CHELTENHAM GIRLS HIGH SCHOOL

YEAR 12
COMMON TEST 2

2007

EXTENSION 2

Time allowed : 120 MINUTES

DIRECTIONS TO CANDIDATES

- Attempt ALL questions.
- ALL questions are of equal value.
- All necessary working should be shown in every question. Marks may be deducted for careless or badly arranged work
- Board approved calculators may be used.
- Each question is to be started on a **new page** and you are to write your name and teachers name on each page.
- The marks allocated for each question are indicated

Name : ________________________ Student Number: ________________

Q	1	2	3	4	Total
Mark	/20	/20	/20	/20	/80

Question 1

a) For the ellipse $\frac{x^2}{16}+\frac{y^2}{4}=1$:

i) determine the value of e (1)

ii) find the coordinates of the foci (1)

iii) find the equations of the directrices (1)

iv) sketch the ellipse (1)

b) Solve the equation $x^2+2x+11=0$. (3)

c) The polynomial $P(x)=x^4-x^3-7x^2+13x-6$ has a root of multiplicity 2.

i) Show that 2 is a zero of *P(x)*. (1)

ii) Determine the multiple root . (2)

iii) Factorise *P(x)* into linear factors. (1)

d) From what external point is the tangent to the ellipse $\frac{x^2}{9}+\frac{y^2}{4}=1$ to be drawn so that $4x-3y+12=0$ is the chord of contact.

You may assume that the chord of contact is from the external point

$T(x_0, y_0)$ is $\frac{xx_0}{a^2}+\frac{yy_0}{b^2}=1$. (2)

e) For the polynomial $P(x)=x^3-2x^2+x-1$ determine the values of:

i) $\alpha+\beta+\gamma$ (1)

ii) $\alpha\beta\gamma$ (1)

iii) $\frac{1}{\alpha\beta}+\frac{1}{\alpha\gamma}+\frac{1}{\beta\gamma}$ (1)

f) i) Write $z=1+i\sqrt{3}$ in modulus argument form. (2)

ii) Hence determine the value of z^{10}. Give your answer in $x+iy$ form. (2)

Question 2

a) i) Divide the polynomial $f(x) = 2x^4 - 10x^3 + 12x^2 + 2x - 3$ by $g(x) = x^2 - 3x + 1$. **(3)**

ii) Hence write $f(x) = g(x)q(x) + r(x)$ where $q(x)$ and $r(x)$ are polynomials. **(1)**

iii) Hence show that $f(x)$ and $g(x)$ have no zeros in common. **(1)**

b) Write $\dfrac{4x^2 - 5x - 7}{(x-1)(x^2 + x + 2)}$ in the form $\dfrac{A}{x-1} + \dfrac{Bx + C}{x^2 + x + 2}$. **(3)**

c) The points P, Q and R represent the complex numbers w_1, w_2 and w_3 where

$$w_2 - w_1 = i(w_3 - w_1).$$

What kind of triangle is ΔPQR ? Explain your answer. **(2)**

d) Show that the condition for $y = mx + c$ to touch $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ is

$$c^2 = b^2 + a^2m^2$$ **(4)**

e) Sketch the locus of z given that $z^2 - \bar{z}^2 = 16i$. **(3)**

f) Find the equation of the tangent to $2x^2 - y^3 = 1$ at $x = -1$. **(3)**

Question 3

a) i) Find the real numbers a and b such that

$$x^4 + x^3 + x^2 + x + 1 = (x^2 + ax + 1)(x^2 + bx + 1).$$ (3)

ii) Find the solutions of $x^4 + x^3 + x^2 + x + 1 = 0$. (3)

iii) Hence determine the exact value of $\cos\frac{2\pi}{5}$. (2)

b) i) Show that $P(a\cos\vartheta, b\sin\vartheta)$ lies on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. (1)

ii) S and S' are the foci of $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. Show that $SP = a - ae\cos\vartheta$. (3)

iii) If $SP' = a + ae\cos\vartheta$ show that $SP + S'P = 2a$ (1)

c) i) Show that the equation of the normal at $P(x_1, y_1)$ to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

, is $\frac{(x - x_1)y_1}{b^2} = \frac{(y - y_1)x_1}{a^2}$. (4)

ii) Show that the only normal that passes through a focus is the x – axis. (3)

Question 4

a) The roots of $x^3 + qx + r = 0$ are α, β and γ.

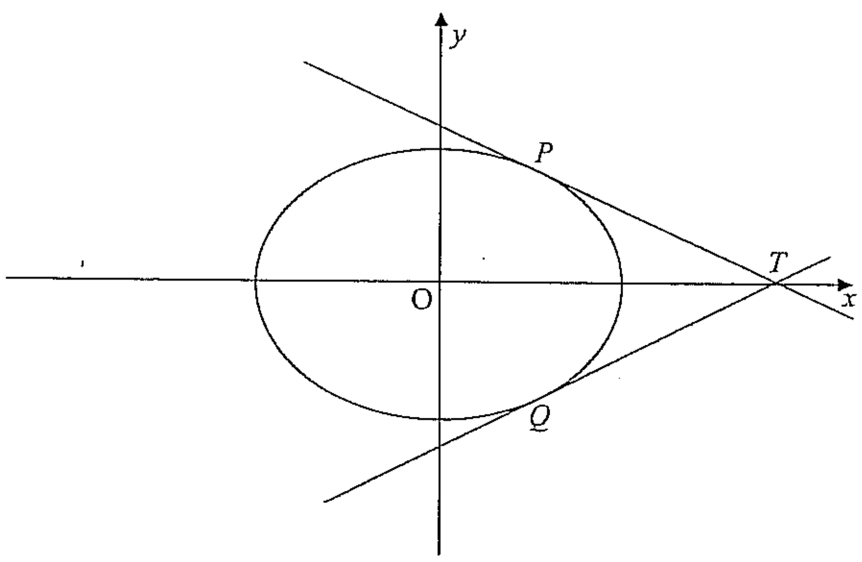
i) Determine $\alpha + \beta + \gamma$. **(1)**

ii) Determine $\alpha\beta + \beta\gamma + \alpha\gamma$. **(1)**

iii) Determine $\alpha\beta\gamma$. **(1)**

iv) Hence prove $(\beta - \gamma)^2 + (\gamma - \alpha)^2 + (\alpha - \beta)^2 = -6q$. **(4)**

b) The point P lies on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. The tangents to the ellipse through P and Q meet at the point $T(\frac{a}{e}, 0)$.



i) Determine the equation of the chord of contact. **(4)**

ii) What is the value of the ratio $\frac{PS}{ST}$? **(1)**

iii) Show that $\angle PTQ$ is less than a right angle. **(1)**

iv) Show that the area of ΔPTQ is $\frac{b^4}{a^2 e}$. **(3)**

Question 4 is continued on the next page.

Question 4 (continued)

c) Let $w = \dfrac{3+4i}{5}$ and $z = \dfrac{5+12i}{13}$, so that $|w| = |z| = 1$. **Do not prove this.**

i) Find wz and $w\bar{z}$ in the form $x + iy$. **(2)**

ii) Hence find 2 distinct ways of writing 65^2 as the sum of a^2 and b^2, where a and b are integers and $0 < a < b$. **(2)**

END OF EXAMINATION

Question 1

a) $\frac{x^2}{16} + \frac{y^2}{4} = 1$

i) $b^2 = a^2(1-e^2)$

$4 = 16(1-e^2)$

$\frac{4}{16} = 1 - e^2$

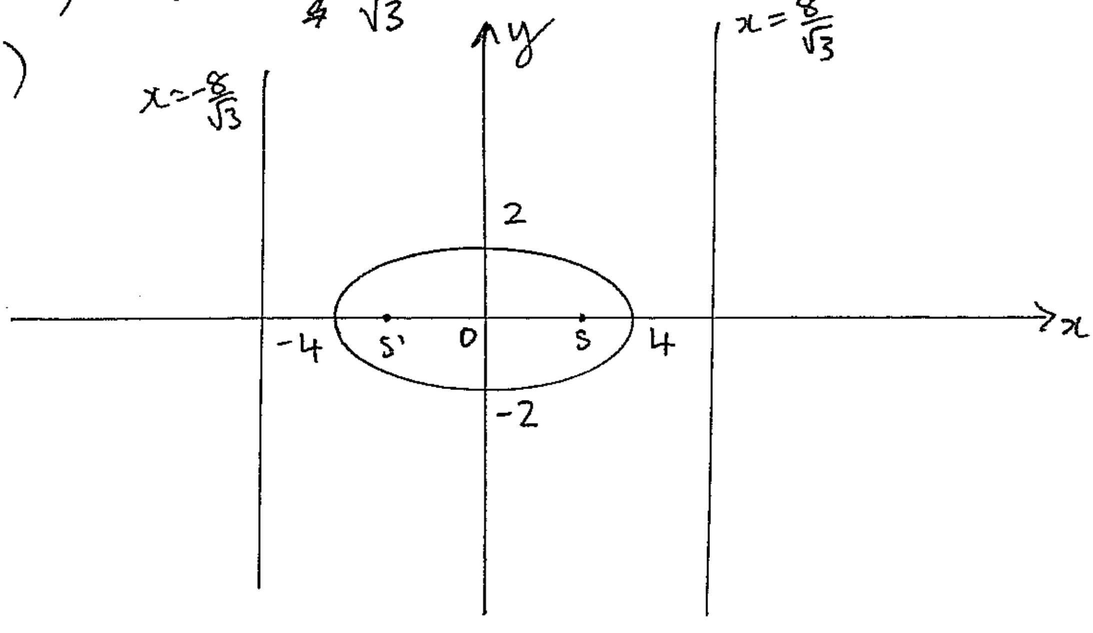
$e^2 = 1 - \frac{4}{16}$

$e = \frac{\sqrt{3}}{2}$

ii) $S(2\sqrt{3}, 0) \quad S'(-2\sqrt{3}, 0)$

iii) $x = \cancel{\pm} \frac{8}{\sqrt{3}}$

iv)



b) $x^2 + 2x + 11 = 0$

$x = \frac{-2 \pm \sqrt{4 - 4.1.11}}{2}$

$= \frac{-2 \pm \sqrt{-40}}{2}$

$= \frac{-2 \pm 2i\sqrt{10}}{2} = -1 \pm i\sqrt{10}$

c) i) $P(2) = 2^4 - 2^3 - 28 + 26 - 6$

$= 0$

$\therefore \quad x = 2$ is a zero.

ii) $P'(x) = 4x^3 - 3x^2 - 14x + 13$

$P'(1) = 4 - 3 - 14 + 13 = 0$

$P(1) = 0$

$\therefore \quad x = 1$ is a repeated root.

$(x-1)^2 (x-2)$ are factors of $P(x)$

$(x^2 - 2x + 1)(x-2) = x^3 - 2x^2 - 2x^2 + 4x + x - 2$

$= x^3 - 4x^2 + 5x - 2$

$$
\begin{array}{r|l}
 & \quad\quad\quad\quad\quad\quad\quad\quad x - 3 \\
x^3 - 4x^2 + 5x - 2 & x^4 - x^3 - 7x^2 + 13x - 6 \\
 & \underline{x^4 - 4x^3 + 5x^2 - 2x} \\
 & \quad 3x^3 - 12x^2 + 15x - 6 \\
 & \quad \underline{3x^3 - 12x^2 + 15x - 6} \\
 & \quad\quad\quad\quad\quad\quad\quad 0
\end{array}
$$

iii) $P(x) = (x-1)^2 (x-2)(x-3)$.

d) chord of contact : $\dfrac{xx_0}{9} + \dfrac{yy_0}{4} = 1$

$\Rightarrow 4xx_0 + 9yy_0 = 36$

eqtn. of chord : $4xx_0 - 3yy_0 = -12 \Rightarrow -12x + 9y = 36$

equating coefficients : $4x_0 = -12$; $9y_0 = 9$

$x_0 = -3$ $\quad$ $y_0 = 1$

the external pt is $(-3, 1)$

e) $\alpha+\beta+\gamma = 2$

$\alpha\beta\gamma = 1$

$$\frac{1}{\alpha\beta} + \frac{1}{\alpha\gamma} + \frac{1}{\beta\gamma} = \frac{\alpha+\beta+\gamma}{(\alpha\beta\gamma)^2}$$

$$= \frac{2}{1}$$

$$= 2$$

Question 2.

a) i)

$$\begin{array}{r|l} & 2x^2 - 4x - 2 \\ x^2 - 3x + 1 & 2x^4 - 10x^3 + 12x^2 + 2x - 3 \\ & 2x^4 - 6x^3 + 2x^2 \\ \hline & \quad -4x^3 + 10x^2 + 2x \\ & \quad -4x^3 + 12x^2 - 4x \\ \hline & \qquad -2x^2 + 6x - 3 \\ & \qquad -2x^2 + 6x - 2 \\ \hline & \qquad\qquad -1 \end{array}$$

ii) $f(x) = (x^2 - 3x + 1)(2x^2 - 4x - 2) - 1$

iii) if they had zeros in common $\Rightarrow$ they had common factors

$\Rightarrow$ $g(x)$ was a factor of $f(x)$.

b) $\dfrac{4x^2-5x-7}{(x-1)(x^2+x+2)} = \dfrac{A}{x-1} + \dfrac{Bx+C}{x^2+x+2}$

$$4x^2-5x-7 = A(x^2+x+2) + (Bx+C)(x-1)$$

when $x=1$, $\quad 4-5-7 = 4A$

$$-8 = 4A$$

$$A = -2$$

when $x=0$, $\quad -7 = 2A - C$

$$-7 = -4 - C$$

$$C = 3$$

when $x=-1$, $\quad 4+5-7 = 2A+(-B+C)(-2)$

$$2 = 2A+2B - 2C$$

$$4+2+6 = 2B$$

$$B = 6$$

c) a right angled isosceles $\triangle$ $\because$ $\left|\dfrac{w_2-w_1}{w_3-w_1}\right| = |i| = 1$ $\therefore$ $|w_2-w_1| = |w_3-w_1|$ isosceles

right angled $\because$ $\times$ by i = as a rotn by $\pi/2$

d). $\dfrac{x^2}{a^2} + \dfrac{(mx+c)^2}{b^2} = 1$

$$b^2x^2 + a^2(m^2x^2+2mxc+c^2) = a^2b^2$$

$$b^2x^2 + a^2m^2x^2 + 2mxca^2 + a^2c^2 - a^2b^2 = 0$$

$\triangle = 0$

$$= (2mca^2)^2 - 4\cdot(b^2+a^2m^2)(a^2c^2-a^2b^2)$$

$$0 = 4m^2c^2a^4 - 4(a^2b^2c^2 - a^2b^4 + a^2m^2c^2$$

$$\frac{x^2}{a^2} + \frac{(mx+c)^2}{b^2} = 1$$

$$b^2x^2 + a^2(m^2x^2 + 2mcx + c^2) = a^2b^2$$

$$b^2x^2 + a^2m^2x^2 + 2a^2mcx + a^2c^2 - a^2b^2 = 0$$

$$(b^2 + a^2m^2)x^2 + 2a^2mcx + (a^2c^2 - a^2b^2) = 0$$

$$\Delta = 0$$

$$0 = (2a^2mc)^2 - 4(b^2 + a^2m^2)(a^2c^2 - a^2b^2)$$

$$= 4a^4m^2c^2 - 4(a^2b^2c^2 - a^2b^4 + a^4m^2c^2 - a^4m^2b$$

$$0 = 4a^4m^2c^2 - 4a^2b^2c^2 + 4a^2b^4 - 4a^4m^2c^2 + 4a^4m^2b$$

$$0 = -4a^2b^2c^2 + 4a^2b^4 + 4a^4m^2b^2$$

$$4a^2b^2c^2 = 4a^2b^4 + 4a^4m^2b^2$$

$$c^2 = b^2 + a^2m^2$$

e) $z^2 - \bar{z}^2 = 16i$

$$(x+iy)^2 = x^2 + 2xyi - y^2$$

$$= x^2 - y^2 + 2xyi$$

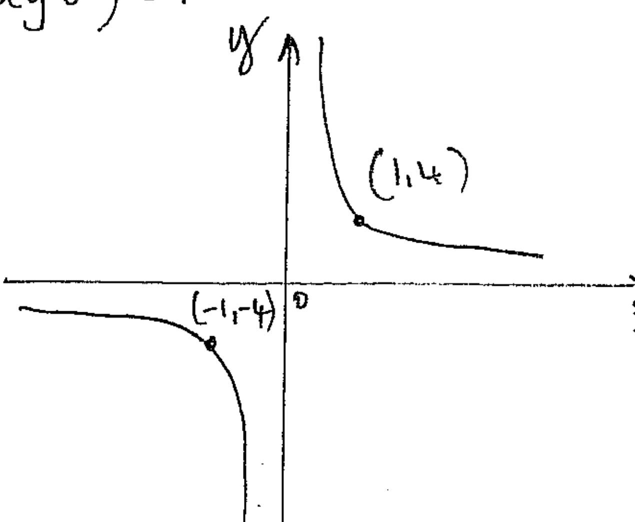
$$\bar{z}^2 = x^2 - y^2 - 2xyi$$

$$x^2 - y^2 + 2xyi - (x^2 - y^2 - 2xyi) = 16i$$

$$4xyi = 16i$$

$$4xy = 16$$

$$xy = 4$$



Q2 f). $2x^2 - y^3 = 1$ at $x = -1$ $\Rightarrow$ $2 - y^3 = 1$

$1 = y^3$

$y = 1$

$$4x - 3y^2 \cdot \frac{dy}{dx} = 0$$

$$4x = 3y^2 \cdot \frac{dy}{dx}$$

$$\frac{4x}{3y^2} = \frac{dy}{dx}$$

at $x = -1$, $\dfrac{dy}{dx} = -\dfrac{4}{3}$

$$\frac{-4}{3} = \frac{y-1}{x+1}$$

$$-4x - 4 = 3y - 3$$

$$4x + 3y + 1 = 0$$

Question 3

a) $(x^2+ax+1)(x^2+bx+1)$

$= x^4+bx^3+x^2+ax^3+abx^2+ax+x^2+bx+1$

$= x^4+(a+b)x^3+(2+ab)x^2+(a+b)x+1$

$a+b=1$, $2+ab=1$

$ab=-1$

$b=-\frac{1}{a}$

$a-\frac{1}{a}=1$

$a^2-a=1$

$a^2-a-1=0$

$a=\frac{1\pm\sqrt{1+4.1.1}}{2}$

$=\frac{1\pm\sqrt{5}}{2}$

$b=\frac{-1}{\frac{1\pm\sqrt{5}}{2}}$ $\rightarrow$

if $a=\frac{1+\sqrt{5}}{2}$

$-1\div\frac{1+\sqrt{5}}{2}$

$=-1\times\frac{2}{1+\sqrt{5}}\cdot\frac{1-\sqrt{5}}{1-\sqrt{5}}$

$=\frac{-2+2\sqrt{5}}{1-5}$

$=\frac{-1+\sqrt{5}}{-2}$

$=\frac{1-\sqrt{5}}{2}$

$a=\frac{1+\sqrt{5}}{2}$, $b=\frac{1-\sqrt{5}}{2}$

ii) $x^5 = 1$

$x^5 = \text{cis}\, 2\pi k$

$x = \text{cis}\, \frac{2\pi k}{5}$

$x_0 = \text{cis}\, 0$

$x_1 = \text{cis}\, \frac{2\pi}{5}$

$x_2 = \text{cis}\, \frac{4\pi}{5}$

$x_3 = \text{cis}\, \frac{6\pi}{5} = \text{cis}\left(-\frac{4\pi}{5}\right)$

$x_4 = \text{cis}\, \frac{8\pi}{5} = \text{cis}\left(-\frac{2\pi}{5}\right)$

sols of $x^4 + x^3 + x^2 + x + 1 = 0$ are $\text{cis}\left(\pm\frac{2\pi}{5}\right), \text{cis}\left(\pm\frac{4\pi}{5}\right)$

$\left(x - \text{cis}\,\frac{2\pi}{5}\right)\left(x - \text{cis}\left(-\frac{2\pi}{5}\right)\right)$ is a quadratic fact

$= x^2 - 2\cos\frac{2\pi}{5} +$

iii) so equating coefficients with

$x^2 + \overset{b}{\cancel{a}}x + 1$, $\cos\frac{2\pi}{5} > 0$

$-2\cos\frac{2\pi}{5} = \frac{1 - \sqrt{5}}{2}$

$\cos\frac{2\pi}{5} = \frac{-1+\sqrt{5}}{4}$

b i) $\dfrac{(a\cos\theta)^2}{a^2} + \dfrac{(b\sin\theta)^2}{b^2}$

$= \dfrac{a^2\cos^2\theta}{a^2} + \dfrac{b^2\sin^2\theta}{b^2}$

$= \cos^2\theta + \sin^2\theta$

$= 1$

$=$ RHS

$\therefore$ P lies on ellipse

ii) $S(ae, 0)$, $P(a\cos\theta, b\sin\theta)$

$SP = \sqrt{(ae - a\cos\theta)^2 + (0 - b\sin\theta)^2}$

$= \sqrt{a^2e^2 - 2a^2e\cos\theta + a^2\cos^2\theta + b^2\sin^2\theta}$

$= \sqrt{a^2e^2 - 2a^2e\cos\theta + a^2\cos^2\theta + b^2 - b^2\cos^2\theta}$

$b^2 = a^2(1-e^2)$

$b^2 = a^2 - a^2e^2$

$a^2e^2 = a^2 - b^2$

$= \sqrt{a^2 - b^2 - 2a^2e\cos\theta + b^2 + (a^2-b^2)\cos^2\theta}$

$= \sqrt{a^2 - 2a^2e\cos\theta + a^2e^2\cos^2\theta}$

$= \sqrt{(a - ae\cos\theta)^2}$

$= a - ae\cos\theta$

iii) $SP' = a + ae\cos\theta$ $\therefore$ $SP + SP' = a + ae\cos\theta + a - ae\cos\theta$

$= 2a$

c) i) $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$

$$\frac{2x}{a^2} + \frac{2y}{b^2} \cdot \frac{dy}{dx} = 0$$

$$\frac{2y}{b^2} \cdot \frac{dy}{dx} = -\frac{2x}{a^2}$$

$$\frac{dy}{dx} = \frac{-2x}{a^2} \cdot \frac{b^2}{2y}$$

$$= \frac{-b^2x}{a^2y}$$

at $P(x_1, y_1)$, $\dfrac{dy}{dx} = \dfrac{-b^2x_1}{a^2y_1}$ the gradient of normal is $\dfrac{a^2y_1}{b^2x_1}$

$$\frac{y - y_1}{x - x_1} = \frac{a^2 y_1}{b^2 x_1}$$

$$b^2x_1y - b^2x_1y_1 = a^2xy_1 - a^2x_1y_1$$

$$b^2x_1y - b^2x_1y_1 = a^2(x - x_1)y_1$$

$$b^2(y - y_1)x_1 = a^2(x - x_1)y_1$$

$$\frac{(y - y_1)x_1}{a^2} = \frac{(x - x_1)y_1}{b^2}$$

ii) if this passes through $(ae, 0)$

then $\dfrac{-y_1x_1}{a^2} = \dfrac{(ae - x_1)y_1}{b^2}$, $\div\, y_1$, assumes that $y_1 \neq$

$$-\frac{x_1}{a^2} = \frac{ae - x_1}{b^2}$$

$$-\frac{b^2}{a^2}x_1 = ae - x_1$$

$$x_1 - \frac{b^2}{a^2}x_1 = ae$$

$$x_1\left(1 - \frac{b^2}{a^2}\right) = ae$$

$$x_1e^2 = ae$$

but x_1 is on the ellipse not the $\qquad x_1 = \frac{a}{e}$

Question 4

a) i) $\alpha+\beta+\gamma = 0$

ii) $\alpha\beta+\beta\gamma+\alpha\gamma = q$

iii) $\alpha\beta\gamma = -r$

iv) $\beta+\gamma = -\alpha$

$$(\beta-\gamma)^2 = \beta^2 - 2\beta\gamma + \gamma^2$$
$$= \beta^2+\gamma^2 - 2\beta\gamma$$
$$= (\beta+\gamma)^2 - 4\beta\gamma$$
$$= \alpha^2 - 4\beta\gamma$$

$$(\beta-\gamma)^2 + (\gamma-\alpha)^2 + (\alpha-\beta)^2$$
$$= \alpha^2-4\beta\gamma + \beta^2-4\gamma\alpha + \gamma^2-4\alpha\beta$$
$$= \alpha^2+\beta^2+\gamma^2 - 4(\alpha\beta+\beta\gamma+\alpha\gamma)$$
$$= (\alpha+\beta+\gamma)^2 - 6(\alpha\beta+\beta\gamma+\alpha\gamma)$$
$$= 0^2 - 6q$$
$$= -6q$$

$(\alpha+\beta+\gamma)^2$

$= (\alpha+\beta)^2 + 2(\alpha+\beta)\gamma$

$= \alpha^2+2\alpha\beta+\beta^2$
$+2\alpha\gamma+2\beta\gamma+\gamma$

b) eqtn of tangent at P (x_1, y_1)

$$\frac{2x}{a^2} + \frac{2y}{b^2} \cdot \frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = \frac{2x}{a^2} \cdot \frac{b^2}{2y}$$

$$= \frac{b^2 x}{a^2 y}$$

at P, $\frac{dy}{dx} = \frac{b^2 x_1}{a^2 y_1}$

eqtn of PT

$$\frac{b^2 x_1}{a^2 y_1} = \frac{y}{x - \frac{a}{e}}$$

$$b^2 x x_1 - b^2 \frac{a}{e} x_1 = y y_1 a^2$$

$$a^2 y y_1 + b^2 x x_1 = \frac{b^2 a}{e} x_1$$

$$\frac{x x_1}{a^2} + \frac{y y_1}{b^2} = \frac{b^2 a}{e a^2 b^2} x_1$$

$$\frac{x x_1}{a^2} + \frac{y y_1}{b^2} = \frac{x_1}{ae}$$

similarly eqtn of QT is

$$\frac{x x_2}{a^2} + \frac{y y_2}{b^2} = \frac{x_2}{ae}$$

chord of contact

$$\frac{x x_0}{a_2} + \frac{y y_0}{b^2} = \frac{x_1}{ae}$$

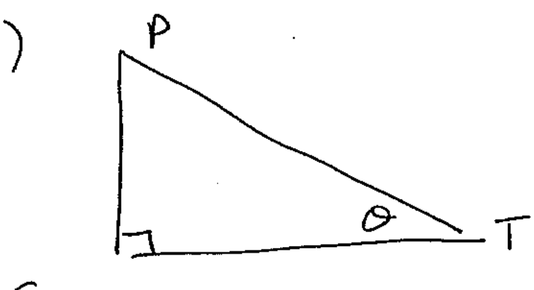
$$\frac{x \cdot \frac{a}{e}}{a^2} = \frac{x_1}{ae}$$

$$x = \frac{x_1}{ae} \times \frac{a^2}{a/e}$$

$$= \frac{x_1 a^2}{a^2}$$

$$x = x_1$$

ii) $\frac{PS}{ST} = \frac{PS}{PM} = e$

iii) 

$$\tan\theta = \frac{PS}{ST}$$

$$= e \quad , \quad e < 1$$

$$\tan\theta < 1$$

$$\theta < \frac{\pi}{4}$$

$$\angle PTQ = 2\theta$$

$$\angle PTQ < \frac{\pi}{2}$$

iv) area $= \frac{1}{2} \times PQ \times ST$, $2PS = PQ$

$PS = eST$

$= \frac{1}{2} \times 2e \times ST \times ST$

$$= \frac{1}{2} \times 2e \times \left(\frac{a}{e} - ae \right)^2$$

$$= e\left(\frac{a - ae^2}{e} \right)^2$$

$$= e\left(a\left(\frac{1}{e} - e \right) \right)^2$$

$$= ea^2\left(\frac{1}{e} - e \right)^2$$

$$= ea^2\left(\frac{1 - e^2}{e} \right)^2$$

$$= \frac{ea^2}{e^2}(1 - e^2)^2$$

$$= \frac{ea^2}{e^2}\left(\frac{b^2}{a^2} \right)^2$$

$$= \frac{b^4}{a^2 e}$$

b) $w = \frac{3+4i}{5}$, $z = \frac{5+12i}{13}$

$$wz = \left(\frac{3+4i}{5}\right)\left(\frac{5+12i}{13}\right)$$

$$= \frac{15+36i+20i-48}{65}$$

$$= \frac{-33+56i}{65}$$

$$w\bar{z} = \left(\frac{3+4i}{5}\right)\left(\frac{5-12i}{13}\right)$$

$$= \frac{15-36i+20i+48}{65}$$

$$= \frac{63-16i}{65}$$

ii) $|wz| = \sqrt{\left(\frac{-33}{65}\right)^2 + \left(\frac{56}{65}\right)^2} = 1$ as $|w| = |z| = 1$

$\Rightarrow \quad 33^2 + 56^2 = 65^2$

$|w\bar{z}| = \sqrt{\left(\frac{63}{65}\right)^2 + \left(\frac{-16}{65}\right)^2} = 1$ as $|w| = |\bar{z}| = 1$

$|z| = |\bar{z}| = 1$

$\Rightarrow \quad 63^2 + 16^2 = 65^2$