BAULKHAM HILLS HIGH SCHOOL

2017

YEAR 12 HALF YEARLY
HIGHER SCHOOL CERTIFICATE
EXAMINATION

Mathematics Extension 2

General Instructions

- Reading time – 5 minutes
- Working time – 2 hours
- Write using black or blue pen
- Black pen is preferred
- Board-approved calculators may be used
- A reference sheet is provided at the back of this paper
- All relevant mathematical reasoning and/or calculations must be shown

Total marks – 70

Section I (Pages 2-4)
10 marks
Attempt Questions 1-10
Allow about 15 minutes for this section

Section II (Pages 5-9)
60 marks
Attempt Questions 11-14
Allow about 1 hours 45 minutes for this section

Section I

10 marks
Attempt Questions 1-10
Allow about 15 minutes for this section

Use the multiple-choice answer sheet for Questions 1 to 10.

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1 For the ellipse with the equation $\frac{x^2}{16} + \frac{y^2}{25} = 1$. What is the eccentricity of the ellipse?

(A) $\frac{9}{25}$

(B) $\frac{3}{5}$

(C) $\frac{1}{5}$

(D) $\frac{1}{\sqrt{5}}$

2 If $w = \sqrt{3}(\cos \pi + i \sin \pi)$, then w^4 is equal to:

(A) -9

(B) 9

(C) 81

(D) -81

3 The polynomial $P(z) = z^3 + (1+i)z^2 + (1+i)z + 1$ has a real zero $z = -1$ and a complex zero $z = \alpha$. The third root is:

(A) $\frac{1}{\alpha}$

(B) $\bar{\alpha}$

(C) $-\alpha$

(D) $1 - \alpha$

4 What is the multiplicity of the root $x = 1$ of the equation $3x^5 - 5x^4 + 5x - 3 = 0$?

(A) 1

(B) 2

(C) 3

(D) 4

5 z and w are two complex numbers. Which of the following statements is ALWAYS TRUE?

(A) $|z+w| \geq |z-w|$

(B) $|z+w| \leq |z-w|$

(C) $|z|-|w| \geq |z+w|$

(D) $|z|+|w| \geq |z-w|$

6 The polynomial $P(x) = x^4 + ax^3 + bx^2 + cx + d$ has real coeffiecients and $P(2i) = P(2+i) = 0$. What is the value of $a+b+c+d$?

(A) 0

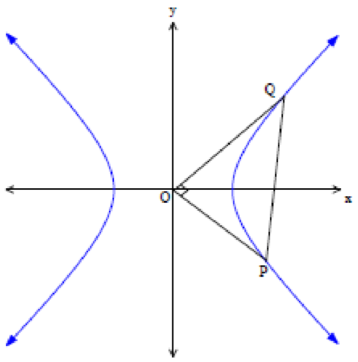
(B) 6

(C) 9

(D) 49

7 The diagram below shows the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ where $a > b > 0$.

The points $P(a\, sec\, \theta\, , b \tan\theta)$ and $Q(a \sec\alpha\, , b\tan\alpha)$ lie on the hyperbola and the chord PQ subtends a right angle at the origin.



Which of the following is true?

(A) $\tan\theta \tan\alpha = -\frac{a^2}{b^2}$

(B) $\tan\theta \tan\alpha = \frac{a^2}{b^2}$

(C) $\sin\theta \sin\alpha = -\frac{a^2}{b^2}$

(D) $\sin\theta \sin\alpha = \frac{a^2}{b^2}$

8 If $x^2 + 3xy = 5y^2$, which of the following is an expresion for $\frac{dy}{dx}$?

(A) $\frac{2x+3y}{10y-3x}$

(B) $\frac{2x}{10y-9xy}$

(C) $\frac{2x+3y}{\sqrt{5(x^2+3xy)}}$

(D) $\frac{2x+3y}{10y}$

9 Seven travellers arrive in a town where there are 4 hotels. If two of the travellers want to stay in the same hotel, how many different accommodation arrangements are there?

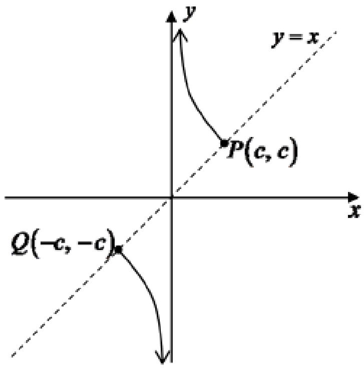
(A) $\frac{6!}{2!2!} \times 2$

(B) $\frac{7!}{2!2!2!} \times 2$

(C) $4^6 \times 2$

(D) 4^6

10 The graph shows a part of the hyperbola $xy = c^2$



Which pair of parametric equations precisely describes the sections of the hyperbola shown?

(A) $x = c(t^2 + 1), y = \frac{c}{t^2+1}$

(B) $x = c(1 - t^2), y = \frac{c}{1-t^2}$

(C) $x = c\sqrt{1 - t^2}, y = \frac{c}{\sqrt{1-t^2}}$

(D) $x = c \sin t\,, y = \frac{c}{\sin t}$

End of Section I

Section II

90 marks
Attempt questions 11 -14
Allow about 1 hours 45 minutes for this section

Answer each question on the appropriate page of your answer booklet
In Questions 11–14, your responses should include relevant mathematical reasoning and/or calculations.

Question 11 (15 marks)		**Marks**
(a)	Let $z = 2 + i$ and $w = i - 1$. Find, in the form $x + iy$:	
	(i) $3z + iw$	**1**
	(ii) $z\overline{w}$	**2**
(b)	(i) Express $\sqrt{3} + i$ in modulus argument form.	**1**
	(ii) If $z = \sqrt{3} + i$, hence show that $z^7 + 64z = 0$.	**2**
(c)	Consider the hyperbola with the equation $\frac{x^2}{16} - \frac{y^2}{9} = 1$.	
	(i) Find the eccentricity of the hyperbola.	**1**
	(ii) Find the coordinates of the foci and the x-intercepts.	**2**
	(iii) Find the equations of the directrices and the equations of the asymptotes.	**2**
(d)	Sketch the region in the complex plane where the inequalities $\lvert z + \bar{z} \rvert \leq 1$ and $\lvert z - i \rvert \leq 1$ hold simultaneously.	**2**
(e)	Let ω be one of the complex roots of the equation $z^3 - 64 = 0$.	
	(i) Show that $\omega^2 = -4(\omega + 4)$.	**1**
	(ii) Hence evaluate $(\omega + 4)^3$.	**1**
	End of Question 11	

Question 12 (15 marks) **Marks**

(a) The roots of $x^3 + 6x^2 + 5x - 2 = 0$ are α, β and γ.

(i) Find the monic polynomial with roots α^2, β^2 and γ^2. **2**

(ii) Find the value of $\alpha^3 + \beta^3 + \gamma^3$. **3**

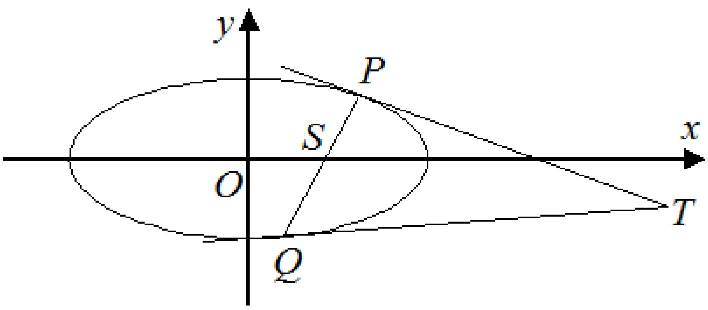
(b) Find a possible polynomial equation $P(x)$, of smallest degree that satisfies the following conditions: **3**

- The polynomial $P(x)$ has rational coefficients.
- Two of its roots are $1 + \sqrt{5}$ and $-6i$.
- When $y = P(x)$ is drawn on a Cartesian plane, there are four x-intercepts.

(c) A complex number ω is such that $|\omega| = 1$. **2**

If $z = \frac{1+\omega}{1-\omega}$, find the locus of z as ω moves on the complex number plane.

(d) The point P lies on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, where $a > b$
The chord through P and the focus $S(ae, 0)$ meets the ellipse again at Q.
The tangents to the ellipse at P and Q meet at the point $T(x_0, y_0)$.
The equation of PQ is $\frac{xx_0}{a^2} + \frac{yy_0}{b^2} = 1$. **[DO NOT PROVE THIS]**



(i) Show that T lies on the directrix. **1**

The point P is now chosen so that T lies on the x-axis.

(ii) What is the value of the ratio $PS : ST$? **1**

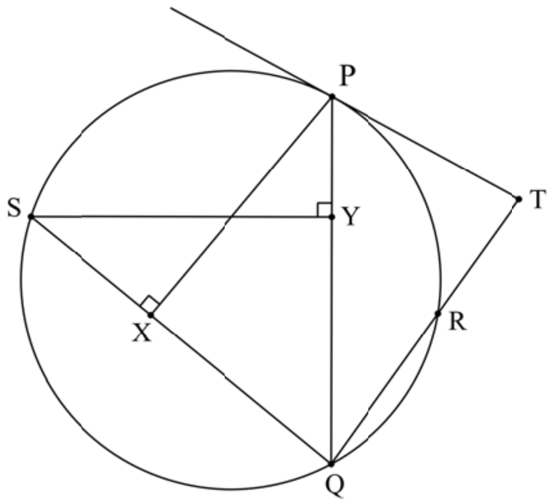
(iii) Show that $\angle PTQ$ is acute. **1**

(iv) Show that the area of the triangle PQT is $b^2\left(\frac{1}{e} - e\right)$. **2**

End of Question 12

Question 13 (15 marks) **Marks**

(a) In the diagram below, TP is the tangent of the circle at P, and TQ is a secant cutting the circle at R.
SQ is a chord of the circle such PX and SY are perpendicular to SQ and PQ respectively.



Copy the diagram in your answer booklet.

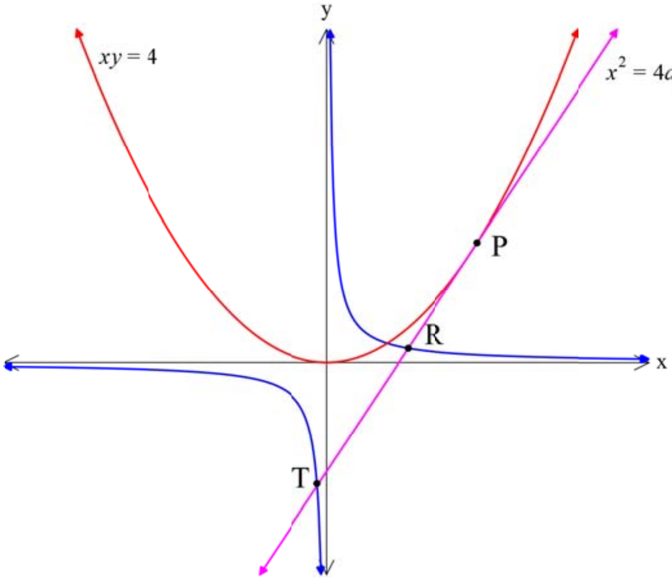
(i) Prove that $\angle TRP = \angle TPQ$. **2**

(ii) Explain why $SPYX$ is a cyclic quadrilateral and state the diameter of the circle $SPYX$. **2**

(iii) Prove $\angle PYX = \angle PRQ$. **2**

Question 13 continues on the next page

(b) $P(8p, 4p^2)$ is a point on the parabola $x^2 = 16y$.
The tangent to the parabola at P cuts the hyperbola $xy = 4$ at two distinct points R and T.



(i) M is the midpoint of RT, show that M has coordinates $(2p, -2p^2)$. **2**
You may assume the tangent to the parabola at P is
$y = px - 4p^2$. **[DO NOT PROVE THIS]**

(ii) Find the equation of the locus of M, as P moves on the parabola $x^2 = 16y$, stating any restrictions. **3**

(c) Let $z = x + iy$ be any non-zero complex number.

(i) Show $z + \frac{1}{z} = x + \frac{x}{x^2+y^2} + i\left(y - \frac{y}{x^2+y^2}\right)$. **1**

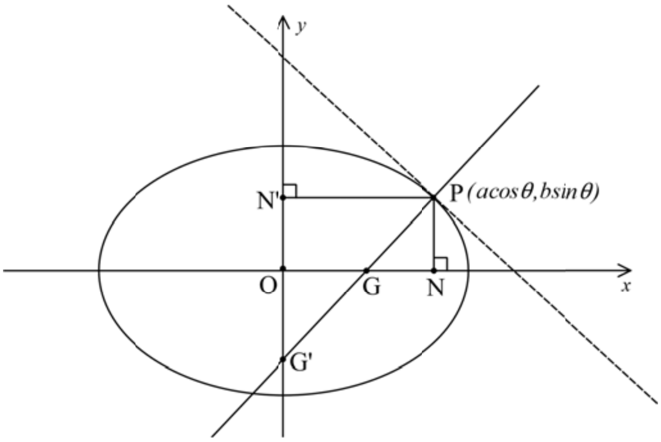
(ii) Given that $z + \frac{1}{z} = k$, where k is real, show that $y = 0$ or $x^2 + y^2 = 1$. **1**

(iii) Show also that if $y = 0$ then $|k| \geq 2$ and that if $x^2 + y^2 = 1$ then $|k| \leq 2$. **2**

End of Question 13

Question 14 (15 marks) **Marks**

(a) The diagram below shows the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, where $a>b$.
The point $P(a\cos\theta\,, b\sin\theta)$ lies on the ellipse.



The normal to the ellipse at P meets the major and minor axes of the ellipse at G and G'respectively. N and N' are the feet of the perpendiculars from P to the major axes and minor axes respectively.

(i) Show that the equation of the normal at P is $ax\sec\theta - by\,\mathrm{cosec}\,\theta = a^2-b^2$. **2**

(ii) Show that the ratio of $OG\colon ON = e^2\colon 1$. **2**

(iii) Given that $\Delta PN'G'///\Delta PNG$, find the ratio of the area of $\Delta PN'G'\colon \Delta PNG$. **1**

(b) (i) Prove $4\cos\frac{\theta}{2}\cos\frac{\theta}{4}\sin\frac{\theta}{4}=\sin\theta$ **1**

(ii) Prove by Mathematical Induction that for any positive integer n **3**

$$\sin\theta = 2^n\cos\frac{\theta}{2}\cos\frac{\theta}{4}\dots\cos\frac{\theta}{2^n}\sin\frac{\theta}{2^n}$$

(iii) Explain why **1**

$$\lim_{n\to\infty}\left(\frac{\sin\frac{\theta}{2^n}}{\theta}\times 2^n\right)=1$$

(iv) Hence using (ii), (iii) and choosing a suitable value for θ, show: **3**

$$\frac{2}{\pi}=\cos\frac{\pi}{4}\times\cos\frac{\pi}{8}\times\cos\frac{\pi}{16}\times\dots\dots$$

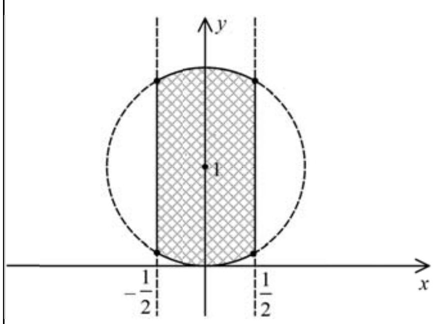
2

(v) Show that $\pi = 2\left(\frac{2}{\sqrt{2}}\times\frac{2}{\sqrt{2+\sqrt{2}}}\times\dots\dots\right)$

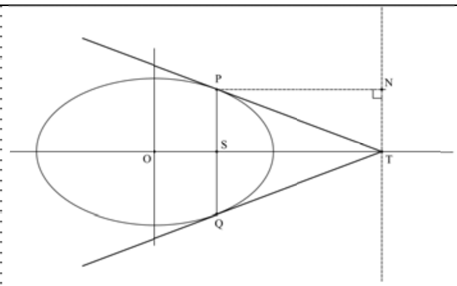
End of Exam

HSC Year12 Extension 2 Half Yearly 2017

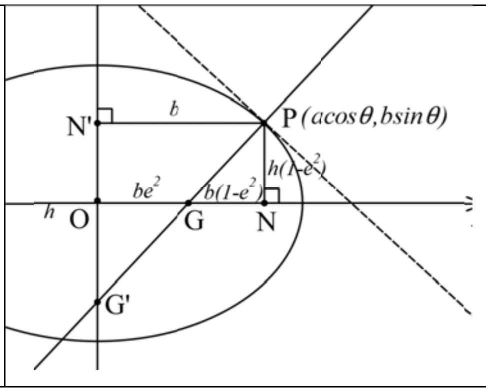
	Solutions	Mks	Comments
1	$b^2 = a^2(1-e^2)$ $16 = 25(e^2-1)$ $e = \frac{3}{5}$	**1**	B
2	$w^4 = \sqrt{3}^4 cis(4\times\pi)$ $w^4 = 9cis\ 4\pi$ $w^4 = 9cis\ 0$ $w^4 = 9$	**1**	B
3	Let the roots be $\alpha, \beta, -1$ Product of roots $= -\frac{b}{a}$ $\alpha \times \beta \times -1 = -1$ $\beta = \frac{1}{\alpha}$	**1**	A
4	$Let\ P(x) = 3x^5 - 5x^4 + 5x - 3$ $P(1) = 0$ $P'(x) = 15x^4 - 20x^3 + 5$ $P'(1) = 0$ $P''(x) = 60x^3 - 60x^2$ $P''(1) = 5$ $P'''(x) = 180x^2 - 120x$ $P'''(1) = 60$ $\therefore P(x)$ has multiplicity 3	**1**	C
5	Triangular inequality	**1**	D
6	The roots are $2i, -2i, 2+i, 2-i$ $P(x) = (x^2+4)(x^2-4x+5)$ $P(x) = x^4 - 4x^3 + 9x^2 - 16x + 20$ $a+b+c+d = -4+9-16+20$ $= 9$	**1**	C
7	$m_{OP} = \frac{b\tan\theta}{a\sec\theta}$ $\qquad m_{OQ} = \frac{b\tan\alpha}{a\sec\alpha}$ $m_{OP} \times m_{OQ} = -1$ $\frac{b\tan\theta}{a\sec\theta} \times \frac{b\tan\alpha}{a\sec\alpha} = -1$ $\frac{\cos^2\theta}{\cos^2\theta} \times \frac{b^2\tan\theta}{a^2\sec\theta} \times \frac{\tan\alpha}{\sec\alpha} = -1$ $\sin\theta\sin\alpha = -\frac{a^2}{b^2}$	**1**	C
8	Derive both side implicitly: $2x + \left(3 \times y + \frac{dy}{dx} \times 3x\right) = 10y \times \frac{dy}{dx}$ $2x + 3y = \frac{dy}{dx}(10y - 3x)$ $\frac{dy}{dx} = \frac{2x+3y}{10y-3x}$	**1**	A
9	If two particular people are together, then there are 6 entities, ie $(AB)CDEFG$. (AB) will have 4 rooms to choose from C will have 4 rooms to choose from … G will have 4 rooms to choose from $\therefore$ there are 4^6 ways to place 6 entities.	**1**	D

	Solutions	Mks	Comments
10	(A) False as x is positive, and there are x-ordiantes that are negative (B) False as $t \to \infty, x \to -\infty$ (C) False as x is positive, and there are x-ordiantes that are negative (D) True as $-1 \le \sin t \le 1, -c \le c\sin t \le c$, ie $-c \le x - \text{ordinate} \le c$	**1**	D
11a(i)	$3(2+i)+i(i-1) = 6+3i-1-i$ $= 5+2i$	**1**	**1 mark** • correct answer.
11a(ii)	$(2+i)(-1-i) = -1-3i$	**2**	**2 mark** • correct solution. **1 mark** • correct conjugate • correct multiplication with a wrong expression of the conjugate.
11b(i)	$z = 2\left[\cos\left(\frac{\pi}{6}\right) + i\sin\left(\frac{\pi}{6}\right)\right]$	**1**	**1 mark** • correct answer.
11b(ii)	$z^7 - 64z = z(z^6 - 64)$ $z^6 = 2^6\left(\cos\frac{6\pi}{6} + i\sin\frac{6\pi}{6}\right)$ $z^6 = 64$ $z^7 - 64z = z(64-64)$ $= 0$	**2**	**2 mark** • correct solution. **1 mark** • correct use of DM's Thm
11c(i)	$9 = -16(1-e^2)$ $e = \frac{5}{4}$	**1**	**1 mark** • correct answer
11c(ii)	$\therefore foci\ are\ (\pm 5, 0)$ $x - \text{int when } y = 0$ $\frac{x^2}{16} = 1$ x-intercepts are $(\pm 4, 0)$	**2**	**2 mark** • correct solution. **1 mark** • correct foci • correct intercept
11c(iii)	Directricies: $x = \pm\frac{16}{5}$ Eqn of asymptote: $y = \pm\frac{3}{4}x$	**2**	**2 mark** • correct solution. **1 mark** • correct directrices • correct asyptote
11d	$\lvert z + \bar{z} \rvert \le 1$ $\lvert 2x \rvert \le 1$ $-\frac{1}{2} \le x \le \frac{1}{2}$  	**2**	**2 mark** • correct solution. **1 mark** • Shows region inside circle at centre (0,1) • Shows region bewtween the lines $-\frac{1}{2} \le x \le \frac{1}{2}$ • Correct boundaries with incorrect region.
11e(i)	$\omega^3 - 64 = 0$ $(\omega - 4)(\omega^2 + 4\omega + 16) = 0$ Since ω is a complex root so $\omega \ne 4$ $\omega^2 + 4\omega + 16 = 0$ $\omega^2 = -4\omega - 16$ $\omega^2 = -4(\omega + 4)$	**1**	**1 mark** • correct solution.

	Solutions	Mks	Comments
11e(ii)	$$(\omega+4)^3=\left(\frac{\omega^2}{-4}\right)^3$$ $$=\frac{(\omega^3)^2}{-64}$$ $$=\frac{(64)^2}{-64}$$ $$=-64$$	**1**	**1 mark** • correct solution.
12a(i)	Let $A=x^2$, where $x=\alpha,\beta$ and γ $$x=\sqrt{A}$$ $$\sqrt{A}^3+6\sqrt{A}^2+5\sqrt{A}-2=0$$ $$A\sqrt{A}+5\sqrt{A}=2-6A$$ Square both sides $$A(A+5)^2=(2-6A)^2$$ $$A^3-26A^2+49A-4=0$$ $$\therefore x^3-26x^2+49x-4=0$$	**2**	**2 mark** • correct solution. **1 mark** • substitutes $x\sqrt{A}$
12a(ii)	$$\alpha^3+6\alpha^2+5\alpha-2=0$$ $$\beta^3+6\beta^2+5\beta-2=0$$ $$\gamma^3+6\gamma^2+5\gamma-2=0$$ $$\alpha^3+\beta^3+\gamma^3+6(\alpha^2+\beta^2+\gamma^2)+5(\alpha+\beta+\gamma)-2\times 3=0$$ $$\alpha^3+\beta^3+\gamma^3+6(26)+5(-6)-2\times 3=0$$ $$a^3+\beta^3+\gamma^3=-120$$	**3**	**3 mark** • correct solution. **2 mark** • significant progress to finding $\alpha^3+\beta^3+\gamma^3$ **1 mark** • Finds $\alpha^2+\beta^2+\gamma^2$ and $\alpha+\beta+\gamma$ • Forms a polynomial with α^2,β^2 and γ^2
12b	If the polynomial has rational coefficients, then the roots must be $-6i, 6i, 1+\sqrt{5}, 1-\sqrt{5}$ As the equation $(x^2+36)(x^2-2x-4)$ has two x-intercepts, it needs any two factors to have 4 x intercepts. The smallest degree must be 6. A possible equation is $x(x-1)(x^2+36)(x^2-2x-4)$	**3**	**3 mark** • correct solution. **2 mark** • Two conditions met. **1 mark** • One condition met
12c	$$z-z\omega=1+\omega$$ $$z-1=\omega(1+z)$$ $$\lvert z-1\rvert=\lvert\omega\rvert\times\lvert 1+z\rvert$$ But $\lvert\omega\rvert=1$ $$\lvert z-1\rvert=\lvert z+1\rvert$$ $$\therefore \text{locus is } x=0$$	**2**	**2 mark** • correct solution. **1 mark** • eliminates ω • simplifies z using $\lvert\omega\rvert=1$
12d (i)	PQ passes through $S(ae,0)$ $$\frac{ae\,x_0}{a^2}+\frac{0\times y_0}{b^2}=1$$ $$\therefore x_0=\frac{a}{e}$$ So $T(x_0,y_0)$ lies on the directrix $x=\frac{a}{e}$	**1**	**1 mark** • correct solution

	Solutions	Mks	Comments
12d (ii)	Since T is on the x-axis, We know that $\frac{PS}{PN} = e$ by the definition of the ellipse. Since $PSTN$ is a rectangle, $PN = ST$ $\therefore \frac{PS}{ST} = e$ 	1	**1 mark** • correct solution
12d(iii)	$\tan \angle PTS = \frac{PS}{PT}$ $= e < 1$ $\angle PTS < 45°$ $\angle PTQ = 2\angle PTS < 90°$	1	**1 mark** • correct solution
12d(iv)	Sub $x = ae$ into $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ $e^2 + \frac{y^2}{b^2} = 1$ $y^2 = b^2(1 - e^2)$ $PS^2 = b^2(1 - e^2)$ $Area\ \Delta PQT = PS \times ST$ $= PS \times \frac{PS}{e}$ $= \frac{b^2(1 - e^2)}{e}$ $= b^2\left(\frac{1}{e} - e\right)$	2	**2 mark** • correct solution. **1 mark** • Finds PS^2 or PS • Correct area using the wrong value of PS.
13a(i)	Let $\angle TRP = \theta$ $\angle TRP = \angle PSQ = \theta$ (Exterior angle of cyclic quad = interior opposite angle) $\angle PSQ = \angle TPQ = \theta$ (alternate segment theorem) $\therefore \angle TPQ = \angle TRP = \theta$	2	**2 marks** • correct solution. **1 mark** • significant progress towards a correct solution.
13a(ii)	$\angle SXP = \angle SYP = 90°$ (angle in a semi circle) $SPYX$ is a cyclic quad with SP as a diameter.	2	**2 marks** • correct solution. **1 mark** • Explains $SPYX$ is a cyclic quad • States SP is a diameter.
13a(iii)	$\angle PYX + \angle PSX = 180°$(opposite angles of a cyclic quad $SPYX$) $\angle PYX = 180° - \theta$ $\angle PRQ + \angle PRT = 180°$ (angles on a straight line) $\therefore\ \angle PRQ = 180° - \theta$ $\angle PYX = \angle PRQ = 180° - \theta$	2	**2 mark** • correct solution. **1 mark** • significant progress towards solution
13b(i)	$y = px - 4p^2 \quad - - - (1)$ $xy = 4 - - - (2)$ Let the intersection of the hyperbola and the parabola be $R(\alpha, y_R)$ and $T(\beta, y_T)$, $x(px - 4p^2) = 4$ $px^2 - 4p^2x - 4 = 0$ $\alpha + \beta = \frac{4p^2}{p}$ $x_M = \frac{4p}{2}$ $x_M = 2p$ $y_M = p(2p) - 4p^2$ $y_M = -2p^2$ $M = (2p, -p^2)$	2	**2 marks** • correct solution. **1 mark** • finds x_M

	Solutions	Mks	Comments
13b(iii)	Sub $p=\frac{x_M}{2}$ into y_M $y_M=-2\times\left(\frac{x_M}{2}\right)^2$ $\therefore$ The locus of M lies on $y=-\frac{x^2}{2}$ At $(0,0)$ the tangent at P does not touch the hyperbola. $\therefore x\neq 0$ Since P and Q is distinct, $P\neq Q$ $(2)\longrightarrow y=\frac{4}{x}$ sub $(2)\to(1)$ $\frac{4}{x}=px-4p^2$ $px^2-4p^2x-4=0$ $\Delta>0$ $16p^4-4\times p\times -4>0$ $16p(p^3+1)>0$ $16p\geq 0$ or $p^3<-1$ $4x>0$ or $\frac{x}{2}<-1$ $x>0$ or $x<-2$ $\underline{\therefore x<-2 \text{ or } x>0}$	3	**3 marks** • correct solution. **2 mark** • finds any part of the correct restriction • correct inequality and solves. **1 mark** • finds the equation of M lies on $y=-\frac{x^2}{2}$ • finds $16p(p^3+1)>0$
13c(i)	$z+\frac{1}{z}=x+iy+\frac{1}{x+iy}$ $=x+iy+\frac{x-iy}{x^2+y^2}$ $=x+\frac{x}{x^2+y^2}+i\left(y-\frac{y}{x^2+y^2}\right)$	1	**1 mark** • correct solution
13c(ii)	If k is real then $Im\left(z+\frac{1}{z}\right)=0$ $y-\frac{y}{x^2+y^2}=0$ $y(x^2+y^2)-y=0$ $y(x^2+y^2-1)=0$ $y=0$ or $x^2+y^2=1$	1	**1 mark** • correct solution
13c(iii)	If $y=0$ $x+\frac{x}{x^2+0^2}=k$ $x+\frac{1}{x}=k$ $x^2+1=kx$ $x^2-kx+1=0$ $\Delta\geq 0$ $k^2-4\geq 0$ $k^2\geq 4$ $\lvert k\rvert\geq 2$ If $x^2+y^2=1$ $x+x=k$ $x=\frac{k}{2}$ Since $x^2+y^2=1$, The x-ordinate must be ≤ 1 $\lvert x\rvert\leq 1$ $\lvert k\rvert\leq 2$	2	**2 mark** • correct solution. **1 mark** • finds one inequality for k

	Solutions	Mks	Comments
14a(i)	Differentiating both sides: $\frac{2x}{a^2}+\frac{2y}{b^2}\times\frac{dy}{dx}=0$ $\frac{dy}{dx}=-\frac{b^2x}{a^2y}$ $m_T=-\frac{b}{a}\cot\theta$ $m_N=\frac{a}{b}\tan\theta$ Equation of normal: $y-b\sin\theta=\frac{a}{b}\tan\theta\,(x-a\cos\theta)$ $by\cos\theta-b^2\sin\theta\cos\theta=ax\sin\theta-a^2\sin\theta\cos\theta$ Multiply both sides by $b\cos\theta$ $by\cos\theta-b^2\sin\theta\cos\theta=ax\sin\theta-a^2\sin\theta\cos\theta$ $\sin\theta\cos\theta\,(a^2-b^2)=ax\sin\theta-by\cos\theta$ $a^2-b^2=ax\sec\theta-by\operatorname{cosec}\theta$	**2**	**2 mark** • correct solution. **1 mark** • finds m_N
14a(ii)	G occurs at the $x$int $a^2-b^2=ax\sec\theta$ $x=\frac{a^2-b^2}{a\sec\theta}$ $OG=\frac{a^2-b^2}{a\sec\theta}$ $ON=a\cos\theta\quad(x-ordinate\ of\ P)$ $\frac{OG}{ON}=\frac{a^2-b^2}{a\sec\theta}\times\frac{1}{a\cos\theta}$ $but\ b^2=a^2(1-e^2)$ $b^2-a^2=a^2e^2$ $\frac{OG}{ON}=\frac{a^2e^2}{a^2}$ $\frac{OG}{ON}=e^2$ $OG:ON=e^2:1$	**2**	**2 mark** • correct solution. **1 mark** • finds m_N
14a(iii)	Since ΔPNG and $\Delta PN'G'$ are similar triangles and $OG:ON=e^2$ $OG=e^2\times ON$ $OG=e^2\times PN'$ $GN=PN'(1-e^2)$ $G'N'=PN\times(1-e^2)$ $\therefore Area\ \Delta PN'G':Area\ \Delta PNG$ $=1:(1-e^2)^2$  	**1**	**1 mark** • correct ratio.

	Solutions	Mks	Comments
14b(i)	$LHS = \cos\frac{\theta}{2}\cos\frac{\theta}{4}\sin\frac{\theta}{4}$ $= \cos\frac{\theta}{2}\left(2\times\cos\frac{\theta}{4}\sin\frac{\theta}{4}\right)\times\frac{1}{2}$ $= \cos\frac{\theta}{2}\sin\frac{\theta}{2}\times 2\times\frac{1}{2}\times\frac{1}{2}$ $= \frac{1}{4}\sin\theta$ $= RHS$	**1**	**1 mark** • correct proof.
14b(ii)	Let $n = 1$ LHS = $\sin\theta$ $RHS = 2^1\cos\frac{\theta}{2^1}\sin\frac{\theta}{2^1}$ $= \sin\theta$ (double angle formula) Assume true for $n = k$ $\sin\theta = 2^k\cos\frac{\theta}{2}\cos\frac{\theta}{4}\ldots\cos\frac{\theta}{2^k}\sin\frac{\theta}{2^k}$ Prove true for $n = k+1$ AIM: $\sin\theta = 2^{k+1}\cos\frac{\theta}{2}\cos\frac{\theta}{4}\ldots\cos\frac{\theta}{2^{k+1}}\sin\frac{\theta}{2^{k+1}}$ $RHS = 2^{k+1}\cos\frac{\theta}{2}\cos\frac{\theta}{4}\ldots\cos\frac{\theta}{2^k}\cos\frac{\theta}{2^{k+1}}\sin\frac{\theta}{2^{k+1}}$ $= 2^k\cos\frac{\theta}{2}\cos\frac{\theta}{4}\ldots\cos\frac{\theta}{2^k}\times 2\cos\frac{\theta}{2^{k+1}}\sin\frac{\theta}{2^{k+1}}$ $= 2^k\cos\frac{\theta}{2}\cos\frac{\theta}{4}\ldots\cos\frac{\theta}{2^k}\times\sin\frac{\theta}{2^k}$ (double angle formula) $= \sin\theta$ (if the assumption is true) $= LHS$ ∴ If it is true for $n = k+1$, if it is true for $n = k$ Since it is also true for $n = 1$, then it is true for positive integer of n by Mathematical Induction.	**3**	**3 mark** • correct solution. **2 mark** • Completes all of the conditions below. **1 mark** • Completes one of the conditions below. **Conditions:** **(A)** Tests for $n = 1$, and writes the correct assumption **(B)** Shows the use of the assumption in a correct expression of $n = k+1$
14b(iii)	$\lim_{n\to\infty}\left(\frac{\sin\left(\frac{\theta}{2^n}\right)}{\theta}\times 2^n\right) = \lim_{n\to\infty}\left(\frac{\sin\left(\frac{\theta}{2^n}\right)}{\left(\frac{\theta}{2^n}\right)}\right)$ if $n\to\infty$, $\frac{\theta}{2^n}\to 0$, so $\frac{\theta}{2^n}$ is a very small angle $\lim_{n\to\infty}\left(\frac{\sin\left(\frac{\theta}{2^n}\right)}{\left(\frac{\theta}{2^n}\right)}\right) = \lim_{\left(\frac{\theta}{2^n}\right)\to 0}\left(\frac{\sin\left(\frac{\theta}{2^n}\right)}{\left(\frac{\theta}{2^n}\right)}\right) = 1$	**1**	**1 mark** • Correct proof and <u>must explain</u> $\frac{\theta}{2^n}$ is a small angle.

	Solutions	Mks	Comments
14b(iv)	Using the expression in part (ii), divide both side by θ $\frac{\sin\theta}{\theta} = 2^n \cos\frac{\theta}{2}\cos\frac{\theta}{4}\dots\ \cos\frac{\theta}{2^n}\ \frac{\sin\frac{\theta}{2^n}}{\theta}$ $As\ n \to \infty$, ans using (iii) $\lim_{n\to\infty}\left(\frac{\sin\theta}{\theta}\right) = \lim_{n\to\infty}\left(\cos\frac{\theta}{2}\cos\frac{\theta}{4}\dots\ \cos\frac{\theta}{2^n}\ \frac{\sin\frac{\theta}{2^n}}{\theta}\times 2^n\right)$ $\frac{\sin\theta}{\theta} = \cos\frac{\theta}{2}\cos\frac{\theta}{4}\dots\dots\times 1$ By substituting $\theta = \frac{\pi}{2}$ $\frac{\sin\frac{\pi}{2}}{\frac{\pi}{2}} = \cos\frac{\left(\frac{\pi}{2}\right)}{2}\cos\frac{\left(\frac{\pi}{2}\right)}{4}\cos\frac{\left(\frac{\pi}{2}\right)}{8}\dots$ $\frac{2}{\pi} = \cos\frac{\pi}{4}\cos\frac{\pi}{8}\cos\frac{\pi}{16}\dots$	3	**3 mark** • correct solution **2 mark** • Completes two of the conditions below. **1 mark** • Completes one of the conditions below. **Conditions:** **(A)** Uses $\theta = \frac{\pi}{2}$ **(B)** Manipulates expression to use the limit in (iii) **(C)** Uses (ii) and divides both sides by θ
14b(v)	$\cos\frac{\pi}{4} = \frac{\sqrt{2}}{2}$ we know: $\cos 2A = 2\cos^2 A - 1$ $\cos A = \sqrt{\frac{\cos 2A + 1}{2}}$ $\cos\frac{\pi}{8} = \sqrt{\frac{\cos\frac{\pi}{4}+1}{2}}$ $= \sqrt{\frac{\frac{\sqrt{2}}{2}+1}{2}}$ $= \sqrt{\frac{\sqrt{2}+2}{4}}$ $= \frac{\sqrt{\sqrt{2}+2}}{2}$ $\cos\frac{\pi}{16} = \sqrt{\frac{\cos\frac{\pi}{8}+1}{2}}$ $= \sqrt{\frac{\frac{\sqrt{\sqrt{2}+2}}{2}+1}{2}}$ $= \sqrt{\frac{\sqrt{\sqrt{2}+2}+2}{4}}$ $= \frac{\sqrt{\sqrt{\sqrt{2}+2}+2}}{2}$ $\therefore \frac{\pi}{2} = \frac{2}{\sqrt{2}}\times\frac{2}{\frac{\sqrt{\sqrt{2}+2}}{2}}\times\frac{2}{\sqrt{\sqrt{\sqrt{2}+2}+2}}\dots\dots$ $\therefore \pi = 2\left(\frac{2}{\sqrt{2}}\times\frac{2}{\frac{\sqrt{\sqrt{2}+2}}{2}}\times\frac{2}{\sqrt{\sqrt{\sqrt{2}+2}+2}}\dots\dots\right)$	2	**2 mark** • correct solution, showing upto $\cos\frac{\pi}{8}$. **1 mark** • Finds the exact value of $\cos\frac{\pi}{4}$ and attempt to use the double angle to find $\cos\frac{\pi}{8}$