GIRRAWEEN HIGH SCHOOL

2006

YEAR 12 HALF YEARLY EXAMINATION

Mathematics Extension 2

General Instructions

- Reading time – 5 minutes
- Working time – 2 hours
- Write using black or blue pen
- Board-approved calculators may be used
- A table of standard integrals is provided at the back of this paper
- All necessary working should be shown in every question
- Marks may be deducted for careless or badly arranged work

Total marks – 100

- Attempt Questions 1 – 4
- All questions are NOT of equal value
- Start a separate piece of paper for each question.
- Put you name and the question number at the top of each sheet.

Total marks – 100
Attempt Questions 1 – 4
All questions are NOT of equal value
Answer each question on a SEPARATE piece of paper clearly marked Question 1, Question 2, etc. Each piece of paper must show your name.

Marks

Question 1 *(29 marks)* Use a *separate* piece of paper

a) Given that for the complex number z, $|z| = 2$ and $\arg z = \frac{2\pi}{3}$.

(i) Express z in the form $a + ib$ *2*

(ii) Express $\bar{z}$ in the form $a + ib$ *1*

(iii) Express z^5 in the form $a + ib$ *2*

b) For the complex equation $(5+3i)z^2 - (1-4i)z + (8-2i) = 0$

(i) Show that the product of the roots is equal to $1-i$ *3*

(ii) Find the modulus of the product of the roots. *1*

(iii) Find the argument of the product of the roots. *1*

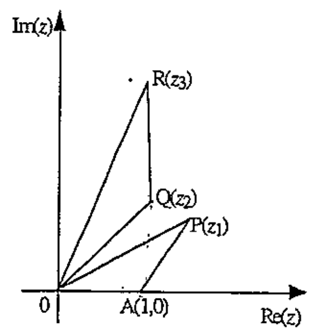
c) Find all the pairs of integers x and y such that $(x+iy)^2 = 5-12i$ *4*

d) (i) On an Argand Diagram shade in the region containing all the points representing complex numbers z such that both $|z| \le 2$ and $\frac{\pi}{4} \le \arg(z+2) \le \frac{\pi}{2}$ *3*

(ii) Find the possible values of $|z|$ and $\arg z$ for such complex numbers z. *3*

e) Interpret geometrically the locus of z given that $\arg\left(\frac{z-2}{z+2}\right) = \frac{\pi}{3}$ *4*

f)



In the diagram above P and Q represent the complex numbers z_1 and z_2 respectively. A is the point (1,0). The triangle OQR is constructed similar to triangle OAP. Let R represent the complex number z_3

(i) Show that $|z_3| = |z_1||z_2|$ *2*

(ii) Show that $\arg z_3 = \arg z_1 + \arg z_2$ *2*

(iii) What does this tell you about the relationship between z_1, z_2 and z_3 ? *1*

Question 2 *(29 marks)* Use a *separate* piece of paper *Marks*

a) $P(x_1, y_1)$ is a point on the ellipse $\frac{x^2}{4}+\frac{y^2}{3}=1$ which has foci at S and S'.

(i) Determine the eccentricity of the ellipse. 2

(ii) Find the coordinates of S and S' and also the equation of the directrices. 2

(iii) Prove that $PS + PS' = 4$ 2

(iv) Show that the equation of the tangent at P is $\frac{xx_1}{4}+\frac{yy_1}{3}=1$ 3

(v) This tangent meets the nearer directrix at R. If S is the nearer focus to P prove that $\angle PSR = 90^\circ$ 4

b) $P(x_1, y_1)$ is a point on the hyperbola $9x^2 - 16y^2 = 144$.

(i) Write down the equation of the tangent at P. 1

(ii) Find the coordinates of Q, the point where the tangent cuts the x axis. 1

(iii) Show that $\frac{SP}{S'P}=\frac{SQ}{S'Q}$ where S and S' are the foci of the hyperbola. 4

c) The hyperbola H has the equation $xy = c^2$. $P\left(cp, \frac{c}{p}\right)$ and $Q\left(cq, \frac{c}{q}\right)$ are points on the hyperbola H where $p > 0$, $q > 0$ and $c > 0$.

(i) Prove that the equation of the tangent at P is $x + p^2y = 2cp$ 3

(ii) Show that the tangents at P and Q meet at $T\left(\frac{2cpq}{p+q}, \frac{2c}{p+q}\right)$ 3

(iii) Suppose further that PQ is perpendicular to OT produced, where O is the origin. Express q in terms of p. 2

(iv) Find the locus of T, stating any restrictions that may exist. 2

Question 3 *(26 marks)* Use a *separate* piece of paper

a) If $2x^4 + 9x^3 + 6x^2 - 20x - 24 = 0$ has a triple root, find all of the roots. 4

b) Given that α, β and γ are the roots of $x^3 - x^2 + 5x - 3 = 0$, find;

(i) $\alpha^2 + \beta^2 + \gamma^2$ 3

(ii) $\alpha^3 + \beta^3 + \gamma^3$ 3

c) The equation $x^3 + 3ax + b = 0$ has two equal roots. 4
Prove that $b^2 + 4a^3 = 0$

d) The equation $2x^3+3x^2+x-5=0$ has roots α, β and γ. 4
Find an equation with roots α^2, β^2 and γ^2

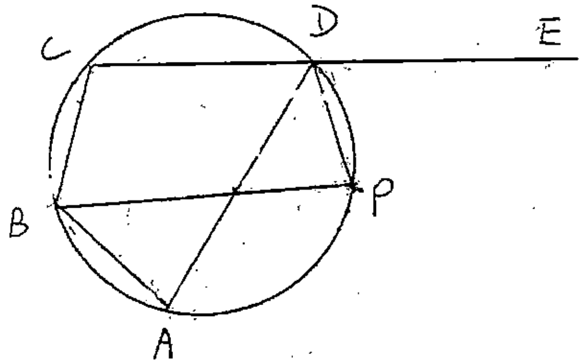
e) (i) Use De Moivre's Theorem to show that $\cos 5\theta = 16\cos^5\theta - 20\cos^3\theta + 5\cos\theta$ 3

(ii) Hence solve $16x^4 - 20x^2 + 5 = 0$ 3

(iii) Hence determine the exact value of $\cos^2\frac{\pi}{10}\cos^2\frac{3\pi}{10}$ 2

Question 4 *(16 marks)* Use a *separate* piece of paper

a) *ABCD* is a cyclic quadrilateral. *CD* is produced to *E*. *P* is a point on the circle such that $PA = PC$.



(i) Prove that *PD* bisects $\angle ADE$ 4

(ii) If $\angle BAP = 90^\circ$ and $\angle APD = 90^\circ$, explain where the centre of the circle is located. 2

b) (i) On the same diagram sketch the graphs of $x^2+y^2=1$ and $x^2-y^2=1$, clearly showing the coordinates of any points of intersection with the axes and the equations of any asymptotes. 3

(ii) Shade the region where the inequality $(x^2+y^2-1)(x^2-y^2-1) \le 0$ 2

c) In the Pascal triangle, a row consists of the integers $\binom{n}{0}, \binom{n}{1}, \binom{n}{2}, \ldots, \binom{n}{n}$.

Let $p_n = \binom{n}{0} \times \binom{n}{1} \times \binom{n}{2} \times \ldots \times \binom{n}{n}$

(i) Show that $\dfrac{\binom{n}{r}}{\binom{n-1}{r}} = \dfrac{n}{n-r}$ 2

(ii) Prove that $\dfrac{p_n}{p_{n-1}} = \dfrac{n^{n-1}}{(n-1)!}$ 3

STANDARD INTEGRALS

$\int x^n dx \qquad = \frac{1}{n+1}x^{n+1}, \; n \neq -1; x \neq 0, \text{ if } \; n < 0$

$\int \frac{1}{x} dx \qquad = \ln x, \; x > 0$

$\int e^{ax} dx \qquad = \frac{1}{a}e^{ax}, \; a \neq 0$

$\int \cos ax \, dx \qquad = \frac{1}{a}\sin ax, \; a \neq 0$

$\int \sin ax \, dx \qquad = -\frac{1}{a}\cos ax, \; a \neq 0$

$\int \sec^2 ax \, dx \qquad = \frac{1}{a}\tan ax, \; a \neq 0$

$\int \sec ax \tan ax \, dx \qquad = \frac{1}{a}\sec ax, \; a \neq 0$

$\int \frac{1}{a^2 + x^2} dx \qquad = \frac{1}{a}\tan^{-1}\frac{x}{a}, \; a \neq 0$

$\int \frac{1}{\sqrt{a^2 - x^2}} dx \qquad = \sin^{-1}\frac{x}{a}, \; a > 0, \; -a < x < a$

$\int \frac{1}{\sqrt{x^2 - a^2}} dx \qquad = \ln\left(x + \sqrt{x^2 - a^2}\right), \; x > a > 0$

$\int \frac{1}{\sqrt{x^2 + a^2}} dx \qquad = \ln\left(x + \sqrt{x^2 + a^2}\right)$

NOTE: $\ln x = \log x, \quad x > 0$

Extension 2 Half Yearly 2006 Solutions

$y = x + 2$
$x - y + 2 = 0$

Question 1 (29)

a) $z = 2(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3})$
$= 2(-\frac{1}{2} + i\frac{\sqrt{3}}{2})$
$= -1 + \sqrt{3}i$ (2)

(ii) $\bar{z} = -1 - \sqrt{3}i$ (1)

(iii) $z^5 = 2^5(\cos\frac{10\pi}{3} + i\sin\frac{10\pi}{3})$
$= 32(-\frac{1}{2} - i\frac{\sqrt{3}}{2})$
$= -16 - 16\sqrt{3}i$ (2)

b) (i) $\alpha\beta = \frac{8-2i}{5+3i} \times \frac{5-3i}{5-3i}$
$= \frac{40 - 24i - 10i - 6}{25 + 9}$
$= \frac{34 - 34i}{34}$
$= 1 - i$ (3)

(ii) $|\alpha\beta| = \sqrt{1^2 + (-1)^2}$
$= \sqrt{2}$ (1)

(iii) $\arg(\alpha\beta) = \tan^{-1}(\frac{-1}{1})$
$= -\frac{\pi}{4}$ (1)

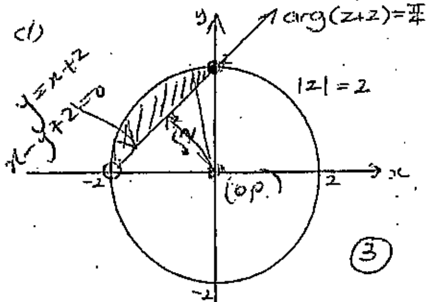
c) $(x + iy)^2 = 5 - 12i$
$x^2 + 2ixy - y^2 = 5 - 12i$

$x^2 - y^2 = 5$ $\quad 2xy = -12$
$x^2 - (\frac{-6}{x})^2 = 5$ $\quad y = \frac{-6}{x}$
$x^4 - 5x^2 - 36 = 0$
$(x^2 - 9)(x^2 + 4) = 0$
$x = \pm 3$ or no real solutions

(4)

$x = 3, y = -2$ or $x = -3, y = 2$

d)

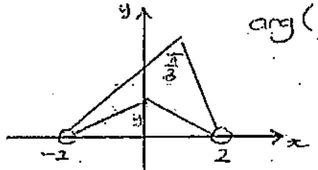


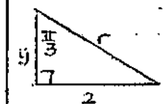
(3)

(ii) $\min|z| = \frac{|0 + 0 + 2|}{\sqrt{1^2 + 1^2}}$
$= \frac{2}{\sqrt{2}}$
$= \sqrt{2}$
$\therefore \sqrt{2} \le |z| \le 2$

$\frac{\pi}{2} \le \arg z < \pi$ (3)

e) $\arg(\frac{z-2}{z+2}) = \frac{\pi}{3}$





$\frac{2}{y} = \tan\frac{\pi}{3}$
$\frac{2}{y} = \sqrt{3}$
$y = \frac{2}{\sqrt{3}}$

$r^2 = 2^2 + (\frac{2}{\sqrt{3}})^2$
$= 4 + \frac{4}{3}$
$= \frac{16}{3}$

$\therefore$ locus is major arc of circle $x^2 + (y - \frac{2}{\sqrt{3}})^2 = \frac{16}{3}$ (4)
cut off by chord joining $(\pm 2, 0)$, excluding these points.

f) $\frac{OQ}{OA} = \frac{OR}{OP}$ (corresponding sides in ||| Δ's)
$\frac{|z_2|}{1} = \frac{|z_3|}{|z_1|}$

$|z_3| = |z_1||z_2|$ (2)

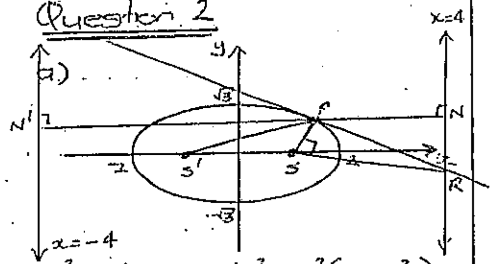
(ii) $\angle QOR = \angle POA$ (corresponding ∠'s in ||| Δ's)
$\angle AOR = \angle AOQ + \angle QOR$ (common ∠s)
$\therefore \angle AOR = \angle AOQ + \angle POA$

$\arg z_3 = \arg z_1 + \arg z_2$ (2)

(iii)
$z_3 = z_1 \times z_2$ (1)

Question 2

a)



$a^2 = 4$

$b^2 = a^2(1 - e^2)$
$3 = 4(1 - e^2)$
$1 - e^2 = \frac{3}{4}$
$e^2 = \frac{1}{4}$
$e = \frac{1}{2}$ (2)

(ii) foci $(\pm 1, 0)$
directrices: $x = \pm 4$ (2)

(iii) $PS + PS' = e(PN + PN')$
$= \frac{1}{2}(8)$
$= 4$ (2)

(iv) $\frac{x^2}{4} + \frac{y^2}{3} = 1$
$\frac{x}{2} + \frac{2y}{3}\frac{dy}{dx} = 0$
$\frac{dy}{dx} = -\frac{3x}{4y}$
at P, $\frac{dy}{dx} = -\frac{3x_1}{4y_1}$
$y - y_1 = -\frac{3x_1}{4y_1}(x - x_1)$
$4yy_1 - 4y_1^2 = -3xx_1 + 3x_1^2$
$3xx_1 + 4yy_1 = 3x_1^2 + 4y_1^2$
$\frac{xx_1}{4} + \frac{4yy_1}{3} = \frac{x_1^2}{4} + \frac{y_1^2}{3}$
$\frac{xx_1}{4} + \frac{4y_1}{3} = 1$ (3)

(v) when $x = 4$, $x_1 + \frac{4y_1}{3} = 1$
$\frac{4y_1}{3} = 1 - x_1$
$y = \frac{3 - 3x_1}{4y_1}$

$R(4, \frac{3 - 3x_1}{4y_1})$

$m_{PS} = \frac{y_1}{x_1 - 1}$
$m_{RS} = \frac{\frac{3 - 3x_1}{4y_1}}{4 - 1}$
$= \frac{1 - x_1}{4y_1}$

$m_{PS} \times m_{RS} = \frac{y_1}{x_1 - 1} \times \frac{1 - x_1}{y_1}$
$= -1$

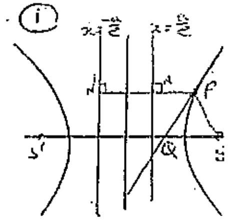
$\therefore PS \perp RS$
$\angle PSR = 90°$ (4)

b) $9x_1x - 16y_1y = 144$ (1)

(ii) $y = 0$, $x = \frac{144}{9x_1}$
$= \frac{16}{x_1}$

$Q(\frac{16}{x_1}, 0)$ (1)

(iii) $SP = ePN$
$= e(x_1 - \frac{4}{e})$



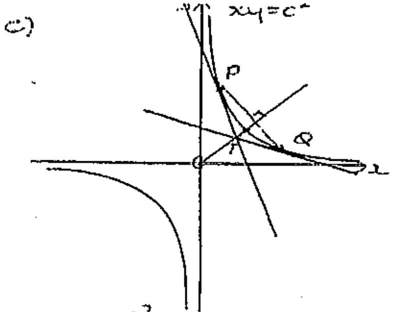
$S'P = ePN'$
$= e(x_1 + \frac{4}{e})$

$\frac{SP}{S'P} = \frac{x_1 - \frac{4}{e}}{x_1 + \frac{4}{e}}$
$= \frac{ex_1 - 4}{ex_1 + 4}$

$SQ = 4e - \frac{16}{x_1}$ $\quad S'Q = 4e + \frac{16}{x_1}$

$\frac{SQ}{S'Q} = \frac{4e - \frac{16}{x_1}}{4e + \frac{16}{x_1}}$
$= \frac{4ex_1 - 16}{4ex_1 + 16}$
$= \frac{ex_1 - 4}{ex_1 + 4}$
$= \frac{SP}{S'P}$ (4)

c)



(i) $y = \frac{c^2}{x}$

$y' = \frac{-c^2}{x^2}$

at P, $y' = \frac{-c^2}{c^2p^2}$

$= -\frac{1}{p^2}$

$y - \frac{c}{p} = -\frac{1}{p^2}(x - cp)$

$p^2y - cp = -x + cp$

$\underline{x + p^2y = 2cp}$ (3)

(ii) $x + p^2y = 2cp$ (−)

$x + q^2y = 2cq$

$(p^2-q^2)y = 2c(p-q)$

$y = \frac{2c}{p+q}$

$x + \frac{2cp^2}{p+q} = 2cp$

$x = \frac{2cp^2 + 2cpq - 2cp^2}{p+q}$

$= \frac{2cpq}{p+q}$

$T\left\{\frac{2cpq}{p+q}, \frac{2c}{p+q}\right\}$ (3)

(iii) $m_{PQ} \times m_{OT} = -1$

$\frac{\frac{c}{p} - \frac{c}{q}}{cp - cq} \times \frac{\frac{2c}{p+q}}{\frac{2cpq}{p+q}} = -1$

$\frac{\frac{q-p}{pq}}{p-q} \times \frac{1}{pq} = -1$

$\frac{-1}{p^2q^2} = -1$

$p^2q^2 = 1$

$pq = 1$ $(p>0, q>0)$

$\underline{q = \frac{1}{p}}$ (2)

(iv) $x = pqy$

$x = y$ (2)

$\underline{\text{locus is } y = x, \; 0 < x \le c}$

Question 3 (26)

a) $P(x) = 2x^4 + 9x^3 + 6x^2 - 20x - 24$

$P'(x) = 8x^3 + 27x^2 + 12x - 20$

$P''(x) = 24x^2 + 54x + 12$

$= 6(4x^2 + 9x + 2)$

$= 6(4x+1)(x+2)$

triple root $x = -\frac{1}{4}$ or $x = -2$

$P'(-2) = 0$, $P(-2) = 0$

∴ $x = -2$ is triple root

$2x^4 + 9x^3 + 6x^2 - 20x - 24 = 0$

$(x+2)^3(2x-3) = 0$

∴ $\underline{\text{roots are } -2, -2, -2, \frac{3}{2}}$ (4)

b) $x^3 - x^2 + 5x - 3 = 0$

(i) $\Sigma\alpha^2 = (\Sigma\alpha)^2 - 2\Sigma\alpha\beta$

$= (1)^2 - 2(5)$

$= \underline{-9}$ (3)

(ii) $\Sigma\alpha^3 - \Sigma\alpha^2 + 5\Sigma\alpha - 9 = 0$

$\Sigma\alpha^3 = \Sigma\alpha^2 - 5\Sigma\alpha + 9$

$= (-9) - 5(1) + 9$

$= \underline{-5}$ (3)

c) $x^3 + 3ax + b = 0$

Let roots be α, α, β

$2\alpha + \beta = 0$ $\quad \alpha^2\beta = -b$

$\beta = -2\alpha$ $\quad -2\alpha^3 = -b$

$\alpha^3 = \frac{b}{2}$

$\alpha = \sqrt[3]{\frac{b}{2}}$

$\alpha^2 + \alpha\beta + \alpha\beta = 3a$

$\alpha^2 - 2\alpha^2 - 2\alpha^2 = 3a$

$-3\alpha^2 = 3a$

$\alpha^2 = -a$

$\left(\sqrt[3]{\frac{b}{2}}\right)^2 = -a$

$\frac{b^2}{4} = -a^3$

$\underline{b^2 + 4a^3 = 0}$ (4)

d) $2x^3 + 3x^2 + x - 5 = 0$

let $y = x^2$

$x = y^{\frac{1}{2}}$

$2y^{\frac{3}{2}} + 3y + y^{\frac{1}{2}} - 5 = 0$

$2y^{\frac{3}{2}} + y^{\frac{1}{2}} = 5 - 3y$

$y(2y+1)^2 = (5-3y)^2$

$4y^3 + 4y^2 + y = 25 - 30y + 9y^2$

$\underline{4y^3 - 5y^2 + 31y - 25 = 0}$ (4)

e) $\text{cis}\,5\theta = (c + is)^5$

$= c^5 + 5ic^4s - 10c^3s^2 - 10ic^2s^3 + 5cs^4 + is^5$

equating reals

$\cos 5\theta$

$= \cos^5\theta - 10\cos^3\theta\sin^2\theta + 5\cos\theta\sin^4\theta$

$= \cos^5\theta - 10\cos^3\theta(1 - \cos^2\theta) + 5\cos\theta(1 - 2\cos^2\theta + \cos^4\theta)$

$= \cos^5\theta - 10\cos^3\theta + 10\cos^5\theta + 5\cos\theta - 10\cos^3\theta + 5\cos^5\theta$ (3)

$\underline{= 16\cos^5\theta - 20\cos^3\theta + 5\cos\theta}$

(ii) $16x^4 - 20x^2 + 5 = 0$

$16x^5 - 20x^3 + 5x = 0$

let $x = \cos\theta$

$\cos 5\theta = 0$

$5\theta = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \frac{7\pi}{2}, \frac{9\pi}{2}$

$\theta = \frac{\pi}{10}, \frac{3\pi}{10}, \frac{\pi}{2}, \frac{7\pi}{10}, \frac{9\pi}{10}$ (3)

$\underline{x = \cos\frac{\pi}{10}, \cos\frac{3\pi}{10}, \cos\frac{7\pi}{10}, \cos\frac{9\pi}{10}}$

NOTE: $\cos\frac{\pi}{2} = 0$ not a solution of original equation

(iii) $\cos\frac{\pi}{10}\cos\frac{3\pi}{10}\cos\frac{7\pi}{10}\cos\frac{9\pi}{10} = \frac{5}{16}$

$\cos\frac{\pi}{10} = -\cos\frac{9\pi}{10}$

$\cos\frac{3\pi}{10} = -\cos\frac{7\pi}{10}$

∴ $\underline{\cos^2\frac{\pi}{10}\cos^2\frac{3\pi}{10} = \frac{5}{16}}$ (2)

Question 4 (16)

a)(i) $PA = PC$ (given)

∴ $\angle CBP = \angle ABP$ (= chords subtend = ∠'s at circumference)

$\angle CBP = \angle PDE$ (exterior ∠ cyclic quad)

$\angle ABP = \angle ADP$ (∠'s in same segment =)

∴ $\angle PDE = \angle ADP$

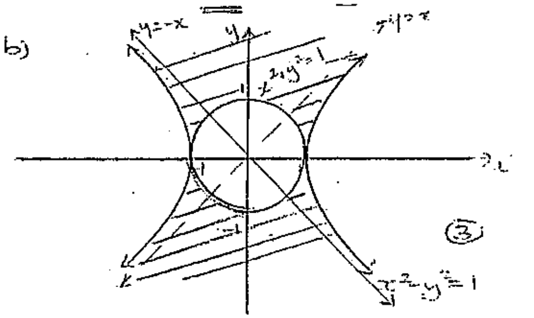
ie $\underline{PD \text{ bisects } \angle ADE}$ (4)

(ii) BP is a diameter ($\angle BAP = 90°$, ∠ in semicircle)

AD is a diameter ($\angle APD = 90°$, ∠ in semicircle)

∴ $\underline{\text{centre is intersection of BP and AD}}$

b)



(3)

(ii) region would be when

$x^2 + y^2 \le 1$ and $x^2 - y^2 \ge 1$ not possible

or

$x^2 + y^2 \ge 1$ and $x^2 - y^2 \le 1$ (2)

c) $\frac{\binom{n}{r}}{\binom{n-1}{r}} = \frac{n!}{r!(n-r)!} \times \frac{r!(n-r-1)!}{(n-1)!}$

$= \underline{\frac{n}{n-r}}$ (2)

(ii) $\frac{P_n}{P_{n-1}} = \frac{\binom{n}{0}\binom{n}{1}\binom{n}{2}\cdots\binom{n}{n-1}\binom{n}{n}}{\binom{n-1}{0}\binom{n-1}{1}\binom{n-1}{2}\cdots\binom{n-1}{n-1}}$

$= \frac{\binom{n}{1}}{\binom{n-1}{1}} \times \frac{\binom{n}{2}}{\binom{n-1}{2}} \times \cdots \times \frac{\binom{n}{n-1}}{\binom{n-1}{n-1}}$

$= \frac{n}{n-1} \times \frac{n}{n-2} \times \cdots \times \frac{n}{n-(n-1)}$

$= \frac{n}{n-1} \times \frac{n}{n-2} \times \cdots \times \frac{n}{1}$

$= \underline{\frac{n^{n-1}}{(n-1)!}}$ (3)