BAULKHAM HILLS HIGH SCHOOL

2013
YEAR 12 HALF YEARLY EXAMINATION

Mathematics Extension 2

General Instructions

- Reading time – 5 minutes
- Working time – 2 hours
- Write using black or blue pen
  Black pen is preferred
- Board-approved calculators may be used
- A table of standard integrals is provided at the back of this paper
- In Questions 11 – 14, show relevant mathematical reasoning and/or calculations.

Total marks – 70

Section I Pages 2 – 5

10 marks

- Attempt Questions 1 – 10
- Allow about 15 minutes for this section

Section II Pages 6 – 10

60 marks

- Attempt Questions 11 – 14
- Allow about 1 hour 45 minutes for this section

Section I

10 marks
Attempt Questions 1 – 10
Allow about 15 minutes for this section

Use the multiple-choice answer sheet for Questions 1 – 10

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1 If $z = a + ib$, where both a and b are non-zero real numbers, which of the following does **not** represent a real number?

(A) $z + \overline{z}$

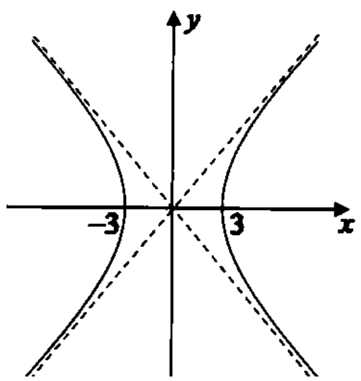
(B) $|z|$

(C) $z^2 - 2abi$

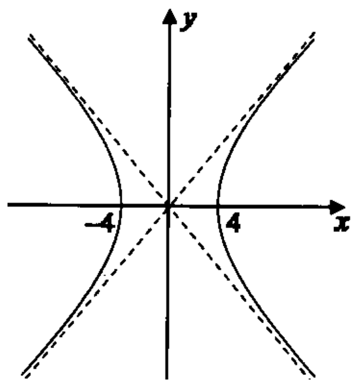
(D) $(z - \overline{z})(z + \overline{z})$

2 Which of the following is the graph of $9x^2 - 16y^2 = 144$?

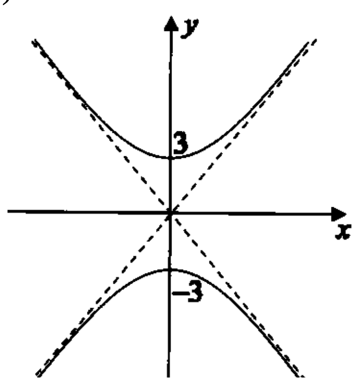
(A)



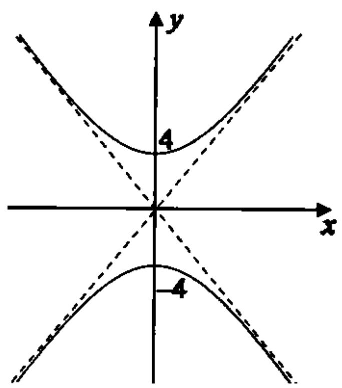
(B)



(C)



(D)



3 Let $z = \text{cis}\left(\frac{5\pi}{6}\right)$, the imaginary part of $z - i$ is

(A) $-\frac{i}{2}$

(B) $-\frac{1}{2}$

(C) $-\frac{3}{2}$

(D) $-\frac{3i}{2}$

4 $P(z)$ is a polynomial of degree 4 with real coefficients.

Which one of the following statements **must** be **false**?

(A) $P(z) = 0$ has two real roots and two non-real roots.

(B) $P(z) = 0$ has one real double root and two non-real roots.

(C) $P(z) = 0$ has one real root and three non-real roots.

(D) $P(z) = 0$ has no real roots.

5 In the complex plane, the ellipse with equation $|z + i| + |z - 3i| = 6$ can be represented by the Cartesian equation

(A) $\frac{x^2}{5} + \frac{(y-1)^2}{9} = 1$

(B) $\frac{(x-1)^2}{5} + \frac{y^2}{9} = 1$

(C) $\frac{x^2}{9} + \frac{(y-1)^2}{5} = 1$

(D) $\frac{(x-1)^2}{9} + \frac{y^2}{5} = 1$

6 If $P(z) = z^3 - 2z^2 + 4z - 8$, then a linear factor of $P(z)$ is

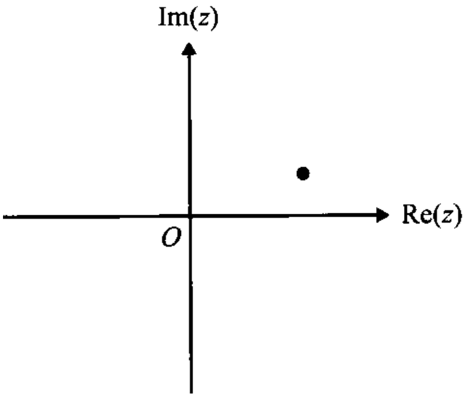
(A) 2

(B) $z + 2$

(C) $z + 2i$

(D) $z^2 + 4i$

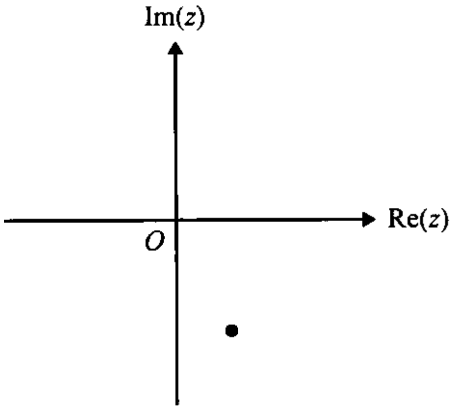
7 The complex number $a + ib$, where a and b are real constants, is represented in the following diagram.



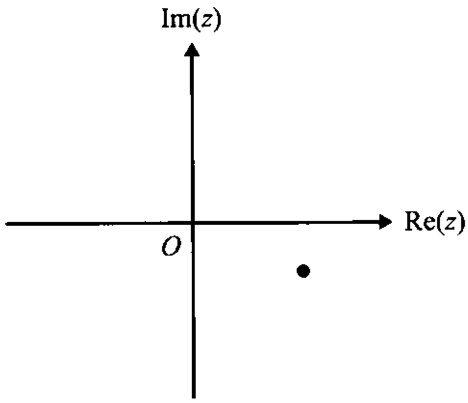
All axes below have the same scale as in the diagram above.

The number $-i(a + ib)$ could be represented by

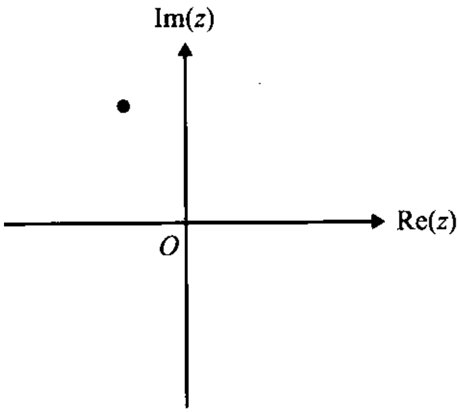
(A)



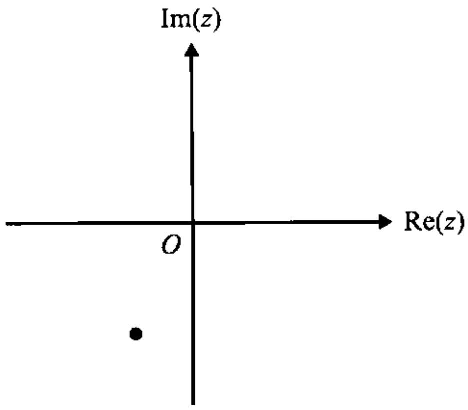
(B)



(C)



(D)



8 P is any point on the hyperbola with equation $x^2 - \frac{y^2}{4} = 1$.

If m is the gradient of the tangent to the hyperbola at P, then m could be

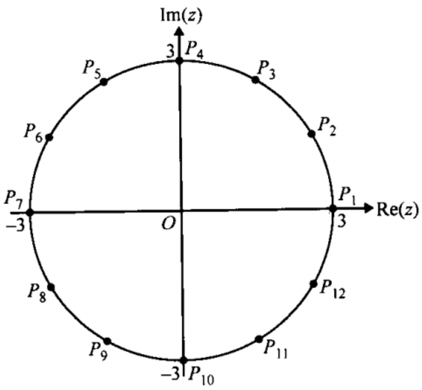
(A) $-\frac{1}{2} < m < \frac{1}{2}$

(B) $m < -\frac{1}{2}$ or $m > \frac{1}{2}$

(C) $-2 < m < 2$

(D) $m < -2$ or $m > 2$

9 On the Argand diagram below, the twelve points $P_1, P_2, P_3, \ldots, P_{12}$ are evenly spaced around the circle of radius 3.



The points which represent complex numbers such that $z^3 = -27i$ are

(A) P_{10} only

(B) P_4 only

(C) P_2, P_6, P_{10}

(D) P_4, P_8, P_{12}

10 Given that the hyperbola with equation $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ has eccentricity e, then the ellipse with equation $\frac{x^2}{a^2 + b^2} + \frac{y^2}{b^2} = 1$ has eccentricity

(A) $-e$

(B) $\frac{1}{e}$

(C) $\sqrt{e}$

(D) e^2

END OF SECTION I

Section II

60 marks
Attempt Questions 11 – 14
Allow about 1 hour 45 minutes for this section

Answer each question on the appropriate answer sheet. Each answer sheet must show your BOS#. Extra paper is available.

In Questions 11 – 14, your responses should include relevant mathematical reasoning and/or calculations.

Marks

Question 11 ***(15 marks)*** Use a *separate* answer sheet

(a) Let $z = 1 - i$ and $w = \sqrt{3} + i$, find

(i) z in modulus-argument form *2*

(ii) $\dfrac{z}{w}$ in the form $a + ib$ *2*

(b) (i) On an Argand diagram, sketch the locus of the points z such that *2*

$$\left| z - \sqrt{2} - i\sqrt{2} \right| \leq 1$$

(ii) Find the maximum value of $|z|$ *2*

(iii) Find the minimum value of $\arg z$ *2*

(c) Given that w is a non-real cube root of unity, evaluate *2*

$$\left(1 - 3w + w^2\right)\left(1 + w - 8w^2\right)$$

(d) In an Argand diagram, $OABC$ is a rhombus, where O is the origin and A is the point (1,2). *3*

If $\angle BAO = 30^\circ$ and B is in the second quadrant, find the complex numbers representing the points B and C.

Marks

<u>Question 12</u> ***(15 marks)*** Use a *separate* answer sheet

(a) Given that the quartic polynomial $x^4 - 5x^3 - 9x^2 + 81x - 108$ has a triple root, completely factorise the polynomial. *3*

(b) For the ellipse $\frac{x^2}{4} + \frac{y^2}{3} = 1$, find

(i) its eccentricity *1*

(ii) the coordinates of the foci *1*

(iii) the equation of the directrices. *1*

(c) Let $\alpha,\ \beta$ and γ be the zeros of the polynomial function

$$P(x) = x^3 + 2x^2 + 19x + 18$$

(i) Find $\alpha + \beta + \gamma$ *1*

(ii) Find $\alpha^2 + \beta^2 + \gamma^2$ *1*

(iii) Find $\alpha^3 + \beta^3 + \gamma^3$ *2*

(iv) Determine how many of the zeros are real. Justify your answer. *2*

(d) Solve the inequality $\frac{3}{x+3} > \frac{x-4}{x}$ *3*

Marks

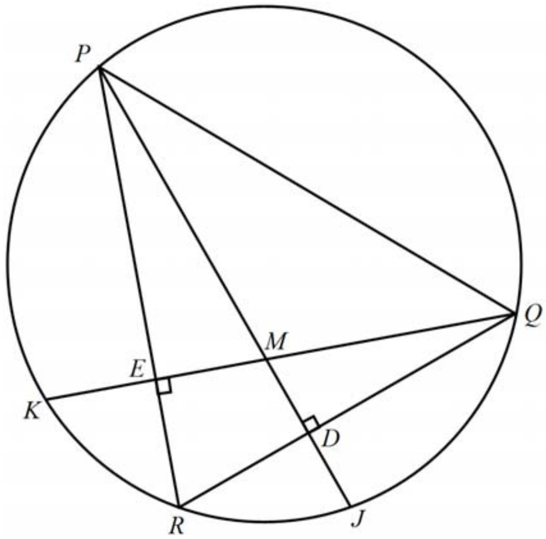
<u>Question 13</u> ***(15 marks)*** Use a *separate* answer sheet

(a) Three pirates are sharing out the contents of a treasure chest containing forty-eight gold coins and two lead coins. The first pirate takes out coins one at a time until a lead coin is taken. The second pirate then takes out coins one at a time until the second lead coin is taken. The third pirate then takes all of the remaining coins.

(i) In how many ways can the coins be distributed? *1*

(ii) What is the probability that all three pirates receive some gold coins? *2*

(b) In the diagram, *PQR* is a triangle inscribed in a circle. The altitude *PD* is produced to meet the circle at *J*, the altitude *QE* is produced to meet the circle at *K* and these two altitudes intersect at *M*.



Copy or trace the diagram into your answer booklet.

(i) Explain why the quadrilaterals *PQDE* and *REMD* are cyclic *2*

(ii) Show that *PR* bisects $\angle KRM$ *2*

(iii) Hence, or otherwise, show that $KR = JR$ *2*

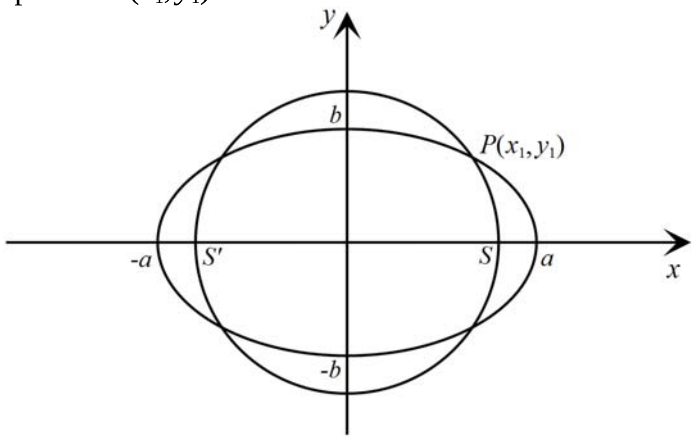
Question 13 continues on page 9

Marks

Question 13 (continued)

(c) A circle, centred at the origin, is drawn through the two foci of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$,

cutting the ellipse at $P(x_1, y_1)$ as shown.



(i) Show that the equation of the normal to the ellipse at P is $\frac{a^2x}{x_1}-\frac{b^2y}{y_1}=a^2e^2$ *2*

(ii) The normal at P meets the y-axis at N. *2*
Show that the y ordinate of N is $-ae$

(iii) Hence deduce that the angle between the tangents to the circle and the ellipse at the point of intersection is equal to the angle of inclination between the normal and the semi-minor axis of the ellipse. *2*

End of Question 13

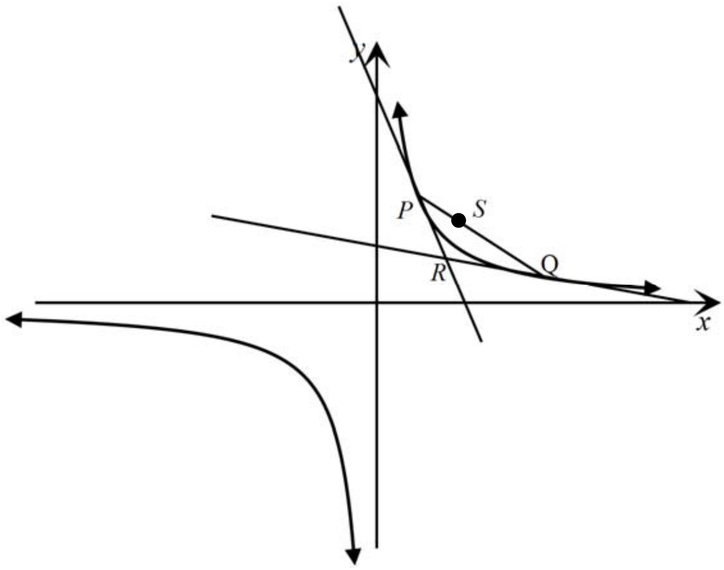
Marks

Question 14 ***(15 marks)*** Use a *separate* answer sheet

(a) (i) On the same diagram, sketch the graphs of $x^2 + y^2 = 1$ and $x^2 - y^2 = 1$, showing clearly the coordinates of any points of intersection with the axes and the equation of any asymptotes. *2*

(ii) Shade the region where the inequality $(x^2 + y^2 - 1)(x^2 - y^2 - 1) \le 0$ holds. *2*

(b) $P\left(cp, \frac{c}{p}\right)$ and $Q\left(cq, \frac{c}{q}\right)$ are two variable points on the rectangular hyperbola $xy = c^2$ which move so that the points *P*, *Q* and $S\left(c\sqrt{2}, c\sqrt{2}\right)$ are always collinear.



(i) Show that the tangent at *P* has the equation $x + p^2 y = 2cp$ *2*

(ii) Hence show that *R* has coordinates $\left(\dfrac{2cpq}{p+q}, \dfrac{2c}{p+q}\right)$ *2*

(iii) Show that $p + q = \sqrt{2}\,(1 + pq)$ *2*

(iv) Hence, or otherwise, find the equation of the locus of *R*. *2*

(c) Prove by induction that the polynomial $x^{2n+1} + a^{2n+1}$, where *a* is a real number, is divisible by $x + a$, for all positive integers *n*. *3*

End of paper

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STANDARD INTEGRALS

$$\int x^n dx = \frac{1}{n+1}x^{n+1},\ n \neq -1; x \neq 0,\ \text{if}\ \ n < 0$$

$$\int \frac{1}{x} dx = \ln x,\ \ x > 0$$

$$\int e^{ax} dx = \frac{1}{a}e^{ax},\ \ a \neq 0$$

$$\int \cos ax\, dx = \frac{1}{a}\sin ax,\ \ a \neq 0$$

$$\int \sin ax\, dx = -\frac{1}{a}\cos ax,\ \ a \neq 0$$

$$\int \sec^2 ax\, dx = \frac{1}{a}\tan ax,\ \ a \neq 0$$

$$\int \sec ax \tan ax\, dx = \frac{1}{a}\sec ax,\ \ a \neq 0$$

$$\int \frac{1}{a^2 + x^2} dx = \frac{1}{a}\tan^{-1}\frac{x}{a},\ \ a \neq 0$$

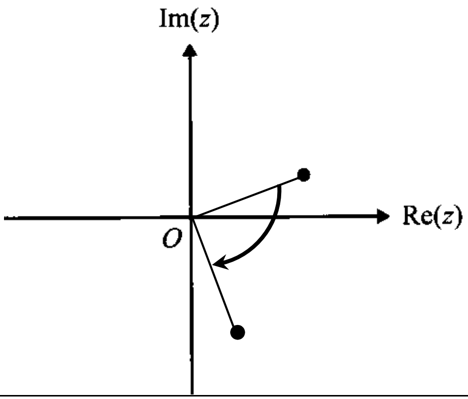
$$\int \frac{1}{\sqrt{a^2 - x^2}} dx = \sin^{-1}\frac{x}{a},\ \ a > 0,\ -a < x < a$$

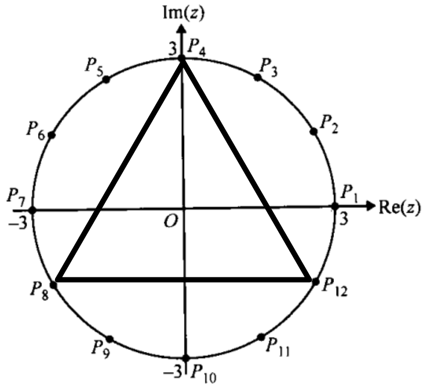
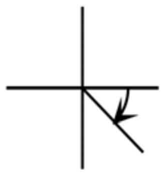
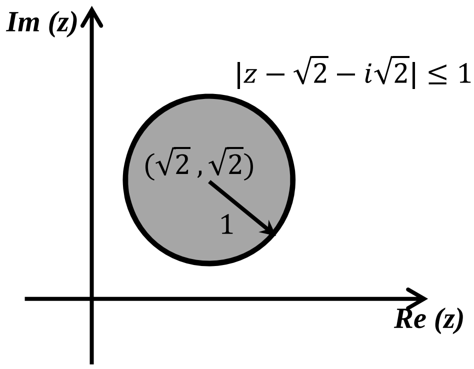
$$\int \frac{1}{\sqrt{x^2 - a^2}} dx = \ln\left(x + \sqrt{x^2 - a^2}\right),\ \ x > a > 0$$

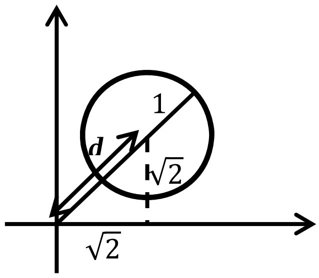
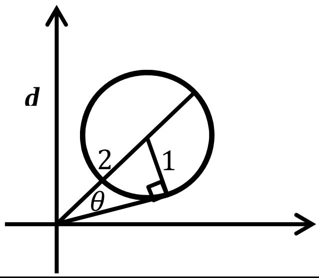
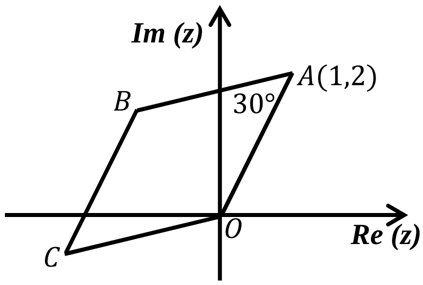
$$\int \frac{1}{\sqrt{x^2 + a^2}} dx = \ln\left(x + \sqrt{x^2 + a^2}\right)$$

NOTE: $\ln x = \log x, \quad x > 0$

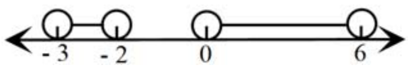
BAULKHAM HILLS HIGH SCHOOL
YEAR 12 EXTENSION 2 HALF YEARLY 2013 SOLUTIONS

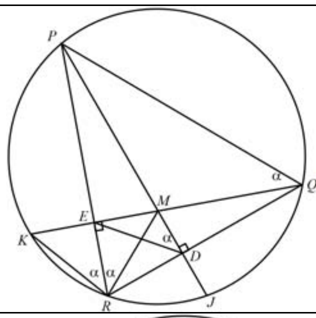
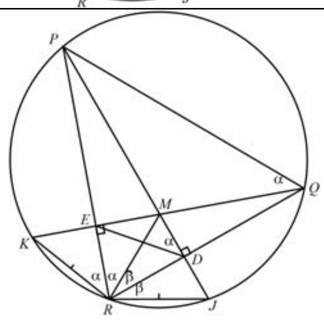
Solution	Marks	Comments
SECTION I		
1. D – $z + \bar{z} = a + ib + a - ib = 2a \Rightarrow$ real $\lvert z \rvert = \sqrt{a^2 + b^2} \Rightarrow$ real $z^2 - 2abi = a^2 + 2abi - b^2 - 2abi = a^2 - b^2 \Rightarrow$ real $(z - \bar{z})(z + \bar{z}) = (a + ib - a + ib)(a + ib + a - ib) = 2ib \times 2a = 4aib \Rightarrow$ imaginary	**1**	
2. B – $9x^2 - 16y^2 = 144 \Rightarrow \frac{x^2}{16} - \frac{y^2}{9} = 1$ Foci are on the x-axis and x-intercepts are $(\pm 4, 0)$	**1**	
3. B - $\text{cis}\left(\frac{5\pi}{6}\right) = \cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6}$ $= -\frac{\sqrt{3}}{2} + \frac{1}{2}i$ $z - i = -\frac{\sqrt{3}}{2} + \frac{1}{2}i - i$ $= -\frac{\sqrt{3}}{2} - \frac{1}{2}i$ $\therefore Im(z - i) = -\frac{1}{2}$	**1**	
4. C – as the coefficients are real, imaginary roots will appear in conjugate pairs $\therefore$ there cannot be an odd number of non-real roots	**1**	
5. A - foci are $(0, -1)$ and $(0, 3) \Rightarrow$ centre shifted to $(0, 1)$ $PS + PS' = 2b \Rightarrow 2b = 6 \Rightarrow b = 3$ NOTE: foci are on y-axis $be = 2$ $e = \frac{2}{3}$ $a^2 = b^2(1 - e^2)$ $= 9\left(1 - \frac{4}{9}\right)$ $= 5$ $\therefore \frac{x^2}{5} + \frac{(y-1)^2}{9} = 1$	**1**	
6. C - 2 would be a root not a linear factor) $P(-2) = (-2)^3 - 2(-2)^2 + 4(-2) - 8 = -28 \neq 0$ $P(-2i) = (-2i)^3 - 2(-2i)^2 + 4(-2i) - 8 = 8i + 8 - 8i - 8 = 0 \therefore (z + 2i)$ is a factor NOTE: $P(z) = (z + 2i)(z - 2i)(z - 2)$	**1**	
7. A - $-i(a + ib) \Rightarrow$ rotate $a + ib$, 90° clockwise $\therefore$ answer should be in the fourth quadrant 	**1**	
8. D – asymptotes are $y = \pm 2x$ $\therefore$ as $x \to \infty$, slope of tangent $\to 2$ at the x-ntercepts , tangent is vertical thus $m < -2$ or $m > 2$	**1**	

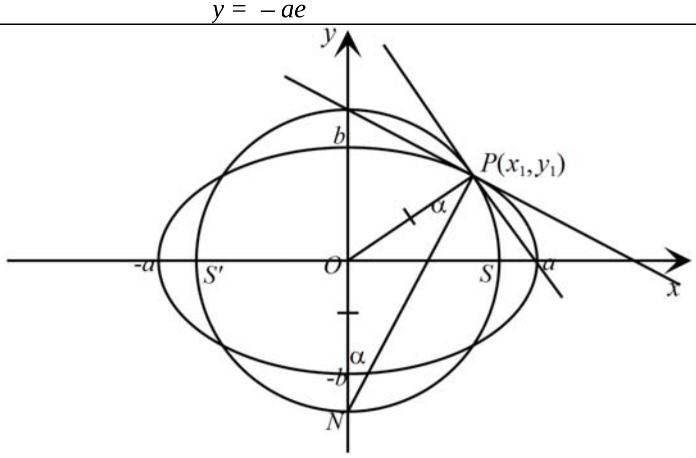
<table>
<tr><th>Solution</th><th>Marks</th><th>Comments</th></tr>
<tr><td>9. D - $z^3 = -27i = -3i \times -3i \times -3i$
$z = 3i$
Solutions are evenly spaced around the circle of radius 3
i.e. form an equilateral triangle. $\Rightarrow P_4, P_8, P_{12}$

</td><td>1</td><td></td></tr>
<tr><td>10. B - hyperbola $e^2 = \dfrac{a^2 + b^2}{a^2}$ ellipse $E^2 = \dfrac{(a^2 + b^2) - b^2}{a^2 + b^2}$
$= \dfrac{a^2}{a^2 + b^2}$
$= \dfrac{1}{e^2}$
$E = \dfrac{1}{e}$</td><td>1</td><td></td></tr>
<tr><th colspan="3">SECTION II</th></tr>
<tr><th colspan="3">QUESTION 11</th></tr>
<tr><td>11(a) (i) $|1 - i| = \sqrt{1^2 + (-1)^2}$ $= \sqrt{2}$
$\arg(1 - i) = \tan^{-1}\dfrac{-1}{1}$ $= -\dfrac{\pi}{4}$

$1 - i = \sqrt{2}\left(\cos\left(-\dfrac{\pi}{4}\right) + i\sin\left(-\dfrac{\pi}{4}\right)\right)$</td><td>2</td><td>2 marks
• Correctly substitutes both the modulus and the argument into the required form.
1 mark
• Finds either the modulus or the argument
• Note: argument should be quoted $-\pi < \arg z \le \pi$</td></tr>
<tr><td>11(a) (ii) $\dfrac{z}{w} = \dfrac{1 - i}{\sqrt{3} + i} \times \dfrac{\sqrt{3} - i}{\sqrt{3} - i}$
$= \dfrac{\sqrt{3} - i - i\sqrt{3} - 1}{(\sqrt{3})^2 - i^2}$
$= \dfrac{(\sqrt{3} - 1) - i(\sqrt{3} + 1)}{3 + 1}$
$= \dfrac{\sqrt{3} - 1}{4} - \dfrac{\sqrt{3} + 1}{4}i$</td><td>2</td><td>2 marks
• Correct solution
1 mark
• Attempts to multiply by the conjugate of the denominator</td></tr>
<tr><td>11(b) (i)

</td><td>2</td><td>2 marks
• Correct solution
1 mark
• Draws the correct boundary circle
• Recognises that the region is inside a circular boundary</td></tr>
</table>

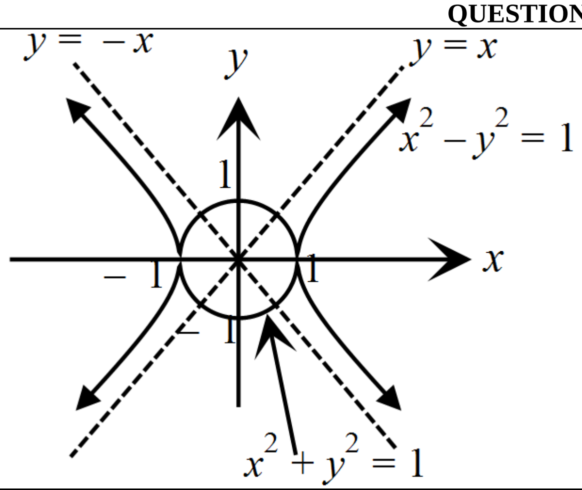
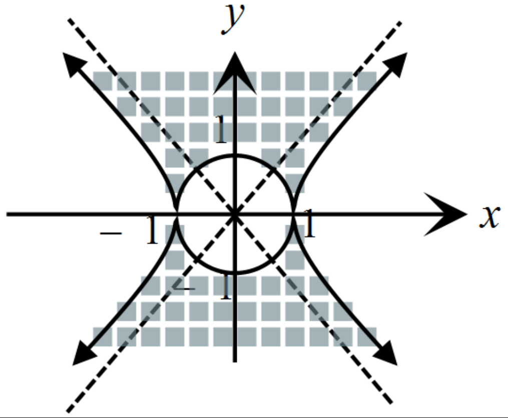
<table>
<tr><th>Solution</th><th>Marks</th><th>Comments</th></tr>
<tr><td>11(b) (ii)
max $|z| = d + 1$
$= \sqrt{\left(\sqrt{2}\right)^2 + \left(\sqrt{2}\right)^2} + 1$
$= \sqrt{4} + 1$
$= 3$

</td><td>2</td><td>2 marks
• Correct solution
1 mark
• Recognises that max $|z|$ is the length of the interval joining the origin to the circumference, passing through the centre of the circle.</td></tr>
<tr><td>11(b) (iii)
min arg$z = \frac{\pi}{4} - \theta$
$= \frac{\pi}{4} - \sin^{-1}\frac{1}{2}$
$= \frac{\pi}{4} - \frac{\pi}{6}$
$= \frac{\pi}{12}$

</td><td>2</td><td>2 marks
• Correct solution
1 mark
• Recognises that the minimum argument involves the tangent drawn to the circle from the origin.</td></tr>
<tr><td>11(c) As w is a non-real cube root of unity, then $w^3 = 1$ and $1 + w + w^2 = 0$
$\left(1 - 3w + w^2\right)\left(1 + w - 8w^2\right) = \left(1 + w + w^2 - 4w\right)\left(1 + w + w^2 - 9w^2\right)$
$= -4w \times -9w^2$
$= 36w^3$
$= 36$</td><td>2</td><td>2 marks
• Correct solution
1 mark
• Makes progress towards a solution using the fact $1 + w + w^2 = 0$
• Obtains 36 by substituting a cube root of unity into the expression</td></tr>
<tr><td>11(d) $\overrightarrow{OC} = \overrightarrow{OA} \times \text{cis}(150^\circ)$
$C = (1 + 2i)\left(-\frac{\sqrt{3}}{2} + \frac{1}{2}i\right)$
$= -\frac{\sqrt{3}}{2} + \frac{1}{2}i - \sqrt{3}\,i - 1$
$= -\frac{2 + \sqrt{3}}{2} + \frac{1 - 2\sqrt{3}}{2}i$


$B = A + \overrightarrow{OC}$
$= 1 + 2i - \frac{2 + \sqrt{3}}{2} + \frac{1 - 2\sqrt{3}}{2}i$
$= -\frac{\sqrt{3}}{2} + \frac{5 - 2\sqrt{3}}{2}i$</td><td>3</td><td>3 marks
• Correct solution
2 marks
• Correctly finds either B or C.
1 mark
• Attempts to find either A or B through the addition or multiplication of vectors.</td></tr>
<tr><td colspan="3">QUESTION 12</td></tr>
<tr><td>12(a) $P(x) = x^4 - 5x^3 - 9x^2 + 81x - 108$
$P'(x) = 4x^3 - 15x^2 - 18x + 81$
$P''(x) = 12x^2 - 30x - 18$
$P''(x) = 0$
$2x^2 - 5x - 3 = 0$
$(2x + 1)(x - 3) = 0$
$x = -\frac{1}{2}$ or $x = 3$
$P(3) = P'(3) = P''(3) = 0$
$\therefore$ 3 is the triple root
$P(x) = (x - 3)^3(x + 4)$</td><td>3</td><td>3 marks
• Correct solution
2 marks
• Correctly identifies the triple root
1 mark
• Attempts to find the multiple root using calculus or equivalent method</td></tr>
</table>

Solution	Marks	Comments
12(b) (i) $e^2 = \dfrac{a^2 - b^2}{a^2}$ $= \dfrac{4-3}{4}$ $= \dfrac{1}{4}$ $e = \dfrac{1}{2}$	**1**	**1 mark** • Correct answer
12(b) (ii) foci: $(\pm ae, 0) = \left(\pm 2 \times \dfrac{1}{2}, 0\right)$ $= (\pm 1, 0)$	**1**	**1 mark** • Correct answer • *Do not penalise for lack of* $\pm$
12(b) (iii) directrices: $x = \pm\dfrac{a}{e}$ $x = \pm\dfrac{2}{1} \times \dfrac{2}{1}$ $x = \pm 4$	**1**	**1 mark** • Correct answer • *Do not penalise for lack of* $\pm$
12(c) (i) $\alpha + \beta + \gamma = -2$	**1**	**1 mark** • Correct answer
12(c) (ii) $\alpha^2 + \beta^2 + \gamma^2 = (\alpha + \beta + \gamma)^2 - 2(\alpha\beta + \alpha\gamma + \beta\gamma)$ $= (-2)^2 - 2(19)$ $= -34$	**1**	**1 mark** • Correct answer
12(c) (iii) $\alpha^3 + 2\alpha^2 + 19\alpha + 18 = 0$ $\beta^3 + 2\beta^2 + 19\beta + 18 = 0$ $\gamma^3 + 2\gamma^2 + 19\gamma + 18 = 0$ $\Sigma\alpha^3 + 2\Sigma\alpha^2 + 19\Sigma\alpha + 54 = 0$ $\Sigma\alpha^3 = -2\Sigma\alpha^2 - 19\Sigma\alpha - 54$ $= -2(-34) - 19(-2) - 54$ $= 52$	**2**	**2 marks** • Correct solution **1 mark** • Uses a valid method in an attempt to find answer.
12(c) (iv) As the coefficients are real, imaginary zeros will appear in conjugate pairs $\therefore$ there is either one or three real zeros $\alpha^2 + \beta^2 + \gamma^2 < 0$ So there must be some imaginary zeros. As the order of the polynomial is three the only possibility is that there is one real zero	**2**	**2 marks** • Correct solution **1 mark** • Mentions some logical relationship between the roots and coefficients that is useful in determining the number of real zeros.
12(d) $\dfrac{3}{x+3} > \dfrac{x-4}{x}$ $x + 3 \neq 0$, $x \neq 0$ $x \neq -3$ $\dfrac{3}{x+3} = \dfrac{x-4}{x}$ $3x = x^2 - x - 12$ $x^2 - 4x - 12 = 0$ $(x+2)(x-6) = 0$ $x = -2$ or $x = 6$  $-3 < x < -2$ *or* $0 < x < 6$	**3**	**3 marks** • Correct graphical solution on number line or algebraic solution, with correct working **2 marks** • Bald answer • Identifies the four correct critical points via a correct method • Correct conclusion to their critical points obtained using a correct method **1 mark** • Uses a correct method • Acknowledges a problem with the denominator. **0 marks** • Solves like a normal equation , with no consideration of the denominator.

Solution	Marks	Comments
QUESTION 13		
13(a) (i) The question is equivalent to how many ways can 2 L's and 48 G's be arranged. $\text{Ways} = \dfrac{50!}{48!2!}$ $= 1225$	1	**1 mark** • Correct solution • *Note: do not penalise for unsimplified answer*
13(a) (ii) If all pirates are to receive some gold coins then the first and the last coin must be **G**, and the two **L**'s cannot be together Ways = Ways begin and end in G – Ways begin and end in G & L's are together $= \dfrac{48!}{46!2!} - \dfrac{47!}{46!}$ $= 1128 - 47$ $= 1081$ $P(\text{all three pirates receive gold}) = \dfrac{1081}{1225}$	2	**2 marks** • Correct solution **1 mark** • Establishes the correct number of ways • *Note: do not penalise for unsimplified answer* • Finds the probability based upon a number of arrangements where some of the restrictions have been considered
13(b) (i) In *PQDE*; $\angle PDQ = \angle PEQ = 90^\circ$ ($\angle$'s in the same segment are =) In *REMD*; $\angle REM + \angle RDM = 180^\circ$ (opposite $\angle$'s in quadrilateral are supplementary)	2	**2 marks** • Correct explanation for both quadrilaterals **1 mark** • Correct explanation for one quadrilateral
13(b) (ii) $\angle KRP = \angle KQP$ ($\angle$'s in same segment =) In *PQDE* $\angle EDP = \angle EQP$ ($\angle$'s in same segment =) In *REMD* $\angle EDM = \angle ERM$ ($\angle$'s in same segment =) $\therefore \angle KRP = \angle ERM$ i.e. *PR* bisects $\angle KRM$  	2	**2 marks** • Correct solution **1 mark** • Significant progress towards correct solution
13(b) (iii) $\angle MRD = \angle JRD$ (similar method to part (ii)) In ΔKER and ΔMER $\angle KER = \angle MER = 90^\circ$ (given) $\angle KRE = \angle MRE$ (proven in part (ii)) *RE* is a common side $\therefore \Delta KER \equiv \Delta MER$ (AAS) $KR = MR$ (matching sides in $\equiv \Delta$'s) In ΔMRD and ΔJRD $MR = JR$ (by a similar method) $\therefore KR = JR$  	2	**2 marks** • Correct solution **1 mark** • Significant progress towards correct solution

Solution	Marks	Comments
13(c) (i) $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$ $\dfrac{2x}{a^2}+\dfrac{2y}{b^2}\times\dfrac{dy}{dx}=0$ $\dfrac{dy}{dx}=-\dfrac{b^2x}{a^2y}$ $\therefore$ at (x_1, y_1); $m_{\text{normal}}=\dfrac{a^2y_1}{b^2x_1}$ $y-y_1=\dfrac{a^2y_1}{b^2x_1}(x-x_1)$ $b^2x_1y-b^2x_1y_1=a^2y_1x-a^2x_1y_1$ $a^2y_1x-b^2x_1y=a^2x_1y_1-b^2x_1y_1$ $\dfrac{a^2x}{x_1}-\dfrac{b^2y}{y_1}=a^2-b^2$ $\dfrac{a^2x}{x_1}-\dfrac{b^2y}{y_1}=a^2e^2$	**2**	**2 marks** • Correct solution **1 mark** • Correctly derives the slope of the normal
13(c) (ii) y-intercept occurs when $x=0$ $-\dfrac{b^2y}{y_1}=a^2e^2$ $y=-\dfrac{a^2e^2y_1}{b^2}$ However P lies on both the ellipse and the circle $x_1^2+y_1^2=a^2e^2 \quad\Rightarrow\quad b^2x_1^2+b^2y_1^2=a^2b^2e^2$ $b^2x_1^2+a^2y_1^2=a^2b^2 \qquad\qquad b^2x_1^2+a^2y_1^2=a^2b^2$ $(a^2-b^2)y_1^2=a^2b^2(1-e^2)$ $a^2e^2y_1^2=b^4$ $y_1^2=\dfrac{b^4}{a^2e^2}$ $y_1=\dfrac{b^2}{ae}$ $y=-\dfrac{a^2e^2}{b^2}\times\dfrac{b^2}{ae}$ $y=-ae$	**2**	**2 marks** • Correct solution **1 mark** • Attempts to eliminate y_1 from the y-intercept using a valid method
13(c) (iii)  $OP=ON$ (= radii) ΔOPN is isosceles (2 = sides) $\angle ONP=\angle OPN$ (base $\angle$'s in isosceles Δ =) $ON\perp$tangent to the circle (radius$\perp$tangent) $\therefore ON$ is the normal to the circle at P $\angle ONP=\angle$ between the two normals $\therefore\angle ONP=\angle$ between the two tangents	**2**	**2 marks** • Correct solution **1 mark** • Significant progress towards correct solution

Solution	Marks	Comments
QUESTION 14		
14(a) (i)  	2	**2 marks** • Correct solution **1 mark** • Two correct graphs with some relevant features not labelled • One correct graph with all relevant labelling included
14(a) (ii) Region must be when ① $x^2 - y^2 \le 1$ and $x^2 + y^2 \ge 1$ as well as ② $x^2 - y^2 \ge 1$ and $x^2 + y^2 \le 1$. However in ② the only common points are $(1, 1)$ and $(-1, -1)$ thus it is only region ①  	2	**2 marks** • Correct solution **1 mark** • Some correct regions indicated, with no more than one incorrect region.
14(b) (i) $y = \dfrac{c^2}{x}$ $\dfrac{dy}{dx} = -\dfrac{c^2}{x^2}$ when $x = 2p$, $\dfrac{dy}{dx} = -\dfrac{c^2}{c^2p^2}$ $= -\dfrac{1}{p^2}$ $\therefore$ required slope $= -\dfrac{1}{p^2}$ $y - \dfrac{c}{p} = -\dfrac{1}{p^2}(x - cp)$ $p^2y - cp = -x + cp$ $x + p^2y = 2cp$	2	**2 marks** • Substitutes into point-slope formula and arrives at the required result **1 mark** • Finds the required slope
14(b) (ii) $x + p^2y = 2cp$ $x + q^2y = 2cq$ $(p^2 - q^2)y = 2c(p - q)$ $\Rightarrow$ $y = \dfrac{2c(p - q)}{(p + q)(p - q)}$ $y = \dfrac{2c}{p + q}$ $x = 2cp - \dfrac{2cp^2}{p + q}$ $x = 2c\left(\dfrac{p(p + q) - p^2}{p + q}\right)$ $x = \dfrac{2c(p^2 + pq - p^2)}{(p + q)}$ $x = \dfrac{2cpq}{p + q}$ $\therefore R\left(\dfrac{2cpq}{p + q}, \dfrac{2c}{p + q}\right)$	2	**2 marks** • Correct solution **1 mark** • Successfully finds the x or y coordinate using a valid method. • Successfully substitutes R into one of the tangents. • Attempts to substitute R into both tangents

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<tr><th>Solution</th><th>Marks</th><th>Comments</th></tr>
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14(b) (ii)…*continued.*

OR

$$x + p^2 y = \frac{2cpq}{p+q} + \frac{2cp^2}{p+q} \qquad\qquad x + q^2 y = \frac{2cpq}{p+q} + \frac{2cq^2}{p+q}$$

$$= \frac{2cp(q+p)}{p+q} \qquad\qquad = \frac{2cq(p+q)}{p+q}$$

$$= 2cp \qquad\qquad = 2cq$$

$\therefore$ as R lies on both tangents, it must be the point of intersection

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14(b) (iii) $m_{PQ} = m_{PS}$

$$\frac{\frac{c}{p} - \frac{c}{q}}{cp - cq} = \frac{\frac{c}{p} - c\sqrt{2}}{cp - c\sqrt{2}}$$

$$\frac{q-p}{pq(p-q)} = \frac{1 - p\sqrt{2}}{p^2 - p\sqrt{2}}$$

$$-\frac{1}{pq} = \frac{1 - p\sqrt{2}}{p\left(p - \sqrt{2}\right)}$$

$$-p + \sqrt{2} = q - pq\sqrt{2}$$

$$p + q = \sqrt{2}\,(1 + pq)$$

</td><td>2</td><td>

2 marks

- Correct solution

1 mark

- Attempts to equate two relevant slopes or equivalent merit

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14(b) (iv)

$$x + y = \frac{2cpq}{p+q} + \frac{2c}{p+q}$$

$$= 2c\left(\frac{pq+1}{p+q}\right)$$

$$= 2c\left(\frac{p+q}{\sqrt{2}\,(p+q)}\right) \qquad \left(\text{from (iii)} \;\; 1 + pq = \frac{p+q}{\sqrt{2}}\right)$$

$$= c\sqrt{2}$$

$\therefore$ locus of R is $x + y = c\sqrt{2}$

OR

S is the focus of the hyperbola i.e. PQ is a focal chord.
In any conic tangents drawn from the extremities of a focal chord meet on the corresponding directrix.

$\therefore$ locus of R is $x + y = c\sqrt{2}$

</td><td>2</td><td>

2 marks

- Correct solution

1 mark

- Attempts to use the relationship found in (iii) in a valid manner
- Correctly states or uses the focal chord property of a conic.

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14(c) When $n = 1$;

$$x^3 + a^3 = (x+a)\left(x^2 - ax + a^2\right)$$

$\therefore x^3 + a^3$ is divisible by $(x + a)$

Hence the result is true for $n = 1$

Assume the result is true for $n = k$ where k is an integer
i.e. $x^{2k+1} + a^{2k+1} = (x+a)Q(x)$ where $Q(x)$ is a polynomial

Prove the result is true for $n = k + 1$
i.e. $x^{2k+3} + a^{2k+3} = (x+a)(R(x)$ where $R(x)$ is a polynomial

PROOF:

$$x^{2k+3} + a^{2k+3} = x^2\left(x^{2k+1}\right) + a^{2k+3}$$

$$= x^2\left[(x+a)Q(x) - a^{2k+1}\right] + a^{2k+3}$$

$$= (x+a)x^2Q(x) - a^{2k+1}x^2 + a^{2k+1}a^2$$

$$= (x+a)x^2Q(x) - a^{2k+1}\left(x^2 - a^2\right)$$

$$= (x+a)x^2Q(x) - a^{2k+1}(x+a)(x-a)$$

$$= (x+a)\left[x^2Q(x) - a^{2k+1}(x-a)\right]$$

$$= (x+a)R(x) \text{ where } R(x) = x^2Q(x) - a^{2k+1}(x-a) \text{, which is a polynomial}$$

Hence the result is true for $n = k + 1$, if it is true for $n = k$

Since the result is true for $n = 1$, then it is true for all positive integers by induction.

</td><td>3</td><td>

There are 4 key parts of the induction;

1. Proving the result true for $n = 1$
2. Clearly stating the assumption and what is to be proven
3. Using the assumption in the proof
4. Correctly proving the required statement

3 marks

- Successfully does all of the 4 key parts

2 marks

- Successfully does 3 of the 4 key parts

1 mark

- Successfully does 2 of the 4 key parts

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