2009
SEMESTER 1
EXAMINATION

Student Number

Mathematics Extension 2

General Instructions

- Reading time – 5 minutes
- Working time – 2 hours
- Write using blue or black pen
- Board-approved calculators may be used
- A table of standard integrals is provided at the back of this paper
- All necessary working should be shown in every question

Total Marks – 75

- Attempt questions 1-5
- All questions are of equal value

Question 1 (15 marks) Start a new writing booklet. Marks

(a) State the domain and range of the function $g(x) = \frac{1}{2}\cos^{-1}\left(\frac{x}{2}\right)$. **2**

(b) Find $\frac{d}{dx}(3x^2 \cos^{-1}x)$. **2**

(c) Differentiate $\ln(\sin^{-1} 2x)$. **2**

(d) Evaluate exactly $\cos^{-1}\left(\frac{1}{2}\right) - \sin^{-1}\left(-\frac{1}{2}\right)$. **2**

(e) **(i)** For what values of x is $\sin^{-1}x$ defined? **1**

(ii) Find the maximum value of $2x(1-x)$. **2**

(iii) Find the range of the function f given by $f(x) = \sin^{-1}\{2x(1-x)\}$ with domain $0 \le x \le 1$. **2**

(f) $\displaystyle\int \frac{2x}{\sqrt{1-x^4}}\,dx$ **2**

End of Question 1

Question 2 (15 marks) Start a new writing booklet.. Marks

(a) A complex number z is given by $1 + \sqrt{3}\,\mathbf{i}$.

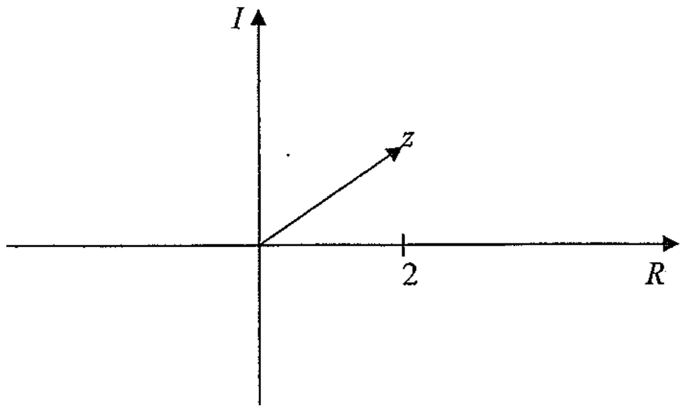
(i) Write down $\bar{z}$. **1**

(ii) Verify that $z\bar{z}$ is real. **1**

(iii) Find $\frac{1}{z}$ in the form $a + ib$, where a and b are real. **1**

(b) Express $1 + \mathbf{i}$ in modulus argument form, and hence simplify $(1 + \mathbf{i})^6$, giving your answer in the form $\mathbf{a + ib}$. **2**

(c) A point z on the Argand Diagram is given below:



Copy this diagram into your examination booklet and use it to plot the following points.

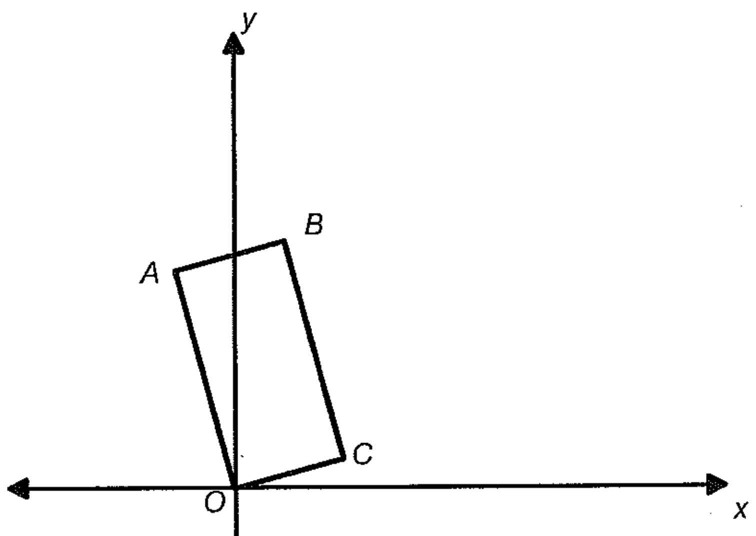
Clearly label each point.

(i) $\bar{z}$ **1**

(ii) $2iz$ **1**

(iii) $\dfrac{1}{z}$ **1**

(d)



The points *OABC* are the vertices of a rectangle on the Argand diagram with $|OA| = 2\,|OC|$. If *OC* represents the complex number $p+iq$, write down the complex numbers represented by:

(i) $\overrightarrow{OA}$ **1**

(ii) $\overrightarrow{OB}$ **1**

(iii) $\overrightarrow{BC}$ **1**

(iv) $\overrightarrow{AC}$ **1**

(e)

(i) If $P(x)=x^4+x^3-3x^2-5x-2$, show that $P(x)=0$ has a multiple root, find this root and its multiplicity. **2**

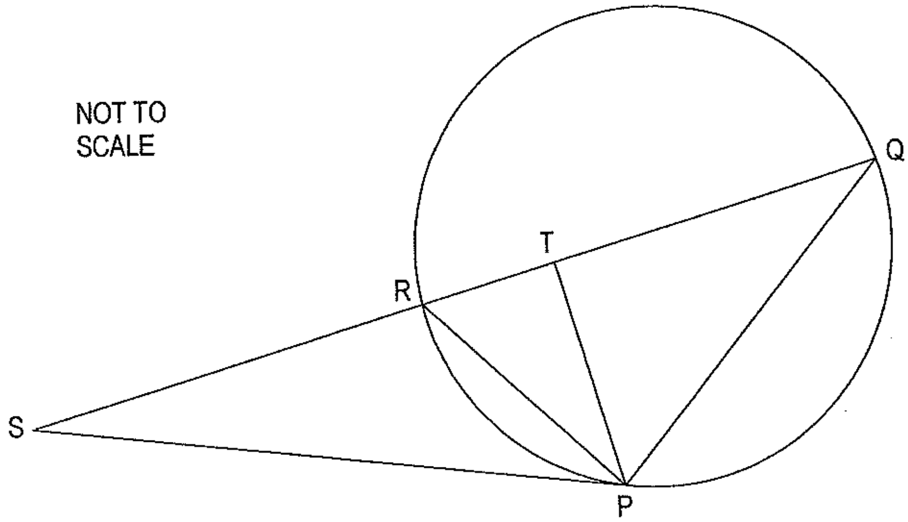
(ii) Hence factorise $P(x)=x^4+x^3-3x^2-5x-2$ into its linear factors. **1**

End of Question 2

Marks

Question 3 (15 marks) Start a new writing booklet..

(a)



QR is the diameter of the circle. The tangent to the circle at P meets QR produced to S.
T is situated on QR such that PR bisects $\angle$TPS.

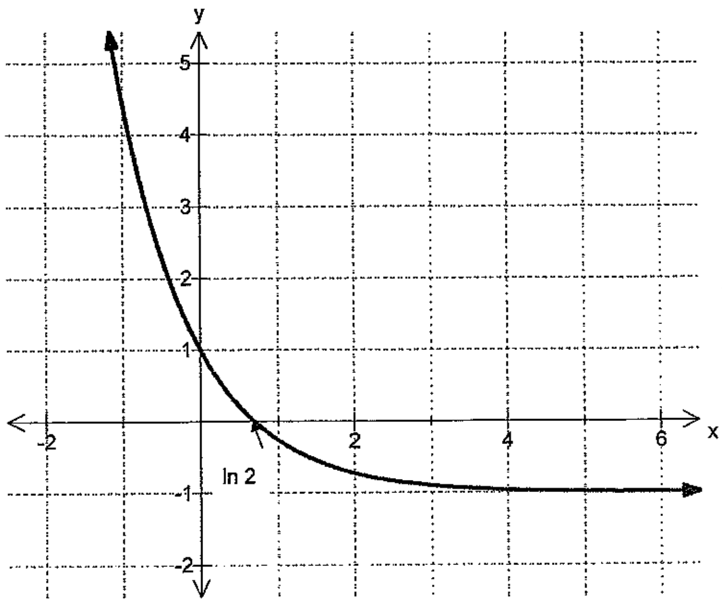
COPY OR TRACE THE DIAGRAM ONTO YOUR PAGE

(i) Give a reason why $\angle RPS = \angle PQR$ **1**

(ii) Hence, show that $PT \perp QR$ **3**

(b) Find the equation of the tangent to the curve $\cos 2x + \sin y = 1$ at the point $x = \frac{\pi}{6}$. **3**

(c)



The diagram shows the graph of $y = 2e^{-x} - 1$. On separate diagrams sketch the following graphs, showing the intercepts on the axes and the equations of any asymptotes:

(i) $y = |f(x)|$ **1**

(ii) $y = \{f(x)\}^2$ **1**

(iii) $y = \dfrac{1}{f(x)}$ **2**

(iv) $y = \ln\{f(x)\}$ **1**

(d) Given $|z + i| \leq 2$ and $0 \leq \arg(z + 1) \leq \frac{\pi}{4}$. Sketch the region in an Argand diagram which contains the point P representing z. **3**

End of Question 3

Question 4 (15 marks) **Start a new writing booklet..** **Marks**

(a) The roots of $x^3 - 3x^2 - 2x + 4 = 0$ are α, β and γ.
Answer the following without finding the actual values of α, β and γ.

(i) Find a cubic equation whose roots are α^2, β^2 and γ^2 **2**

(ii) Hence or otherwise find the value of $\alpha^2 + \beta^2 + \gamma^2$ **1**

(iii) Determine the value of $\alpha^3 + \beta^3 + \gamma^3$ **2**

(b) **(i)** Given $z = \cos\theta + i\sin\theta$, prove $Z^n + \frac{1}{z^n} = 2\cos n\theta$. **2**

(ii) Hence by considering the expansion $\left(z + \frac{1}{z}\right)^4$ show that

$$\cos^4\theta = \frac{1}{8}\cos 4\theta + \frac{1}{2}\cos 2\theta + \frac{3}{8}$$ **3**

(iii) Hence evaluate $\int_0^{\frac{\pi}{2}} \cos^4\theta \, d\theta$. **2**

(c) The roots of the polynomial $P(x) = x^3 + ax^2 + bx + c = 0$ are three consecutive terms of an arithmetic series.
Prove that the relationship between the coefficients is given by

$$2a^3 - 9ab + 27c = 0$$ **3**

Hint: make an appropriate choice for the roots in arithmetic progression

End of Question 4

Question 5 (15 marks) **Start a new writing booklet..** **Marks**

(a) $z_1 = 1 + i$ and $z_2 = \sqrt{3} - i$

(i) Find $\frac{z_1}{z_2}$ in the form $a+ib$ where a and b are real. **1**

(ii) Write z_1 and z_2 in modulus – argument form. **2**

(iii) By equating equivalent expressions for $\frac{z_1}{z_2}$ write $\cos\frac{5\pi}{12}$ as a surd. **1**

(iv) Explain why there is no positive integer n such $z_1(z_2)^n$ is real. **2**

(b)

(i) Prove that $y = \frac{1}{x}$ is concave up for all $x > 0$. **3**

(ii) Sketch $y = \frac{1}{x}$ for $x > 0$.

Suppose $0 < a < b$ and consider the points $A\left(a, \frac{1}{a}\right)$ and $B\left(b, \frac{1}{b}\right)$ on the graph of $y = \frac{1}{x}$. **1**

(iii) Plot two such points on your sketch. **1**

(iv) Find the coordinates of the point P that divides the line segment AB in the ratio 2:1. **2**

(v) With reference to areas and by careful argument, deduce that

$$\ln\frac{b}{a} < \frac{b^2 - a^2}{2ab}$$ **2**

End of Paper

STANDARD INTEGRALS

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1}, \quad n \neq -1; \; x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x}\,dx = \ln x, \quad x > 0$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax}, \quad a \neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax, \quad a \neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax, \quad a \neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax\,dx = \frac{1}{a}\sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2 + x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2 - x^2}}\,dx = \sin^{-1}\frac{x}{a}, \quad a > 0, \; -a < x < a$$

$$\int \frac{1}{\sqrt{x^2 - a^2}}\,dx = \ln\left(x + \sqrt{x^2 - a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2 + a^2}}\,dx = \ln\left(x + \sqrt{x^2 + a^2}\right)$$

NOTE : $\ln x = \log_e x, \quad x > 0$