BAULKHAM HILLS HIGH SCHOOL

2014

YEAR 12 HALF-YEARLY

Mathematics Extension 2

General Instructions

- Reading time – 5 minutes
- Working time – 120 minutes
- Write using black or blue pen
- Board-approved calculators may be used
- All necessary working should be shown in every question
- Marks may be deducted for careless or badly arranged work
- Attempt all questions
- Start a new page for each question

Total marks – 70
Exam consists of 11 pages.

This paper consists of TWO sections.

<u>Section 1</u> – Pages 4-6
Multiple Choice
Question 1-10 (10 marks)

<u>Section 2</u> – Pages 7-10
Extended Response
Question 11- 14 (60 marks)

Standard integrals provided on page 11

Section I - 10 marks
Allow about 15 minutes for this section
Use the multiple choice answer sheet for question 1-10

1. Which of the following shapes is the locus of the point P representing the complex number z moving in an Argand diagram such that

$$|z - 2i| + |z + 2i| = 6$$

(A) A circle

(B) A parabola

(C) A hyperbola

(D) An ellipse

2. What is the multiplicity of the root $x = -1$ of the equation $3x^5 - 5x^4 - 35x - 27 = 0$?

(A) one

(B) two

(C) three

(D) four

3. In modulus argument form $-\sqrt{2}(1 - i)$ is

(A) $2\left(\cos\left(\frac{3\pi}{4}\right) - i\sin\left(\frac{3\pi}{4}\right)\right)$

(B) $2\left(\cos\left(\frac{3\pi}{4}\right) + i\sin\left(\frac{3\pi}{4}\right)\right)$

(C) $\cos\frac{\pi}{4} - i\sin\frac{\pi}{4}$

(D) $-\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}$

4. If $y = \cos^{-1}(e^x)$, the expression for $\frac{dy}{dx}$ is

(A) $-\text{cosec}\, y$

(B) $-\tan y$

(C) $-\cot y$

(D) $-\sec y$

5. If α, β and γ are roots of the equation $x^3 + 5x^2 + 4 = 0$, then the cubic equation with roots α^2, β^2 and γ^2 is

(A) $4x^3 + 5x^2 + 1 = 0$

(B) $x^3 - 25x^2 - 40x - 16 = 0$

(C) $x^3 + 25x^2 + 40x + 16 = 0$

(D) $x^3 + 25x^2 - 40x + 16 = 0$

6. The equation of the normal to the hyperbola $x = 2\sec\theta, y = \tan\theta$ at the point where $\theta = \frac{\pi}{4}$ is

(A) $\sqrt{2}x + y - 5 = 0$

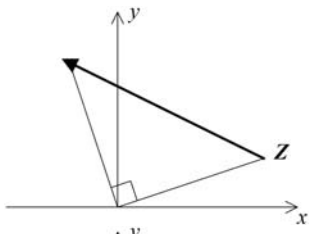
(B) $\sqrt{2}x + 2y - 2 = 0$

(C) $\sqrt{2}x - 2y - 2 = 0$

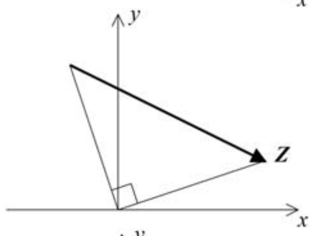
(D) $\sqrt{2}x - y + 5 = 0$

7. On an Argand diagram, point Z is shown to represent the complex number z. Which diagram below shows the vector that represents $(1 - i)z$?

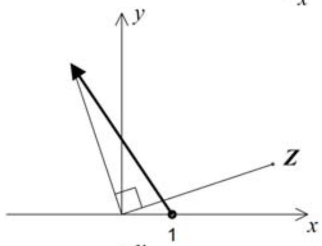
(A)



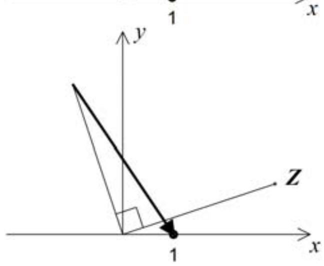
(B)



(C)



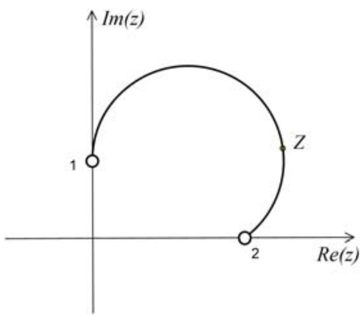
(D)



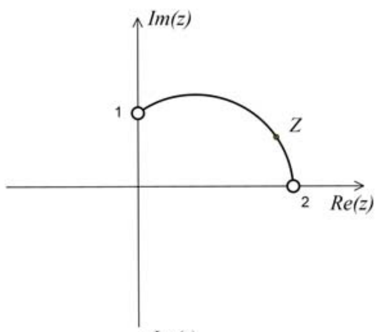
8.	Three of the six letters of the word ***R O M A R O*** are selected and arranged in a row. How many different arrangements are possible? (A) 120 (B) 30 (C) 42 (D) ${}^{6}P_3 \times 3!$
9.	Given the hyperbola $\frac{x^2}{9} - \frac{y^2}{16} = 1$, which of the following would be the equation of the asymptotes? (A) $y = \pm\frac{3}{4}x$ (B) $y = \pm\frac{4}{3}x$ (C) $y = \pm\frac{16}{9}x$ (D) $y = \pm\frac{16}{9}x$

10. The locus of z in the Argand plane where $\arg(z-2) - \arg(z-i) = \frac{\pi}{3}$ is

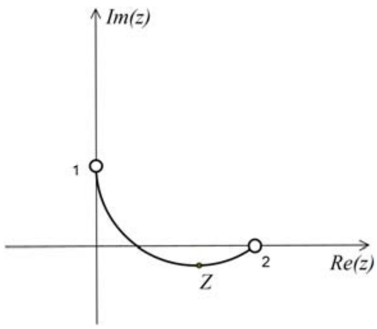
(A)



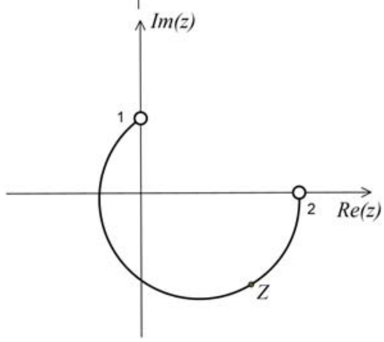
(B)



(C)



(D)

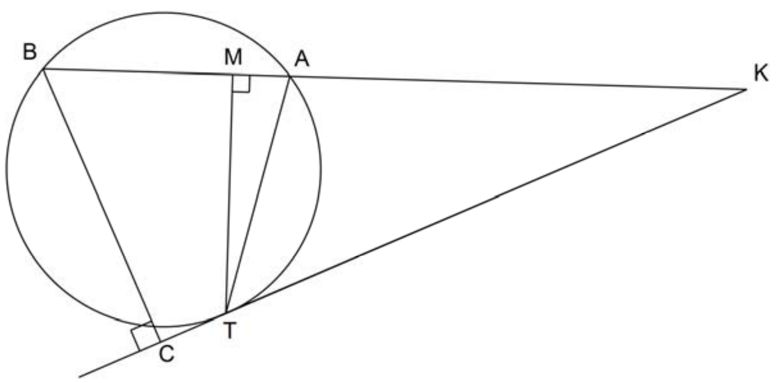


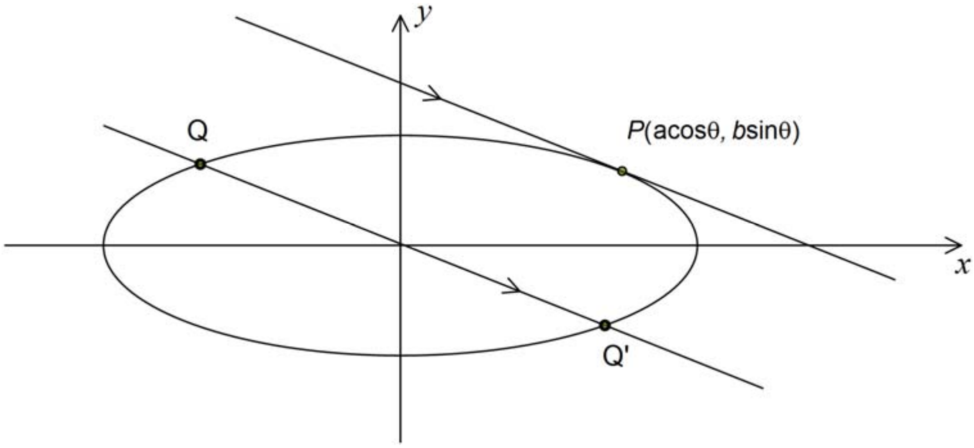
End of Section I

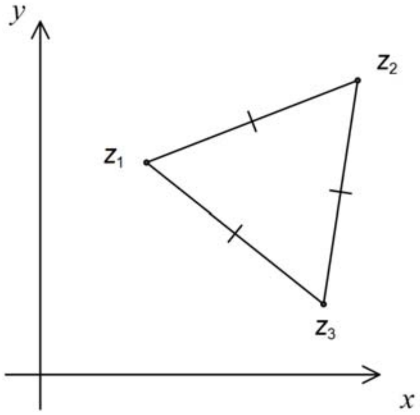
Section II – Extended Response

Attempt questions 11-14. Show all necessary working.
Answer each question on a SEPARATE PAGE Clearly indicate question number.
Each piece of paper must show your BOS number.
All necessary working should be shown in every question.

Question 11 (15 marks)		**Marks**
(a)	Let $z = i - \sqrt{3}$ and $w = 1 - i$, find	
	(i) $\bar{z} + w$	**1**
	(ii) zw in modulus /argument form.	**3**
(b)	(i) Find all the complex numbers $z = a + ib$ such that $(a + ib)^2 = 8 + 6i$, where a, b are real numbers.	**2**
	(ii) Hence solve $z^2 + 2z(1 + 2i) - (11 + 2i) = 0$	**2**
(c)	Solve $x^4 + 4x^3 - 16x - 16 = 0$ given that it has a root of multiplicity 3	**3**
(d)	Find the equation of the locus of all z, such that $\lvert z - 2 \rvert = Re(z)$.	**2**
(e)	For the hyperbola $xy = 12$, find its	
	(i) foci	**1**
	(ii) equations of directrices	**1**
	End of Question 11	

Question 12 (15 marks)		**Marks**
a)	A nine member committee consists of 4 male students, 3 female students and 2 teachers. The committee meets around a circular table so that the male students sit together as a group, and so do the female students, but no female student sits next to a male student.	
	(i) How many different arrangements are possible?	**2**
	(ii) One particular male student does not wish to sit next to one particular teacher. How many ways can this be arranged?	**2**
b)	The points represented by the complex number $z_1 = \sqrt{3} + i$ and two other complex numbers z_2 and z_3, lie on the circumference of a circle with centre O and radius 2. These three points are vertices of an equilateral triangle. Find the complex numbers z_2 and z_3 in the form $a + ib$ where a and b are real.	**2**
(c)	Solve $z^4 - z^3 + 6z^2 - z + 15 = 0$ for z given that $z = 1 - 2i$ is a root of the equation.	**2**
(d)	The locus of z is represented by $\lvert z - 1\rvert = 1$	
	(i) Sketch the locus.	**1**
	(ii) Show that $\lvert z^2 - z\rvert = \lvert z\rvert$	**1**
	(iii) Show that $\arg(z - 1) = \frac{2}{3}\arg(z^2 - z)$.	**2**
(e)	 The diagram shows points A, B and T marked on a circle. A tangent to the circle at T, KC is drawn such that BC is perpendicular to KC. TM is perpendicular to AB. Show that MC is parallel to AT.	**3**
	End of Question 12	

Question 13 (15 marks)		Marks
a)	Solve $\|3x^2 - 2x - 2\| < 3x$	3
b)	Given that ω is a non-real root of the equation $z^5 = 1$ and that $\alpha = \omega + \omega^4$ is a root of the quadratic equation $x^2 + bx + c = 0$, where b and c are real.	
	(i) Prove that $1 + \omega + \omega^2 + \omega^3 + \omega^4 = 0$.	1
	(ii) Find the second root β of the equation $x^2 + bx + c = 0$ in terms of positive powers of ω.	1
	(iii) Find the values of the coefficients a and b.	2
	(iv) Deduce that the exact value of $$\cos\frac{2\pi}{5} + \cos\frac{4\pi}{5} = -\frac{1}{2}$$	2
(c)		
	The point $P(a\cos\theta, b\sin\theta)$ lies on $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. A line is drawn through the origin parallel to the tangent at P. The line meets the ellipse at Q and Q'.	
	(i) Show that the equation of the tangent at P is $\frac{x\cos\theta}{a} + \frac{y\sin\theta}{b} = 1$	2
	(ii) Show that the equation of QQ' is $xb\cos\theta + ya\sin\theta = 0$	1
	(iii) Hence find the coordinates of Q and Q'	1
	(iv) Prove that the area of the $\Delta QPQ'$ is independent of the position of P.	2
	End of Question 13	

Question 14 (15 marks)		Marks
a)	Given that $cos\theta + isin\theta \neq 1$, prove by mathematical induction that $$1 + cis\theta + cis2\theta + \cdots + cis(n\theta) = \frac{1 - cis(n+1)\theta}{1 - cis\theta}$$ *where* $n \geq 0$, and $cis\theta = \cos\theta + i\sin\theta$	3
b)	 If complex numbers z_1, z_2, z_3 represent vertices of an equilateral triangle	
	(i) Express $z_2 - z_1$ in terms of $z_3 - z_1$	1
	(ii) Hence prove $z_1{}^2 + z_2{}^2 + z_3{}^2 = z_1 \cdot z_2 + z_2 \cdot z_3 + z_3 \cdot z_1$	3
c)	The points $P\left(cp, \frac{c}{p}\right)$ and $Q\left(cq, \frac{c}{q}\right)$, where, $\lvert p\rvert \neq \lvert q\rvert$, lie on the rectangular hyperbola with equation $xy = c^2$.	
	(i) Show that the equation of the tangent to the hyperbola $xy = c^2$ at the point $P\left(cp, \frac{c}{p}\right)$ is $x + p^2y = 2cp$	2
	(ii) If the tangents at $P\left(cp, \frac{c}{p}\right)$ and $Q\left(cq, \frac{c}{q}\right)$ meet at the point $R(x_0, y_0)$, prove that $p + q = \frac{2c}{y_0}$ and $pq = \frac{x_0}{y_0}$	3
	(iii) If the length of chord PQ is d units, show that $d^2 = c^2(p-q)^2\left\{1 + \frac{1}{p^2q^2}\right\}$	1
	(iv) If d remains fixed, deduce that the locus of R has equation $$4c^2(x^2 + y^2)(c^2 - xy) = x^2y^2d^2$$	2
	End of Question 14	

End of Paper

Yr. 12 Extension 2 - Half Yearly 2014 Solutions

Multiple choice

1. $\boxed{D}$ ellipse

2. $\boxed{B}$ $P(-1)=0$
$P'(x)=15x^4-20x^3-35 \therefore P'(-1)=0$
$P''(x)=60x^3-60x^2 \quad \therefore P''(-1)\neq 0$
$\therefore$ multiplicity $=2$

---

3. $\boxed{A}$ $-\sqrt{2}(1-i)=-\sqrt{2}+\sqrt{2}i=z$
modulus $=\sqrt{(-\sqrt{2})^2+(\sqrt{2})^2}=2 \qquad \arg z=\tan^{-1}\left(\frac{-\sqrt{2}}{\sqrt{2}}\right)=\frac{3\pi}{4}$
$\therefore z=2\left(\cos\left(\frac{3\pi}{4}\right)+i\sin\left(\frac{3\pi}{4}\right)\right)$

---

4. $\boxed{A}$ $(2\sec\theta, \tan\theta)$ at $\theta=\frac{\pi}{4} \qquad a=2 \quad b=1$
$=\left(2\sec\frac{\pi}{4}, \tan\frac{\pi}{4}\right)=(2\sqrt{2}, 1)$
$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1 \therefore \frac{x^2}{4}-y^2=1 \therefore \frac{2x}{4}-2y\cdot\frac{dy}{dx}=0$
$\therefore \frac{dy}{dx}=\frac{x}{4y}$
$\therefore m_T=\frac{2\sqrt{2}}{4\times 1}=\frac{\sqrt{2}}{2} \therefore m_N=-\frac{2}{\sqrt{2}}=-\sqrt{2}$
$\therefore$ eqn. of normal $\therefore$ $y-1=-\sqrt{2}(x-2\sqrt{2}) \therefore \sqrt{2}x+y-5=0$

---

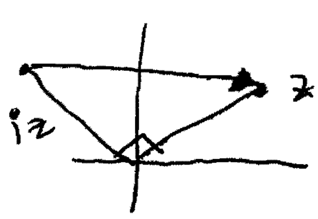
5. $\boxed{B}$ let $y=x^2 \therefore x=\sqrt{y} \quad \therefore P(x)=0 \quad \therefore P(\sqrt{y})=0$
$\therefore (\sqrt{y})^3+5(\sqrt{y})^2+4=0$
$y\sqrt{y}=-5y-4 \quad \therefore y^3=(-5y-4)^2=25y^2+40y+16$
$\therefore$ cubic equation is $y^3-25y^2-40y+16=0$ or $x^3-25x^2-40x+16=0$

---

6. $\boxed{C}$ $y=\cos^{-1}(e^x) \quad \therefore e^x=\cos y$
$\frac{dy}{dx}=\frac{-e^x}{\sqrt{1-(e^x)^2}}=\frac{-\cos y}{\sqrt{1-\cos^2 y}}=\frac{-\cos y}{\sin y}=-\cot y$

---

7. $\boxed{B}$ $(1-i)z=z-iz$



8. $\boxed{C}$

choose R, R and one of O, M, A $\therefore$ ${}^3C_1 \times \frac{3!}{2!} = 9$

choose O, O, and one of R, M, A $\therefore$ 9

arrange $\boxed{R}$ $\boxed{O}$ M, A $\therefore$ ${}^4P_4 = 4! = 24$

$\therefore$ total $= 9 + 9 + 24 = 42$

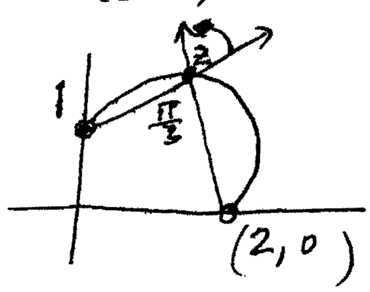
9. $\boxed{B}$

$$\frac{y^2}{16} = \frac{x^2}{9} - 1$$

$$\frac{y^2}{x^2} = \frac{16}{9} - \frac{16}{x^2} \quad \therefore \text{ asymptote } \frac{y}{x} = \pm\frac{4}{3}$$

$$\therefore y = \pm\frac{4}{3}x$$

10. $\boxed{A}$ $\arg\frac{(z-2)}{(z-i)} = \frac{\pi}{3}$



Question 11 **BOS#:** ______________

(a) $z = i - \sqrt{3}$, $w = 1 - i$

(i) $\bar{z} + w = -\sqrt{3} - i + 1 - i = (-\sqrt{3}+1) - 2i$ ①

(ii) $z \cdot w = (-\sqrt{3}+i)(1-i) = (-\sqrt{3}+1) + i(1+\sqrt{3})$ ①/3

OR $|z \cdot w| = |-\sqrt{3}+i||1-i| = |2| \times |\sqrt{2}| = 2\sqrt{2}$ ①

$\arg(z \cdot w) = \arg(z) + \arg(w) = \frac{5\pi}{6} + \left(-\frac{\pi}{4}\right) = \frac{7\pi}{12}$ ①

$\therefore zw = 2\sqrt{2} \operatorname{cis} \frac{7\pi}{12}$ ①

(b)(i) $(a+ib)^2 = 8+6i$

$a^2 - b^2 + 2iab = 8+6i$

(1) $a^2 - b^2 = 8$ ①

(2) $2ab = 6 \quad \therefore b = \frac{3}{a}$

$\therefore a^2 - \left(\frac{3}{a}\right)^2 = 8 \quad \therefore a^4 - 8a^2 - 9 = 0$

$(a^2-9)(a^2+1) = 0$

$a = \pm 3$ no solution in real

$\therefore$ $a = 3$, $b = 1$; $a = -3$, $b = -1$ $\quad \therefore z = \pm(3+i)$ ①

(ii) $z^2 + 2z(1+2i) - (11+2i) = 0$

$$z = \frac{-2(1+2i) \pm \sqrt{4(1+2i)^2 - (-4(11+2i))}}{2}$$

$$= \frac{-\cancel{2}(1+2i) \pm \cancel{2}\sqrt{(1+2i)^2 + (11+2i)}}{\cancel{2}}$$ ①

$= -(1+2i) \pm \sqrt{8+6i} = -(1+2i) \pm (\pm 3+i)$

Question 11 **BOS#:** ____________

11b) ii) cont. $\therefore z = 2 - i$ or $-4 - 3i$ (1)

c) $x^4 + 4x^3 - 16x - 16 = 0$

let $P(x) = x^4 + 4x^3 - 16x - 16$

$P'(x) = 4x^3 + 12x^2 - 16 = 4(x^3 + 3x^2 - 4)$

$P''(x) = 12x^2 + 24x = 0 = 12x(x+2)$

when $x = 0$, $x = -2$

but $P(0) \neq 0 \therefore 0$ is not a root

$P(-2) = 0$, $P'(-2) = 0$ & $P''(-2) = 0$ (1)

$\therefore$ $x = -2$ is a root multiplicity 3 (1)

$\therefore (x^4 + 4x^3 - 16x - 16) \div (x+2)^3 = x - 2$

$\therefore x = -2$ multiplicity 3 and $x = 2$ (single root) (1)

d) $|z - 2| = Re(z)$ let $z = x + iy$

$\therefore |x + iy - 2| = x$

$\therefore \sqrt{(x-2)^2 + y^2} = x$ (1)

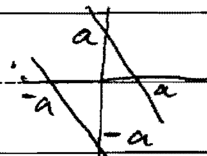
$(x-2)^2 + y^2 = x^2$

$\cancel{x^2} - 4x + 4 + y^2 = \cancel{x^2}$

$\therefore y^2 = 4x - 4$ (1) $\therefore$ locus of z

e) $xy = 12 = c^2 = \frac{a^2}{2}$ $\therefore 12 = \frac{a^2}{2}$ $\therefore a = \sqrt{24} = 2\sqrt{6}$

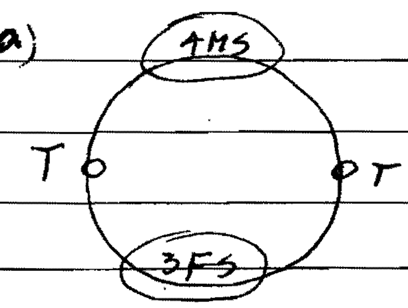
$S, S'(\pm a, \pm a) \therefore S(+2\sqrt{6}, +2\sqrt{6}), S'(-2\sqrt{6}, -2\sqrt{6})$ (1)

directrices:  $\therefore x + y = 2\sqrt{6}$ or $x + y = -2\sqrt{6}$ (1)

You may ask for extra writing paper if you need more space to answer question 11

Question 12 **BOS#:**________________________

a)



i) $4! \, 3! \times 2 = \underline{\underline{288 \text{ ways}}}$ (1)

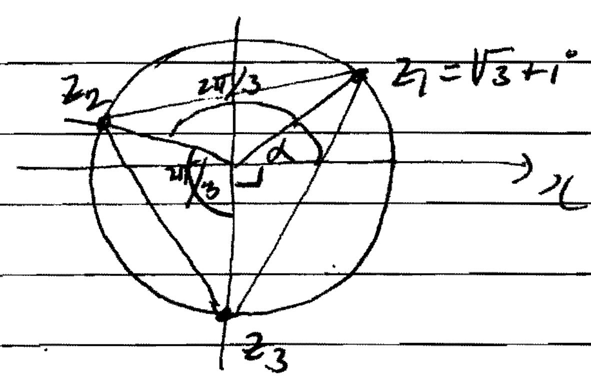
teachers

ii) a particular male student M_1 sits next to a particular teacher T_1 in $3! \, 3! \times 2 = 72$ ways (1)

$\therefore$ number of ways M_1 doesn't sit next to $T_1 = 288 - 72$ (1)

$= \underline{\underline{216}}$ ways

b)



$z_1 = \sqrt{3} + i = 2 \operatorname{cis} \frac{\pi}{6}$

$\alpha = \frac{\pi}{6}$

$\therefore z_2 = 2 \operatorname{cis}\left(\frac{\pi}{6} + \frac{2\pi}{3}\right)$

$= 2 \operatorname{cis} \frac{5\pi}{6}$

$\therefore z_2 = 2\left(-\frac{\sqrt{3}}{2} + i\frac{1}{2}\right) = -\sqrt{3} + i$ (1)

$z_3 = z_2 \cdot \operatorname{cis}\left(\frac{2\pi}{3}\right) = 2 \operatorname{cis} \frac{3\pi}{2} = 2 \operatorname{cis}\left(-\frac{\pi}{2}\right) = -2i$ (1)

c) ω is a root of $z^5 = 1$ $\therefore$ $\omega^5 = 1$ or $\omega^5 - 1 = 0$

but $\omega^5 - 1 = (\omega - 1)(\omega^4 + \omega^3 + \omega^2 + \omega + 1) = 0$

since ω - non real $\therefore \omega \neq 1$ $\therefore$ $\omega^4 + \omega^3 + \omega^2 + \omega + 1 = 0$

Now quadratic equation $x^2 + bx + c = 0$

where $\alpha = \omega + \omega^4$, $\beta = \omega^2 + \omega^3$. But $\alpha + \beta = -\frac{b}{a}$ (1)

$\therefore \omega + \omega^4 + \omega^2 + \omega^3 = -b = -1$

$\therefore b = 1$

Question 12 **BOS#:**______________________

12c) cont:. $\alpha \cdot \beta = \frac{c}{a}$

$\therefore (\omega + \omega^4)(\omega^2 + \omega^3) = \omega^3 + \omega^4 + \omega^6 + \omega^7 = c$

since $\left.\begin{array}{l}\omega^6 = \omega^5 \cdot \omega = \omega \\ \omega^7 = \omega^5 \cdot \omega^2 = \omega\end{array}\right\}$ $\therefore \underbrace{\omega^3 + \omega^4 + \omega + \omega^2}_{-1} = c = -1$ (1)

$\therefore$ quad. equation is $x^2 + x - 1 = 0$

d) solve $z^4 - z^3 + 6z^2 - z + 15 = 0$ if $z = 1 - 2i$ is root

$\therefore$ coeff. are real $\therefore$ $1 + 2i = \bar{z}$ is root also (1)

$\therefore$ quadratic factor is $[z - (1-2i)][z - (1+2i)] = z^2 - 2z + 5$

now $(z^4 - z^3 + 6z^2 - z + 15) \div (z^2 - 2z + 5) = z^2 + z + 3$

$$\begin{array}{r}
-(z^4 - 2z^3 + 5z^2) \\
\hline
z^3 + z^2 - z + 15 \\
-(z^3 - 2z^2 + 5z) \\
\hline
3z^2 - 6z + 15 \\
-(3z^2 - 6z + 15) \\
\hline
0
\end{array}$$

$\therefore$ the other roots are solutions of $z^2 + z + 3 = 0$

$\therefore z = \frac{-1 \pm \sqrt{11}}{2}$ (1)

$\therefore$ roots are : $1 - 2i$, $1 + 2i$, $-\frac{1}{2} + \frac{\sqrt{11}}{2}$, $-\frac{1}{2} - \frac{\sqrt{11}}{2}$

e) $3x^2 - 2x - 2 < |3x|$

$x > 0$ $\therefore$ $3x^2 - 2x - 2 < 3x$ or $x < 0$ $\therefore$ $3x^2 - 2x - 2 < -3x$

$3x^2 - 5x - 2 < 0$ $3x^2 + x - 2 < 0$

$(3x + 1)(x - 2) < 0$ $(3x - 2)(x + 1) < 0$

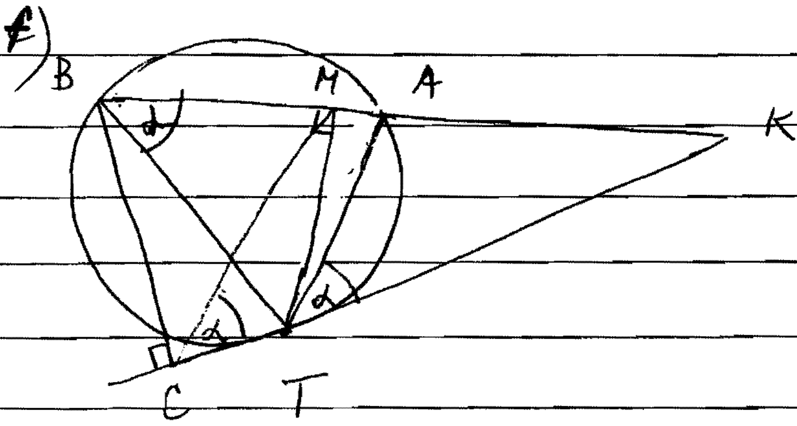
$\to -\frac{1}{3} < x < 2$ $\to -1 < x < \frac{2}{3}$

$\therefore$ $0 < x < 2$ $\leftarrow$ solutions $\rightarrow$ $\therefore -1 < x < 0$

You may ask for extra writing paper if you need more space to answer question 12

Question 12 **BOS#:**

12f)



CBMT is a cyclic quadrilateral → $\angle BMT = \angle BCT = 90°$

① ∴ opposite ∠'s in quadrilateral are supplementary

Now $\angle ATK = \alpha = \angle TBM$ ① (∠ between tangent & chord at point of contact = ∠ in alternate segment subtended by that chord)

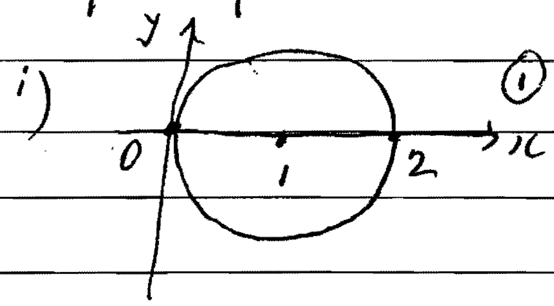
But $\angle MCT = \angle MBT = \alpha$ (angles standing on the same arc TM of circle CBMT)

∴ $AT \parallel MC$ (since $\angle ATK = \angle MCT$ and they're corresponding ∠'s) ①

Question 13 **BOS#:** ____________

a) $|z-1|=1$

i)



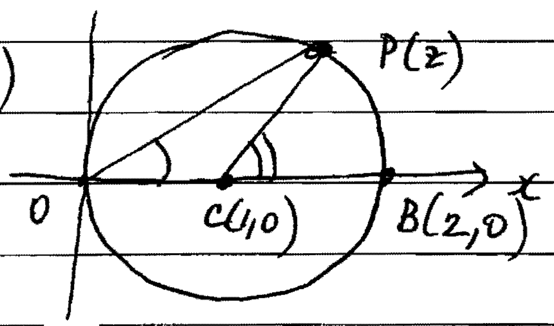
1

Locus is a circle centre $(1,0)$ radius $r=1$

1

ii) $|z^2-z| = |z(z-1)| = |z|\,|z-1| = |z|\times 1 = |z|$ ∴ shown

iii)



Let P be the pt. representing z
centre $C(1,0)$, pt. $B(2,0)$

Now $\angle PCB = 2\times\angle POC$ (angle at the centre = double angle at circumf.)

but $\angle POC = \arg z$ and $\angle PCB = \arg(z-1)$

$\therefore \arg(z-1) = 2\cdot\arg z$ 1

Now $\frac{2}{3}\arg(z^2-z) = \frac{2}{3}\arg[z(z-1)] = \frac{2}{3}[\arg z + \arg(z-1)]$

1 $= \frac{2}{3}[\arg z + 2\cdot\arg z] = \frac{2}{3}(3\arg z) = 2\arg z = \arg(z-1)$

b) i) since $z^5-1 = (z-1)(z^4+z^3+z^2+z+1)$

$\therefore$ roots of $z^4+z^3+z^2+z+1=0$ are amongst roots of $z^5-1=0$, where $z\neq 1$. 1

$\therefore$ roots are $z_1 = cis\,\frac{2\pi}{5}$, $z_2 = cis\,\frac{4\pi}{5}$

$z_3 = \bar{z}_2 = cis\left(-\frac{4\pi}{5}\right)$

$z_4 = \bar{z}_1 = cis\left(-\frac{2\pi}{5}\right)$ 1

Question 13 **BOS#:** ______________________

13b) ii) Now $z^4+z^3+z^2+z+1 = [(z-z_1)(z-z_2)(z-z_3)(z-z_4)]$

$= (z-z_1)(z-z_2)(z-\bar{z}_2)(z-\bar{z}_1)$

$= (z^2-2z.\text{Re}z_1+1)(z^2-2z.\text{Re}z_2+1)$

$\therefore z^4+z^3+z^2+z+1 = [z^2-2z.\cos\frac{2\pi}{5}+1][z^2-2z.\cos\frac{4\pi}{5}+1]$

Now match up coefficients of z

$\therefore z = -2z.\cos\frac{2\pi}{5} + (-2z.\cos\frac{4\pi}{5})$

$\therefore 1 = -2(\cos\frac{2\pi}{5}+\cos\frac{4\pi}{5})$

$\therefore -1/2 = \cos\frac{2\pi}{5} + \cos\frac{4\pi}{5}$ $\therefore$ proven

c) i) $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ $\therefore \frac{2x}{a^2}+\frac{2y}{b^2}.\frac{dy}{dx}=0$ $\therefore \frac{dy}{dx}=\frac{-b^2}{a^2}.\frac{x}{y}$

$\therefore$ at $P(a\cos\theta, b\sin\theta)$ $m_T = \frac{-b^2}{a^2}.\frac{a\cos\theta}{b\sin\theta} = \frac{-b\cos\theta}{a\sin\theta}$ (1)

$\therefore$ equation of tangent $\therefore$ $y-b\sin\theta = \frac{-b\cos\theta}{a\sin\theta}(x-a\cos\theta)$

$\frac{y\sin\theta}{b} - \sin^2\theta = \frac{-\cos\theta}{a}x + \cos^2\theta$ (1)

$\therefore \frac{x\cos\theta}{a} + \frac{y\sin\theta}{b} = \cos^2\theta+\sin^2\theta = 1$

$\therefore$ shown

ii) QQ' ∥ tangent at P $\therefore m_{QQ'} = \frac{-b\cos\theta}{a\sin\theta}$

and $(0,0)\in QQ'$ $\therefore$ equation of QQ' $\therefore$ $y = -\frac{b\cos\theta}{a\sin\theta}x$ (1)

$\therefore y\,a\sin\theta = -b\cos\theta\,x$

$\therefore QQ'$ is : $xb\cos\theta + y\,a\sin\theta = 0$

You may ask for extra writing paper if you need more space to answer question 13

Question 13 **BOS#:**______________________

13c) iii)

$$Q, Q' \begin{cases} xb\cos\theta + ya\sin\theta = 0 & (1) \\ \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1 & (2) \end{cases}$$

$$(1)\ x = \frac{-ya\sin\theta}{b\cdot\cos\theta} \quad \therefore (2)\ \frac{\frac{y^2 a^2\sin^2\theta}{b^2\cos^2\theta}}{a^2} + \frac{y^2}{b^2} = 1$$

$$\therefore \frac{y^2\sin^2\theta}{b^2\cdot\cos^2\theta} + \frac{y^2}{b^2} = 1 \quad \therefore \frac{y^2}{b^2}\left[\frac{\sin^2\theta}{\cos^2\theta} + 1\right] = 1$$

$$\therefore y^2\left[\frac{\sin^2\theta + \cos^2\theta}{\cos^2\theta}\right] = b^2 \quad \therefore y^2 = b^2\cos^2\theta \quad \therefore y = \pm b\cos\theta \quad (1)$$

$$\therefore x_{Q,Q'} = -y\frac{a\sin\theta}{b\cos\theta} = \begin{cases} -(+b\cos\theta)\dfrac{a\sin\theta}{b\cos\theta} = -a\sin\theta \\ -(-b\cos\theta)\dfrac{a\sin\theta}{b\cos\theta} = +a\sin\theta \end{cases}$$

$$\therefore Q(-a\sin\theta, +b\cos\theta) \quad Q'(+a\sin\theta, -b\cos\theta) \quad (1)$$

iv) $\text{Area}_{\Delta QPQ'} = \frac{1}{2}\cdot QQ' \times h$

$$QQ' = \sqrt{(2a\sin\theta)^2 + (2b\cos\theta)^2} = 2\sqrt{a^2\sin^2\theta + b^2\cos^2\theta}$$

h = perp. distance of QQ': $xb\cos\theta + ya\sin\theta = 0$ from $P(a\cos\theta, b\sin\theta)$

$$\therefore h = \frac{|a\cos\theta\cdot b\cos\theta + b\sin\theta\cdot a\cdot\sin\theta|}{\sqrt{b^2\cos^2\theta + a^2\sin^2\theta}} \ (1) = \frac{|ab|\cdot|\cos^2\theta + \sin^2\theta|}{\sqrt{b^2\cos^2\theta + a^2\sin^2\theta}}$$

$$\therefore \text{Area}_{\Delta QPQ'} = \frac{1}{\cancel{2}}\,\frac{|ab|}{\cancel{\sqrt{b^2\cos^2\theta + a^2\sin^2\theta}}}\cdot \cancel{2}\cancel{\sqrt{a^2\sin^2\theta + b^2\cos^2\theta}} = \underbrace{|ab|}_{\text{constant}}$$

which is independent of position of P.

Question 14 **BOS#:** ______

a) STEP1 test $n=0$ $\therefore LHS = 1$

$$RHS = \frac{1-\text{cis}\,\theta}{1-\text{cis}\,\theta} = 1$$

$\therefore$ the statement is true for $n=0$.

STEP 2: Assume that statement is true for $n=k$

ie. $1+\text{cis}\,\theta+\text{cis}\,2\theta+\dots+\text{cis}\,k\theta = \dfrac{1-\text{cis}(k+1)\theta}{1-\text{cis}\,\theta}$

STEP 3: Now prove statement true for $n=k+1$ ie. $1+\text{cis}\,\theta+\dots+\text{cis}(k+1)\theta = \dfrac{1-\text{cis}(k+2)\theta}{1-\text{cis}\,\theta}$

$LHS = \underbrace{1+\text{cis}\,\theta+\dots+\text{cis}\,k\theta}_{\text{assumption}}+\text{cis}(k+1)\theta = \dfrac{1-\text{cis}(k+1)\theta}{1-\text{ci}\,\theta}+\text{cis}(k+1)\theta$

$$= \frac{1-\text{cis}(k+1)\theta+(1-\text{cis}\,\theta)(\text{cis}(k+1)\theta)}{1-\text{cis}\,\theta}$$

$$= \frac{1-\cancel{\text{cis}(k+1)\theta}+\cancel{\text{cis}(k+1)\theta}-\text{cis}\,\theta\cdot\text{cis}(k+1)\theta}{1-\text{cis}\,\theta}$$

$$= \frac{1-\text{cis}[\theta+(k+1)\theta]}{1-\text{cis}\,\theta} = \frac{1-\text{cis}(k+2)\theta}{1-\text{cis}\,\theta} = RHS$$

$\therefore$ the statement is true for $n=k+1$ if it's true for $n=k$, and since proven true for $n=0$ $\therefore$ by math. induction the statement is true for all integers $n\geq 0$.

You may ask for extra writing paper if you need more space to answer question 15

Question 14 **BOS#:** ______________________

b) i) $z_2 - z_1 = (z_3 - z_1)\,\text{cis}\,\frac{\pi}{3}$ 1

ii) $z_1 - z_3 = (z_2 - z_3)\,\text{cis}\,\frac{\pi}{3}$ 1

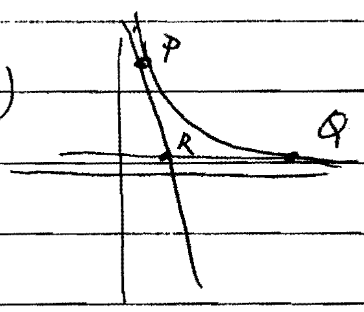
$$\therefore \frac{z_2 - z_1}{z_1 - z_3} = \frac{(z_3 - z_1)\cancel{\text{cis}\frac{\pi}{3}}}{(z_2 - z_3)\cancel{\text{cis}\frac{\pi}{3}}} \quad 1$$

$$\therefore (z_2 - z_1)(z_2 - z_3) = (z_3 - z_1)(z_1 - z_3) \quad 1$$

$$z_2^2 - z_1 z_2 + \cancel{z_1 z_3} - z_2 z_3 = z_3 z_1 - z_1^2 - z_3^2 + \cancel{z_1 z_3}$$

$$\therefore z_1^2 + z_2^2 + z_3^2 = z_1 z_2 + z_3 z_2 + z_1 z_3 \quad \therefore \text{proven}$$

c) i)



$$xy = c^2 \;\therefore\; y = \frac{c^2}{x}$$

$$\frac{dy}{dx} = \frac{-c^2}{x^2}$$

at $P\left(cp, \frac{c}{p}\right)$ $\therefore m_T = \frac{-c^2}{c^2p^2} = \frac{-1}{p^2}$ 1

eqn. of tangent at P is

$$y - \frac{c}{p} = -\frac{1}{p^2}(x - cp) \quad 1$$

$$p^2\left(y - \frac{c}{p}\right) = -(x - cp)$$

$$p^2 y - cp = -x + cp$$

$$\therefore x + p^2 y = 2cp \quad (i) \text{ proven}$$

ii) similarly at $Q\left(cq, \frac{c}{q}\right)$ tangent is $\therefore x + q^2 y = 2cq$ (ii)

$R(x_0 y_0)$ $\begin{cases} x_0 + p^2 y_0 = 2cp \\ x_0 + q^2 y_0 = 2cq \end{cases}$ (i) $-$ (ii)

$$p^2 y_0 - q^2 y_0 = 2cp - 2cq \quad 1$$

$$(p^2 - q^2)y_0 = 2c(p - q)$$

$$(p-q)(p+q)y_0 = 2c(p-q) \quad \div(p-q)$$

if $|p| \neq |q|$ $\therefore p - q \neq 0$ $\therefore (p+q)y_0 = 2c$

$$\therefore p + q = \frac{2c}{y_0} \quad 1$$

Question 14 **BOS#:**

14c) cont.

From (i) $x_0 = 2cp - p^2 y_0$ $\div y_0$

$\therefore \frac{x_0}{y_0} = \frac{2cp}{y_0} - \frac{p^2 \cancel{y_0}}{\cancel{y_0}}$ ① but $\frac{2c}{y_0} = p+q$

$\therefore \frac{x_0}{y_0} = (p+q).p - p^2 = \cancel{p^2} + qp - \cancel{p^2} = pq$

$\therefore \frac{x_0}{y_0} = pq$ (proven)

iii) $PQ^2 = d^2$ (since $PQ = d$)

$\therefore d^2 = (cp - cq)^2 + \left(\frac{c}{p} - \frac{c}{q}\right)^2$

$= c^2(p-q)^2 + c^2\left(\frac{1}{p} - \frac{1}{q}\right)^2$

$= c^2\left[(p-q)^2 + \frac{(q-p)^2}{p^2q^2}\right]$ ① $\{(p-q)^2 = (q-p)^2\}$

$= c^2(p-q)^2\left[1 + \frac{1}{p^2q^2}\right]$

iv) but $(p-q)^2 = (p+q)^2 - 4pq$

sub. this into (iii)

$\therefore d^2 = c^2\left[(p+q)^2 - 4pq\right]\left[1 + \frac{1}{p^2q^2}\right]$

also $(p+q) = \frac{2c}{y_0}$ and $pq = \frac{x_0}{y_0}$

$\therefore d^2 = c^2\left[\left(\frac{2c}{y_0}\right)^2 - 4\left(\frac{x_0}{y_0}\right)\right]\left[1 + \frac{1}{\left(\frac{x_0}{y_0}\right)^2}\right]$

$d^2 = c^2\left[\frac{4c^2}{y_0^2} - \frac{4x_0}{y_0}\right]\left[1 + \frac{y_0^2}{x_0^2}\right]$

$d^2 = \frac{4c^2}{y_0^2}(c^2 - x_0y_0)\left[\frac{x_0^2 + y_0^2}{x_0^2}\right]$

Question 14 **BOS#:**________________________

14 c) iv) cont.

$\therefore d^2 x_0^2 y_0^2 = 4c^2(c^2 - x_0 y_0)(x_0^2 + y_0^2)$

but $R(x_0, y_0) \in$ on the locus

$\therefore$ locus of $R(x_0, y_0)$ is: $d^2 x^2 y^2 = 4c^2(c^2 - xy)(x^2 + y^2)$

$\therefore 4c^2(x^2 + y^2)(c^2 - xy) = x^2 y^2 d^2.$

You may ask for extra writing paper if you need more space to answer question 16