

5. If $1, w, w^2$ are the cube roots of unity, simplify $w^7(1 + w)(-w^2 - 1)$

a) $-w$ b) w c) w^2 d) $-w^2$

6. Which of the following could describe the locus of z shown in the diagram.



a) $\arg\left(\frac{z-v}{z-w}\right) = \frac{\pi}{3}$ b) $\arg\left(\frac{z-v}{z-w}\right) = \frac{-\pi}{3}$

c) $\arg\left(\frac{z-v}{z-w}\right) = \frac{2\pi}{3}$ d) $\arg\left(\frac{z-v}{z-w}\right) = \frac{-2\pi}{3}$

7. $z^2 - z^{-2}$ is equivalent to

a) $2\cos\theta$ b) $2\cos 2\theta$ c) $2i\sin\theta$ d) $4i\sin\theta\cos\theta$

8. For $\dfrac{x^2}{16} + \dfrac{y^2}{25} = 1$, which of the following is true?

a) foci are $\left(\pm\dfrac{12}{5}, 0\right)$ and directrices $x = \pm\dfrac{25}{3}$

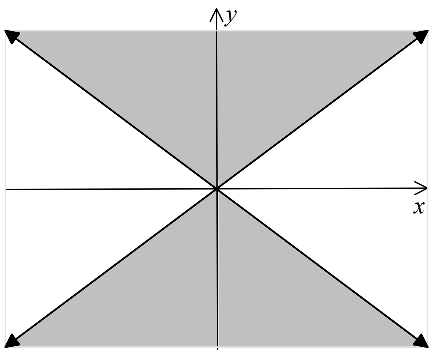
b) foci are $(0, \pm 3)$ and directrices $y = \pm\dfrac{20}{3}$

c) foci are $\left(\pm\dfrac{12}{5}, 0\right)$ and directrices $x = \pm\dfrac{20}{3}$

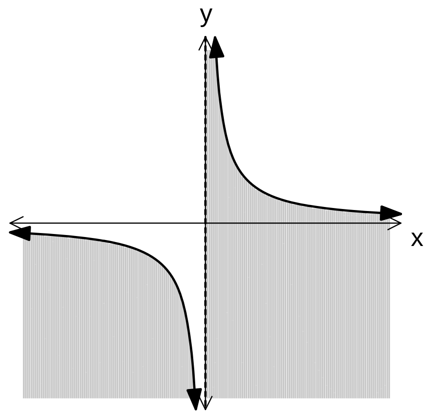
d) foci are $(0, \pm 3)$ and directrices $y = \pm\dfrac{25}{3}$

9. If $z = x + iy$, the locus of $Re\,(z^2) \le 0$ is given by

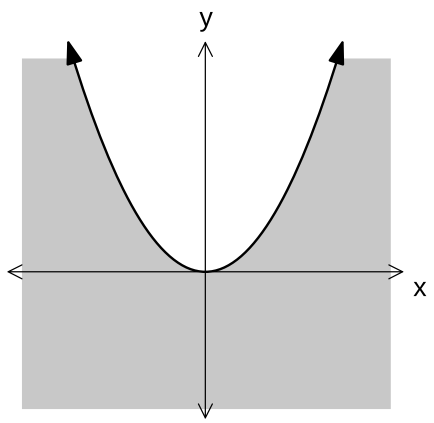
(A)



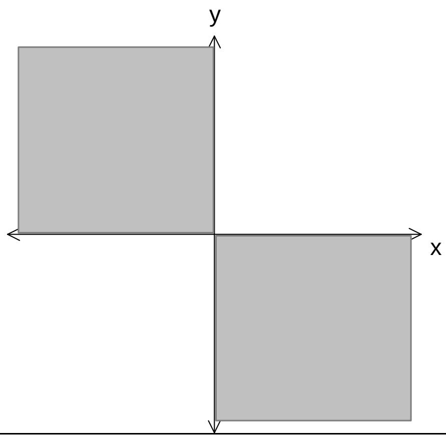
(B)



(C)



(D)



10. If $|\, z - (1+2i)\,| = 1$ the maximum value of $|\,z\,|$ is

a) 2 b) 3 c) $1 + \sqrt{5}$ d) $2 + \sqrt{5}$

End of Section I

Section II – Extended Response

Attempt questions 11-14. Show all necessary working.
Answer each question on a SEPARATE PAGE Clearly indicate question number.
Each piece of paper must show your BOS number.
All necessary working should be shown in every question.

Question 11 (15 marks) – Start a new page **Marks**

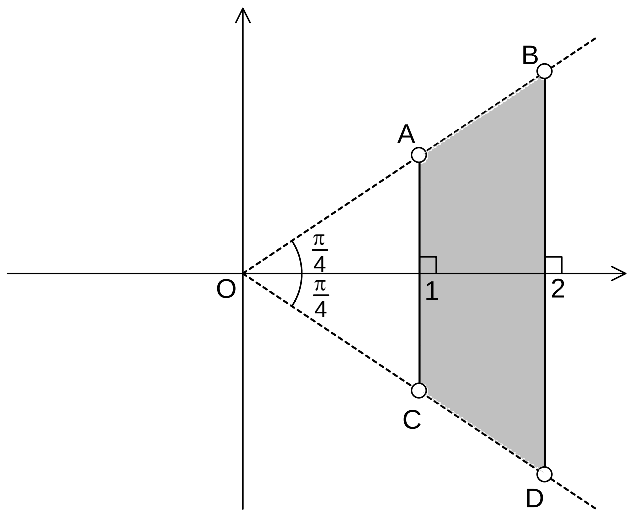
a) The complex numbers **3**

$w = \dfrac{a}{1-2i}$ and $z = \dfrac{b}{2+i}$ (where a and b are real numbers) are such that $w + z = 1$.

Find the values of a and b

b) If $z = 3 + i\sqrt{3}$ find the smallest positive integer, n, for which $Im(z^n) < 0$ and the number is purely imaginary. **3**

c) i) Write down a pair of inequalities which describe the region in the Argand Diagram below. **2**



ii) The region in i) is rotated anticlockwise by $\dfrac{\pi}{4}$ radians to A'B'C'D' **3**

Find the complex number represented by the point of intersection of the diagonals of A'B'C'D' in the form $z = a + ib$

d) i) Show that the locus of z represented by $z\bar{z} + 10(z + \bar{z}) = 21$ is a circle centred at $(-10, 0)$ with radius 11. **2**

ii) Hence find which two purely imaginary numbers satisfy the equation $z\bar{z} + 10(z + \bar{z}) = 21$ **2**

Question 12 (15 marks) – Start a new page **Marks**

a) The roots of $x^3 + 6x^2 + 5x - 8 = 0$ are α, β *and* γ.

Find:

i) the value of $\alpha^2 + \beta^2 + \gamma^2$ **2**

ii) the value of $\alpha^3 + \beta^3 + \gamma^3$ **2**

iii) the monic polynomial with roots α^2, β^2, γ^2 **3**

b) i) Find the values of b and c in the polynomial **3**

$P(z) = z^2 + bz + c$ if b and c are real and $P(3 + i) = 0$

ii) Hence or otherwise, solve $z^3 - 7z^2 + 16z - 10 = 0$ over C. **2**

c) The equation $8x^4 + 12x^3 - 30x^2 + 17x - 3 = 0$ has a root of multiplicity three. **3**
Solve the equation completely.

Question 13 (15 marks) – Start a new page **Marks**

a) α) Determine the real values of p for which

$$\frac{x^2}{4+p} + \frac{y^2}{9+p} = 1 \text{ defines}$$

i) an ellipse **1**

ii) a hyperbola **2**

β) Let $p = -5$ in the equation $\dfrac{x^2}{4+p} + \dfrac{y^2}{9+p} = 1$.

i) Find the eccentricity **1**

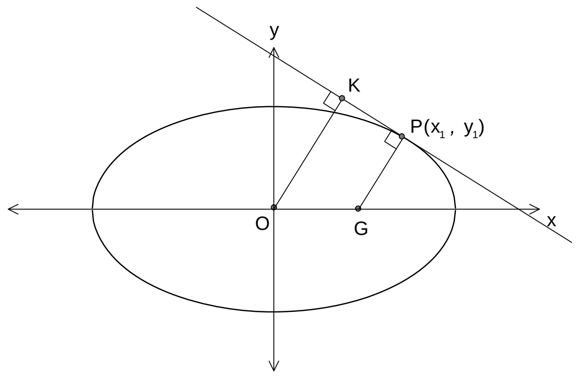
ii) Find the coordinates of foci **1**

iii) Find the equations of the directrices **1**

iv) Find the equations of the asymptotes **1**

v) Hence draw a neat sketch showing all important features **1**

b) i) For the ellipse $x^2 + 4y^2 = 100$ show that the equation of the tangent at at a point $P(x_1, y_1)$ on the curve is $y - y_1 = \dfrac{-x_1}{4y_1}(x - x_1)$ **2**



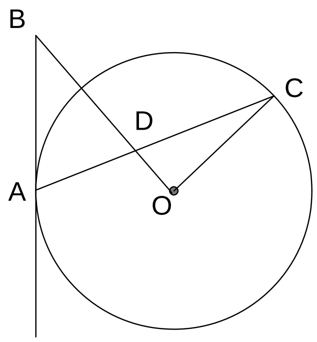
ii) If the normal at P meets the major axis at G show that the distance **3**

$$PG = \frac{\sqrt{x_1^2 + 16y_1^2}}{4}$$

iii) The perpendicular from the centre O to the tangent at P meets the tangent at K. Show that $PG.OK$ is equal to the square of the length of the semi minor axis. **2**

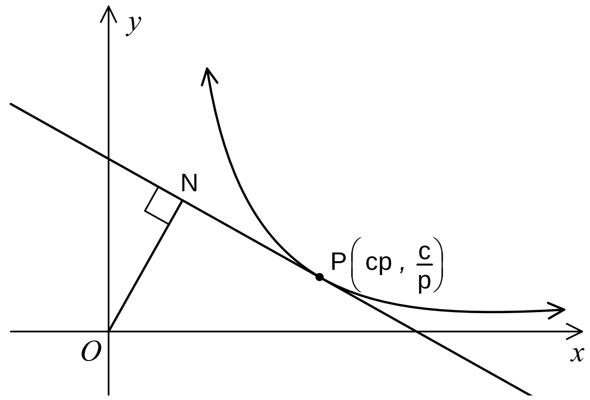
Question 14 (15 marks) – Start a new page **Marks**

a) AB is a tangent to a circle centred at O and AB=BD. **3**



Prove that $\angle COB = 90°$

b) The line through O perpendicular to the tangent at $P\left(cp, \dfrac{c}{p}\right)$ on the rectangular hyperbola $xy = c^2$ meets the tangent at N



i) Show that the coordinates of N are $\left(\dfrac{2cp}{1+p^4}, \dfrac{2cp^3}{1+p^4}\right)$ **3**

ii) Find, in fully simplified form, the equation of the locus of N. **2**

c) If $P(x)$ is divided by $(x-a)(x-b)$ a reminder of $R(x)$ is obtained. **3**
Show that the remainder is given by

$$R(x) = \left(\frac{P(a) - P(b)}{a-b}\right)x + \frac{a\,P(b) - bP(a)}{a-b}$$

d) The points P, Q and R represent three different non zero complex numbers, p, q and r respectively. **4**

Show that these points must satisfy the condition $q^2 + r^2 + 2p^2 = 2(pq + pr)$ in order that ΔPQR forms a right angled isosceles triangle with $PQ=PR$.

End of Paper

STANDARD INTEGRALS

$$\int x^n dx = \frac{1}{n+1}x^{n+1}, \; n \neq -1; \; x \neq 0, \; if \; n < 0$$

$$\int \frac{1}{x} dx = \ln x, \qquad x > 0$$

$$\int e^{ax} dx = \frac{1}{a}e^{ax}, \; a \neq 0$$

$$\int \cos axdx = \frac{1}{a}\sin ax, \; a \neq 0$$

$$\int \sin axdx = -\frac{1}{a}\cos ax, \; a \neq 0$$

$$\int \sec^2 axdx = \frac{1}{a}\tan ax, \; a \neq 0$$

$$\int \sec ax \tan axdx = \frac{1}{a}\sec ax, \; a \neq 0$$

$$\int \frac{1}{a^2 + x^2} dx = \frac{1}{a}\tan^{-1}\frac{x}{a}, \; a \neq 0$$

$$\int \frac{1}{\sqrt{a^2 - x^2}} dx = \sin^{-1}\frac{x}{a}, \; a > 0, \; -a < x < a$$

$$\int \frac{1}{\sqrt{x^2 - a^2}} dx = \ln(x + \sqrt{x^2 - a^2}), \; x > a > 0$$

$$\int \frac{1}{\sqrt{x^2 + a^2}} dx = \ln(x + \sqrt{x^2 + a^2})$$

NOTE: $\ln x = \log_e x, \; x > 0$

Ext 2 Half Yearly 2012

1. b
2. c
3. a
4. b
5. a
6. d
7. d
8. d
9. a
10. c

Q11a) $\frac{a}{1-2i}+\frac{b}{2+i}=1$

$a(2+i)+b(1-2i)=(1-2i)(2+i)$ ✓

$2a+b+i(a-2b)=2-3i+2$

$2a+b+i(a-2b)=4-3i$

$2a+b=4$ ①

$a-2b=-3$ ② ✓

②×2 $2a-4b=-6$ ③

①−③ $5b=10$

$b=2$

$\therefore a=-3+4$

$a=1$

$\therefore a=1, b=2$ ✓

11 b) $z=\sqrt{12}\,\text{cis}\left(\frac{\pi}{6}\right)$ ✓

$z^n=(2\sqrt{3})^n\text{cis}\left(\frac{\pi n}{6}\right)$ ✓

$\frac{\pi n}{6}=\frac{3\pi}{2}$

$n=\frac{18\pi}{2\pi}$

$n=9$ ✓

11 c) i) $1\leq Re(z)\leq 2$ ✓

$-\frac{\pi}{4}<\arg(z)<\frac{\pi}{4}$ ✓

ii) A is $(1,1)$

D is $(2,-2)$

Eqn of AD is $y-1=\frac{-2-1}{2-1}(x-1)$

$y-1=-3(x-1)$

$y=-3x+4$ ✓

By symmetry of shape diagonals intersect on x axis

when $y=0$

$0=-3x+4$

$3x=4$

$x=\frac{4}{3}$

$\therefore$ Pt intersection is $\left(\frac{4}{3},0\right)$ ✓

When rotated pt of int given by $\left(\frac{4}{3}+0i\right)\left(\text{cis}\frac{\pi}{4}\right)$

ie $\frac{4}{3}\left(\frac{1}{\sqrt{2}}+\frac{i}{\sqrt{2}}\right)=\frac{4}{3\sqrt{2}}+\frac{4i}{3\sqrt{2}}$

OR $\frac{2\sqrt{2}}{3}+\frac{2\sqrt{2}\,i}{3}$ ✓

11 d i) let $z=x+iy$

$z\bar{z}+10(z+\bar{z})=21$

$|z|^2+20x=21$ ✓

$x^2+y^2+20x=21$

$x^2+20x+100+y^2=21+100$ ✓

$(x+10)^2+y^2=121$

$\therefore$ Locus is circle centre $(-10,0)$ radius 11.

ii) when $x=0$ $100+y^2=121$ ✓

$y^2=21$

$y=\pm\sqrt{21}$

$\therefore$ Numbers are $\pm i\sqrt{21}$ ✓

12.

a i) $\alpha^2+\beta^2+\gamma^2 = (\alpha+\beta+\gamma)^2 - 2(\alpha\beta+\alpha\gamma+\beta\gamma)$ ✓

$= (-6)^2 - 2\times 5$

$= 26$ ✓

ii) $\alpha^3 = -6\alpha^2 - 5\alpha + 8$ since α, β, γ are roots ✓

$\beta^3 = -6\beta^2 - 5\beta + 8$

$\gamma^3 = -6\gamma^2 - 5\gamma + 8$

$\alpha^3+\beta^3+\gamma^3 = -6(\alpha^2+\beta^2+\gamma^2) - 5(\alpha+\beta+\gamma) + 24$

$= -6\times 26 - 5\times(-6) + 24$

$= -102$ ✓

iii) let $y = x^2$ $\therefore x = \sqrt{y}$

$y\sqrt{y} + 6y + 5\sqrt{y} - 8 = 0$ ✓

$\sqrt{y}(y+5) = 8 - 6y$ ✓

$y(y^2+10y+25) = 64 - 96y + 36y^2$

$y^3 - 26y^2 + 121y - 64 = 0$ ✓

ie $x^3 - 26x^2 + 121x - 64 = 0$ is req'd eqn.

b i) since coeff are real $P(3-i)=0$ ✓

$3+i+3-i = -b$ $\quad (3+i)(3-i) = c$

$b = -6$ ✓ $\quad c = 10$ ✓

ii) let $Q(z) = z^3 - 7z^2 + 16z - 10$

$Q(1) = 1 - 7 + 16 - 10$

$\therefore Q(1) = 0$ ✓

$\therefore 1$ is a root.

Now $z^3 - 7z^2 + 16z - 10 = (z-1)(z^2 - 6z + 10)$

$\therefore$ Roots $1, 3+i, 3-i$ ✓

c) let $P(x) = 8x^4 + 12x^3 - 30x^2 + 17x - 3$ have root mult 3.

$P'(x) = 32x^3 + 36x^2 - 60x + 17$ " " 2

$P''(x) = 96x^2 + 72x - 60$ " " 1 ✓

$96x^2 + 72x - 60 = 0$

$8x^2 + 6x - 5 = 0$

$(4x+5)(2x-1) = 0$

$x = \frac{1}{2}, -\frac{5}{4}$

Now $P(\frac{1}{2}) = \frac{8}{16} + \frac{12}{8} - \frac{30}{4} + \frac{17}{2} - 3$

$= 0$

$\therefore \frac{1}{2}$ is root of mult 3. ✓

let roots be $\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \alpha$

sum of roots $\alpha + \frac{3}{2} = -\frac{12}{8}$

$\alpha = -3$.

$\therefore x = \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, -3$. ✓

13 a) α) i) For ellipse $4+p>0$ and $9+p>0$

$p>-4$ and $p>-9$

$\therefore p>-4$ for an ellipse ✓

ii) For hyperbola

$4+p>0$ and $9+p<0$ OR $4+p<0$, $9+p>0$

$p>-4$ and $p<-9$ $\quad p<-4$ $p>-9$

impossible ✓ $\quad -9<p<-4$

$\therefore$ for hyperbola $-9<p<-4$.

(① for $p>-9$ or ① for $p<-4$)

β) $\frac{x^2}{-1} + \frac{y^2}{4} = 1$

$\frac{y^2}{4} - \frac{x^2}{1} = 1$

i) $a=1$, $b=2$

$a^2 = b^2(e^2-1)$

$1 = 4(e^2-1)$

$\frac{1}{4} = e^2 - 1$

$e^2 = \frac{5}{4}$

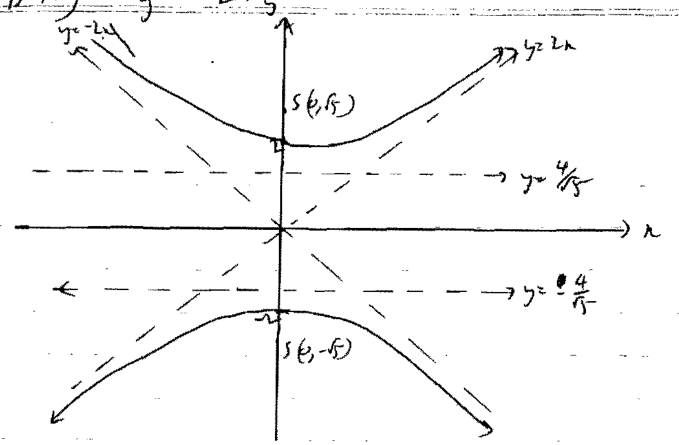
$e = \frac{\sqrt{5}}{2}$ ✓

ii) Foci $(0, \pm\sqrt{5})$ ✓

iii) Directrices $y = \pm\frac{2}{\sqrt{5}/2}$

$y = \pm\frac{4}{\sqrt{5}}$ or $y = \pm\frac{4\sqrt{5}}{5}$ ✓

13 b iv) $y = \pm 2x$



✓ ✓

b i) Diff wrt x $\quad 2x + 8y\frac{dy}{dx} = 0$

$$\frac{dy}{dx} = \frac{-2x}{8y}$$

$$\frac{dy}{dx} = \frac{-x}{4y}$$ ✓

At $P(x_1, y_1)$ $\frac{dy}{dx} = \frac{-x_1}{4y_1}$

eqn of tangent is $y - y_1 = \frac{-x_1}{4y_1}(x - x_1)$ ✓

ii) Eqn of normal is $y - y_1 = \frac{4y_1}{x_1}(x - x_1)$

At G: when $y = 0$ $\quad -y_1 x_1 = 4y_1(x - x_1)$

$$x = \frac{-x_1}{4} + x_1 = \frac{3x_1}{4}$$

$\therefore$ G is $\left(\frac{3x_1}{4}, 0\right) =$ ✓

$$PG = \sqrt{\left(x_1 - \left(\frac{3x_1}{4}\right)\right)^2 + (y_1 - 0)^2}$$ ✓ (① if uses dist or perp dist formula)

$$= \sqrt{\left(\frac{x_1}{4}\right)^2 + y_1^2}$$

$$= \sqrt{\frac{x_1^2}{16} + \frac{16y_1^2}{16}}$$

$$= \frac{\sqrt{x_1^2 + 16y_1^2}}{4} = \frac{\sqrt{x_1^2 + 16y_1^2}}{4}$$ as reqd. ✓

13 b iii) tangent: $y - y_1 = \frac{-x_1}{4y_1}(x - x_1)$

$$4y_1 y - 4y_1^2 = -x_1 x + x_1^2$$

$$x x_1 + 4y_1 y - (x_1^2 + 4y_1^2) = 0$$

$$x_1 x + 4y_1 y - 100 = 0$$

Perp dist O to tangent

$$PO = \frac{|0 + 0 - 100|}{\sqrt{x_1^2 + 16y_1^2}}$$

$$= \frac{|-100|}{\sqrt{x_1^2 + 16y_1^2}}$$

$$OK = \frac{100}{\sqrt{x_1^2 + 16y_1^2}}$$ ✓

$$PG \cdot OK = \frac{\sqrt{x_1^2 + 16y_1^2}}{4} \times \frac{100}{\sqrt{x_1^2 + 16y_1^2}}$$

$$= 25$$

$$= 5^2$$

$\frac{x^2}{100} + \frac{y^2}{25} = 1$ has semi minor axis 5 ✓

$PG \cdot OK = (\text{length of semi minor axis})^2$

Q14 a) Join OA

Let $\angle BAO = x^\circ$

$\angle BOA = x^\circ$ (base angles of isosceles ΔABO equal, $AB = BO$) ✓

$\angle ODC = x^\circ$ (vertically opposite angles) ✓

$\angle OAD = 90^\circ - x^\circ$ (tangent ⊥ radius at point of contact)

ΔOAC is isosceles ($OA = OC$, radii of circle)

$\angle OCA = 90^\circ - x^\circ$ (base angles of isosceles ΔOCA)

$\angle COB = 180^\circ - (90^\circ - x) - x^\circ$ ($\angle$ sum of ΔCOD) ✓

$\therefore \angle COB = 90^\circ$

b) i)

$xy = c^2$

$y = c^2 x^{-1}$

$\frac{dy}{dx} = \frac{-c^2}{x^2}$

At P $\frac{dy}{dx} = \frac{-c^2}{c^2p^2}$

$= \frac{-1}{p^2}$

Eqn of tangent $y - \frac{c}{p} = \frac{-1}{p^2}(x - cp)$

$p^2y - cp = -x + cp$

$x + p^2y = 2cp$ (1) ✓

$m_{ON} = p^2$

Eqn ON: $y = p^2 x$ (2) ✓

sub (2) in (1) $x + p^2(p^2x) = 2cp$

$x(1+p^4) = 2cp$

$x = \frac{2cp}{1+p^4}$

$\therefore y = p^2 \frac{2cp}{1+p^4}$

$y = \frac{2cp^3}{1+p^4}$

$\therefore N$ is $\left(\frac{2cp}{1+p^4}, \frac{2cp^3}{1+p^4}\right)$ ✓

14 b ii)

$y = p^2 x$

$p^2 = \frac{y}{x}$

$y = \frac{2cp^3}{1+p^4}$

$y(1+p^4) = 2cp^3$

$\left[y\left(1+\frac{y^2}{x^2}\right)\right]^2 = \left[\frac{2cy}{x}p\right]^2$ ✓

$y^2\left(1+\frac{2y^2}{x^2}+\frac{y^4}{x^4}\right) = \frac{4c^2y^2}{x^2}\cdot\frac{y}{x}$

$x^4 + 2x^2y^2 + y^4 = 4c^2yx$

$(x^2+y^2)^2 = 4c^2xy$ ✓

14 c) Let $P(x) = Q(x)(x-a)(x-b) + R(x)$ (where $R(x) = cx + d$) ✓

$P(a) = ac + d$ (1)

$P(b) = bc + d$ (2)

(1)−(2) $P(a) - P(b) = (a-b)c$

$c = \frac{P(a) - P(b)}{a-b}$ ✓

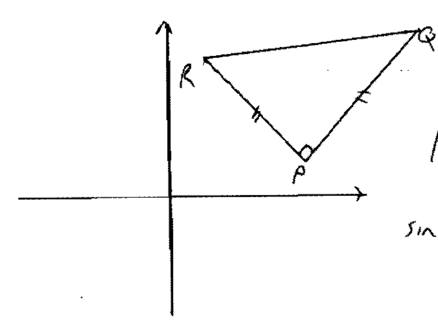
when $c = \frac{P(a)-P(b)}{a-b}$ $P(a) = a\frac{(P(a)-P(b))}{a-b} + d$

$aP(a) - bP(a) = aP(a) - aP(b) + d$

$d = \frac{aP(b) - bP(a)}{a-b}$ ✓

$\therefore R(x) = \frac{P(a)-P(b)}{a-b}x + \frac{aP(b) - bP(a)}{a-b}$

14 d)



$|\overrightarrow{PQ}| = |q - p|$

$|\overrightarrow{PR}| = |r - p|$

since $|\overrightarrow{PQ}| = |\overrightarrow{PR}|$

$|q - p| = |r - p|$

$$\therefore \left|\frac{q-p}{r-p}\right| = 1 \qquad (1) \quad ✓$$

$$\arg(q-p) - \arg(r-p) = \pm\frac{\pi}{2}$$

$$\arg\left(\frac{q-p}{r-p}\right) = \pm\frac{\pi}{2} \qquad (2) \quad ✓$$

$$\therefore \frac{q-p}{r-p} = \operatorname{cis}\frac{\pi}{2} \quad \text{or} \quad \frac{q-p}{r-p} = \operatorname{cis}\left(\frac{-\pi}{2}\right)$$

$$\frac{q-p}{r-p} - i = 0 \quad (3) \qquad \frac{q-p}{r-p} + i = 0 \quad (4) \quad ✓$$

$(3) \times (4)$

$$\left(\frac{q-p}{r-p}\right)^2 + 1 = 0$$

$$(q-p)^2 + (r-p)^2 = 0$$

$$q^2 - 2pq + p^2 + r^2 - 2pr + p^2 = 0$$

$$\therefore q^2 + r^2 + 2p^2 = 2(pq + pr)$$

✓ multiplies & finishes proof.