FINAL MARK

GIRRAWEEN HIGH SCHOOL
MATHEMATICS EXTENSION 2
HALF-YEARLY EXAMINATION
ANSWERS COVER SHEET

Name: ____

QUESTION	MARK	E2	E3	E4	E5	E6	E7	E8	E9
MC	/5								✔
Q6	/25		✔						✔
Q7	/19			✔					✔
Q8	/10		✔	✔					✔
Q9	/8		✔	✔					✔
Q10	/10		✔	✔					✔
Q11	/8	✔	✔						✔
TOTAL									
	/85	/8	/61	/47	/0	/0	/0	/0	/85

HSC Outcomes **Mathematics Extension 2**

E2 chooses appropriate strategies to construct arguments and proofs in both concrete and abstract settings.

E3 uses the relationship between algebraic and geometric representations of complex numbers and of conic sections.

E4 uses efficient techniques for the algebraic manipulation required in dealing with questions such as those involving conic sections and polynomials.

E5 uses ideas and techniques from calculus to solve problems in mechanics involving resolution of forces, resisted motion and circular motion

E6 combines the ideas of algebra and calculus to determine the important features of the graphs of a wide variety of functions.

E7 uses the techniques of slicing and cylindrical shells to determine volumes.

E8 applies further techniques of integration, including partial fractions, integration by parts and recurrence formulae, to problems.

E9 communicates abstract ideas and relationships using appropriate notation and logical argument.

GIRRAWEEN HIGH SCHOOL

HALF YEARLY EXAMINATION

2016

MATHEMATICS

EXTENSION 2

Time Allowed: Two hours
(Plus 5 minutes reading time)

DIRECTIONS TO CANDIDATES

- Attempt ALL questions. Write using ***Blue*** or ***Black*** pen only.

- Board-approved calculators may be used.
  BOSTES Reference Sheets are provided.

- All necessary working should be shown in Questions 6 - 10.
  Marks may be deducted for careless or badly arranged work.

- For Questions 1 - 5, colour in the circle corresponding to the correct answer on the answer sheet provided.
  For Questions 6 – 10, each question is to be returned on a *separate* piece of paper clearly marked Question 6, Question 7, etc.
  Write **'End of Solutions'** at the conclusion of your solutions to the examination.

- You may ask for extra pieces of paper if you need them.

Questions 1 - 5 (5 marks)
Colour in the circle corresponding to the correct answer on the answer sheet provided.

1. Given that $z = 1 + i$, what is the value of z^8 ?

   A. -16 B. -8 C. 8 D. 16

2. What are the coordinates of the foci of the ellipse $\frac{x^2}{4} + \frac{y^2}{9} = 1$?

   A. $(\pm\sqrt{5}, 0)$ B. $\left(\frac{\pm 2\sqrt{5}}{3}, 0\right)$ C. $(0, \pm\sqrt{5})$ D. $\left(0, \pm\frac{2\sqrt{5}}{3}\right)$

3. What is the eccentricity of the hyperbola $\frac{x^2}{16} - \frac{y^2}{9} = 1$?

   A. $\frac{3}{4}$ B. $\frac{5}{4}$ C. $\frac{9}{16}$ D. $\frac{25}{16}$

4. α, β, γ are the roots of $x^3 - 4x^2 + x - 5 = 0$.
   An equation that has the roots $\frac{1}{\alpha}, \frac{1}{\beta}, \frac{1}{\gamma}$ is

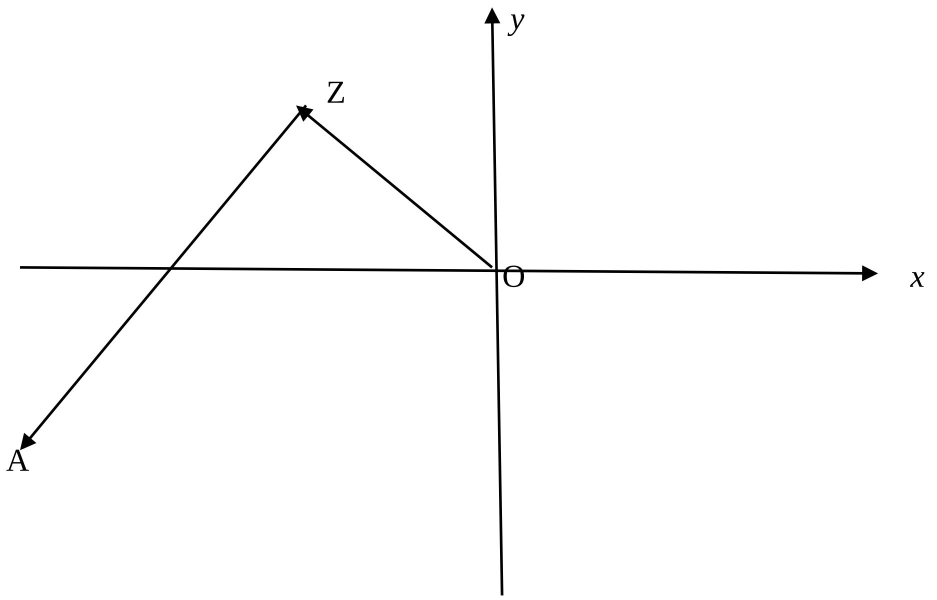
   A. $5x^3 + x^2 - 4x - 1 = 0$

   B. $1 - 4x + x^2 - 5x^3 = 0$

   C. $5x^3 + 4x + x^2 - 1 = 0$

   D. $1 + 4x + x^2 - 5x^3 = 0$

5. The point A represents the complex number $-4 - 3i$.
   $\angle OZA = 90°$ and $|ZA| = 2|z|$
   Find the complex number represented by the point Z.



   A. $-1 + \sqrt{2}i$ B. $-1 + 2i$ C. $-2 + i$ D. $-\sqrt{2} + i$

Question 6 (25 marks)

a. For the complex number $w = 1 - i\sqrt{3}$,

[2]

(i) Find $|w|$ and arg (w).

(ii) Express $\overline{w}$, w^2, $\frac{1}{w}$ and $\sqrt{w}$ in the form $\boldsymbol{a} + \boldsymbol{i}\,\boldsymbol{b}$. [6]

b. Describe and sketch, showing all the important features, the locus of the point z such that

$|z + 3i| + |z - 3i| = 10$ [4]

c. Sketch the region in the complex plane where the inequalities

$|z + 1 - 2i| \leq 3$ and $-\frac{\pi}{3} \leq \arg z \leq \frac{\pi}{4}$ both hold. [3]

d. Let α be a real number and suppose z is a complex number such that $z + \frac{1}{z} = 2\cos\alpha$.

(i) By reducing the above equation to a quadratic equation in z, solve for z and use de Moivre's theorem to show that $z^n + \frac{1}{z^n} = 2\cos n\,\alpha$. [3]

(ii) Let $w = z + \frac{1}{z}$. Prove that $w^3 + w^2 - 2w - 2 = \left(z + \frac{1}{z}\right) + \left(z^2 + \frac{1}{z^2}\right) + \left(z^3 + \frac{1}{z^3}\right)$. [3]

(iii) Hence, or otherwise, find all solutions of $\cos\alpha + \cos 2\alpha + \cos 3\alpha = 0$ for $0 \leq \alpha \leq 2\pi$. [4]

Question 7 (19 marks)

a. The polynomial $P(x) = x^4 + 7x^3 + 9x^2 - 27x + C$ has a triple zero.

(i) Determine the value of the triple zero. [3]

(ii) Hence, find the value of C. [1]

(iii)Factorise $P(x)$. [1]

b. The equation $x^3 - x^2 + 3 = 0$ has roots α, β and γ.

(i) Find the polynomial equation that has roots α^2, β^2 and γ^2.
Express with integral powers. [2]

(ii) Find the value of $\alpha^4 + \beta^4 + \gamma^4$. [2]

c. The polynomial $f(x) = x^4 + px^3 + qx^2 + rx + s$ has four zeros α, β, γ and δ
Such that the sum of α and β equals the sum of γ and δ.
Let $C = \alpha + \beta = \gamma + \delta$; $P = \alpha\beta$; $Q = \gamma\delta$

(i) Find p, q, r and s in terms of C, P and Q. [4]

(ii) Show that the coefficients of $f(x)$ satisfy the equation
$p^3 + 8r = 4pq$ [2]

(iii) It is given that the polynomial $g(x) = x^4 - 18x^3 + 79x^2 + 18x - 440$ has the property that the sum of two of the zeros equals the sum of the other two zeros.
Using the results from (i), or otherwise, find all four zeros of $g(x)$. [4]

Question 8 (10 marks)

An ellipse has the equation $\frac{x^2}{81} + \frac{y^2}{49} = 1$

(i) Sketch the ellipse showing the foci and the directrices. [4]

(ii) Prove that the tangent to the ellipse at the point $P(9\cos\theta, 7\sin\theta)$ has the equation $\frac{x\cos\theta}{9} + \frac{y\sin\theta}{7} = 1$. [3]

(iii) The ellipse meets the *y*-axis at *B* and *B*'. The tangents at *B* and *B*' meet the tangent at P at the points *Q* and *Q*'.

Show that *BQ.B'Q*'= 81 [3]

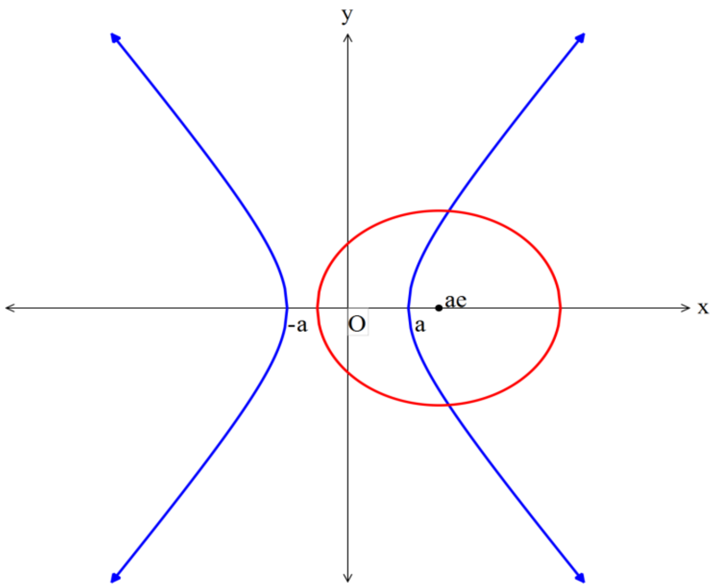
Question 9 (8 marks)

a. For the hyperbola, $5x^2 - 4y^2 = 20$

(i) Find the eccentricity and the coordinates of the foci. [3]

(ii) Find the equations of the asymptotes. [1]

b.



The tangent at $P(a\sec\theta, b\tan\theta)$ on the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ has equation $\frac{x\sec\theta}{a} - \frac{y\tan\theta}{b} - 1 = 0$. (DO NOT PROVE THIS)

Show that if the tangent at P is also tangent to the circle with centre (ae, 0) and radius $a\sqrt{e^2+1}$, then $\sec\theta = -e$. [4]

Question 10 (10 marks)

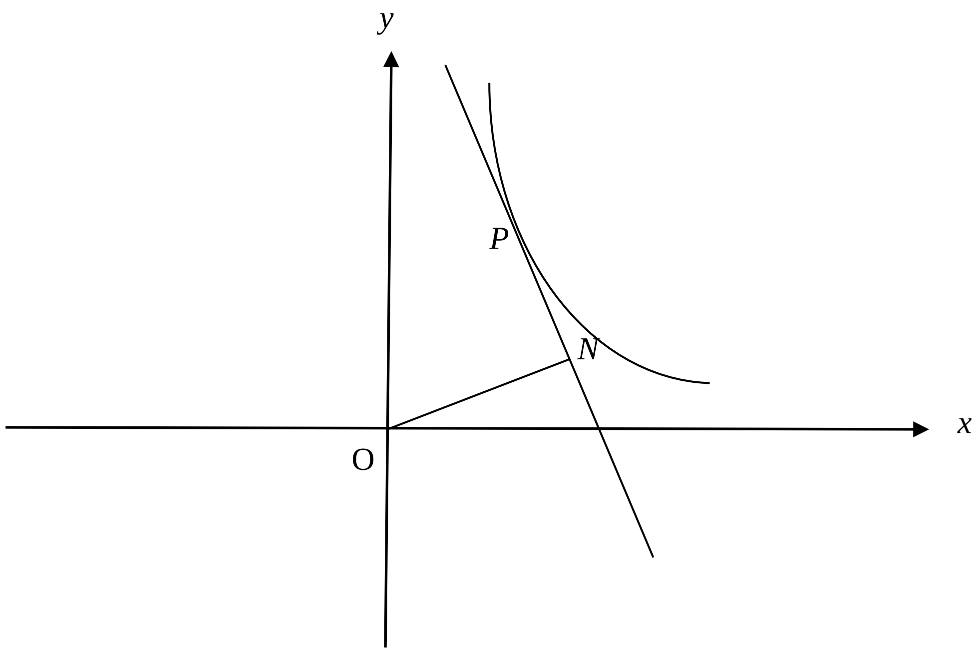
a. $P\left(pc\ ,\ \frac{c}{p}\right)$ and $Q\left(qc\ ,\frac{c}{q}\right)$ are two points on the rectangular hyperbola $xy = c^2$, where p and q are constants.

(i) Show that the gradient of PQ is $\frac{-1}{pq}$. [1]

(ii) Show that the gradient of the tangent to the hyperbola at P is $\frac{-1}{p^2}$. [1]

(iii) Hence, or otherwise, find an expression for q in terms of p that will make PQ a normal to the hyperbola at P. [2]

b. The tangent at $P\left(6p\ ,\frac{6}{p}\right)$ to the rectangular hyperbola $y = \frac{36}{x}$ has equation $x + p^2y - 12p = 0$. The line through the origin, O, perpendicular to the tangent at P meets the tangent at N.

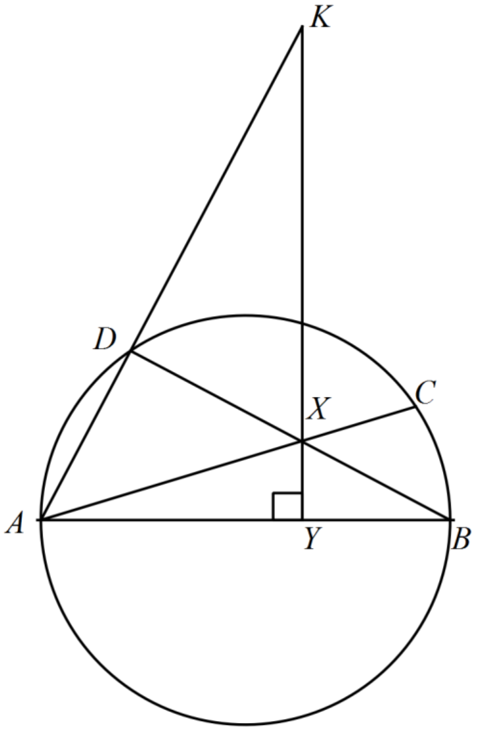


(i) Find the coordinates of N. [3]

(ii) Show that as p varies, the locus of N is $(x^2 + y^2)^2 = 144xy$ [3]

Question 11 (8 marks)

a. In the diagram, AB is the diameter of the circle. The chords AC and BD intersect at X. The point Y lies on AB such that XY is perpendicular to AB. The point K is the Intersection of AD produced and YX produced.



Copy or trace the diagram into your writing paper.

(i) Show that $\angle AKY = \angle ABD$. [2]

(ii) Show that $CKDX$ is a cyclic quadrilateral. [3]

(iii) Show that B, C and K are collinear. [3]

End of Examination

Please write 'End of Solutions' on your answer paper.

Year 12 HY Examination 2016 - Mathematics Extension 2

Multiple Choice Answer Sheet

Student Name: ________________ Teacher: __________________

Select the alternative A, B, C or D that best answers the question. Fill in the response oval completely.

Sample: 2 + 4 = (A) 2 (B) 6 (C) 8 (D) 9

A ○ B ● C ○ D ○

If you think you have made a mistake, put a cross through the incorrect answer and fill in the new answer.

A ● B ⊗ C ○ D ○

If you change your mind and have crossed out what you consider to be the correct answer, then indicate the correct answer by writing the word *correct* and drawing an arrow as follows.

A ⊗ B ⊗ (correct ↙) C ○ D ○

1.	A	○	B	○	C	○	D	○
2.	A	○	B	○	C	○	D	○
3.	A	○	B	○	C	○	D	○
4.	A	○	B	○	C	○	D	○
5.	A	○	B	○	C	○	D	○

YEAR 12 HALF-YEARLY SOLUTIONS EXT 2.
2016

MC 1. D 2. C 3. B 4. B 5. C

1. $z = 1+i$

$z^8 = (1+i)^8$

$= \left(\sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)\right)^8$

$= 2^4(\cos 2\pi + i\sin 2\pi)$ **D**

$= 16$

2. $\frac{x^2}{4} + \frac{y^2}{9} = 1$ $a=2, b=3$

$e = \sqrt{1 - \left(\frac{2}{3}\right)^2}$

$= \frac{\sqrt{5}}{3}$

$F(0, \pm be)$ **C**

$= (0 \pm \sqrt{5})$

3. $\frac{x^2}{16} - \frac{y^2}{9} = 1$

$e^2 = 1 + \frac{9}{16}$

$= \frac{25}{16}$ **B**

$e = \frac{5}{4}$

4. $x^3 - 4x^2 + x - 5 = 0$

$x = \frac{1}{y}$

$\therefore \left(\frac{1}{y}\right)^3 - 4\left(\frac{1}{y}\right)^2 + \left(\frac{1}{y}\right) - 5 = 0$ $\times y^3$

$1 - 4y + y^2 - 5y^3 = 0$ **B**

$\Rightarrow 1 - 4x + x^2 - 5x^3 = 0$

5. $A = -4-3i$; $|ZA| = 2|Z|$

$Z = x+iy$

$|OA| = |OZ| + 2|ZA|$

$-4-3i = x+iy + 2i(x+iy)$

$-4-3i = x-2y + (2x+y)i$

Equating real & imaginary parts,

$x-2y = -4$; $2x+y = -3$

$\Rightarrow x = -2, y = 1$ **C**

$\therefore z = -2+i$

Question 6 (25 marks)

a) $w = 1 - i\sqrt{3}$

i) $|w| = \sqrt{1^2 + (-\sqrt{3})^2}$ ①

$= 2$

$\arg(w) = \tan^{-1}(-\sqrt{3})$ ①

$= \frac{-\pi}{3}$

ii) $\overline{w} = 1 + \sqrt{3}i$ ①

$w^2 = (1 - \sqrt{3}i)^2$

$= 1 - 2\sqrt{3}i - 3$ ①

$= -2 - 2\sqrt{3}i$

$\frac{1}{w} = \frac{1}{1-\sqrt{3}i} \times \frac{1+\sqrt{3}i}{1+\sqrt{3}i}$

$= \frac{1+\sqrt{3}i}{4}$ ①

$\sqrt{w} = \sqrt{1-\sqrt{3}i}$

$w^2 = 1 - \sqrt{3}i = (a+ib)^2$

$a^2 - b^2 + 2iab = 1 - \sqrt{3}i$

Equating real & imaginary parts.

$a^2 - b^2 = 1$; $2ab = -\sqrt{3}$

$b = -\frac{\sqrt{3}}{2a}$

Substituting $b = -\frac{3}{2a}$ into $a^2 - b^2 = 1$,

$a^2 - \frac{3}{4a^2} = 1$

$4a^4 - 4a^2 - 3 = 0$

$a^2 = \frac{4 \pm \sqrt{16 - 4(4)(-3)}}{8}$

$= \frac{4 \pm \sqrt{64}}{8}$ ③

$= \frac{3}{2}$ or $-\frac{1}{2}$

$\therefore a^2 = \frac{3}{2}$ or $a^2 = -\frac{1}{2}$ ← no real solution.

$a = \pm\frac{\sqrt{3}}{\sqrt{2}} = \pm\frac{\sqrt{6}}{2}$; $b = \pm\frac{\sqrt{2}}{2}$

$\therefore w = \frac{\sqrt{6} - \sqrt{2}i}{2}, \frac{-\sqrt{6} + \sqrt{2}i}{2}$

6)b.

$|z+3i| + |z-3i| = 0$

represents an ellipse with major axis along the y-axis, since

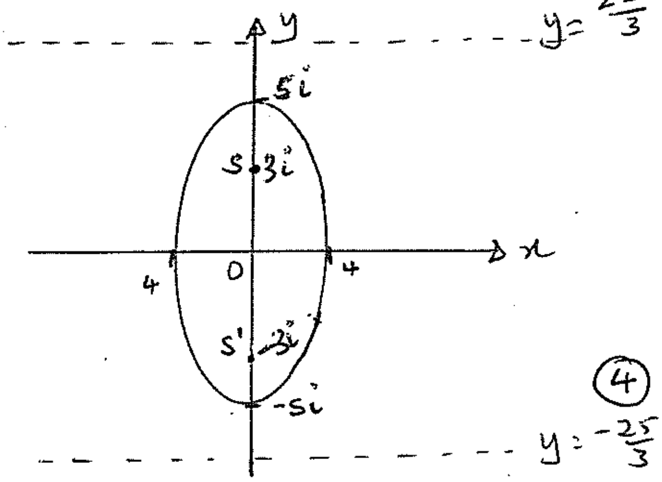
$PS + PS' = 2b$ foci $(0, \pm 3i)$

$b = 5$ $be = 3$

$a^2 = b^2(1-e^2)$ centre $(0,0)$

$a = 4$ $e = \frac{3}{5}$

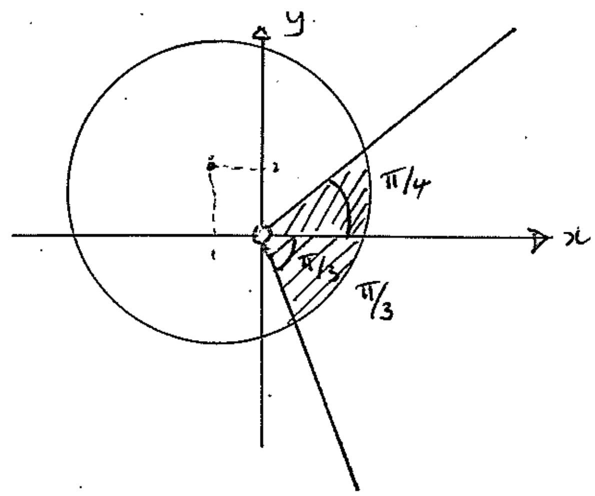
$\therefore$ Locus is an ellipse with equation $\frac{x^2}{16} + \frac{y^2}{25} = 1$



(4)

c. $|z+1-2i| \leq 3 \cap -\frac{\pi}{3} \leq \arg z \leq \frac{\pi}{4}$

$|z+1-2i| = 3$ represents a circle with centre $(-1, 2)$ and radius 3.



(3)

d) $z + \frac{1}{z} = 2\cos\alpha$

i) $z + \frac{1}{z} = 2\cos\alpha$ $(\times z)$

$\therefore z^2 - 2z\cos\alpha + 1 = 0$

$$z = \frac{2\cos\alpha \pm \sqrt{4\cos^2\alpha - 4}}{2}$$

$$= \frac{2\cos\alpha \pm 2i\sqrt{1-\cos^2\alpha}}{2}$$

$\therefore z = \cos\alpha \pm i\sin\alpha$

Using de Moivre's theorem,

$z^n = \cos n\alpha \pm i\sin n\alpha$ and

$z^{-n} = \cos(-n\alpha) \pm i\sin(-n\alpha)$

$= \cos n\alpha \mp \sin n\alpha$ [As sin is odd cos is even]

$\therefore z^n + \frac{1}{z^n} = 2\cos n\alpha$

(3)

ii) $w = z + \frac{1}{z}$

$w^3 + w^2 - 2w - 2 =$

$\left(z+\frac{1}{z}\right)^3 + \left(z+\frac{1}{z}\right)^2 - 2\left(z+\frac{1}{z}\right) - 2$

$= z^3 + 3z + \frac{3}{z} + \frac{1}{z^3} + z^2 + 2 + \frac{1}{z^2} - 2z - \frac{2}{z} - 2$

$= z^3 + z + \frac{1}{z} + \frac{1}{z^3} + z^2 + \frac{1}{z^2}$

$= \left(z + \frac{1}{z}\right) + \left(z^2 + \frac{1}{z^2}\right) + \left(z^3 + \frac{1}{z^3}\right)$

(3)

iii) $\cos\alpha + \cos 2\alpha + \cos 3\alpha = 0, 0 \leq \alpha \leq 2\pi$

$\therefore 2\cos\alpha + 2\cos 2\alpha + 2\cos 3\alpha = 0$

$\left(z+\frac{1}{z}\right) + \left(z^2+\frac{1}{z^2}\right) + \left(z^3+\frac{1}{z^3}\right) = 0$

$\therefore w^3 + w^2 - 2w - 2 = 0$

$w^2(w+1) - 2(w+1) = 0$

$(w^2-2)(w+1) = 0$

$\therefore w = \pm\sqrt{2}$; $w = -1$ $\left(w = z + \frac{1}{z}\right)$

$\therefore 2\cos\alpha = \pm\sqrt{2}$ or $2\cos\alpha = -1$

$\alpha = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$ $\alpha = \frac{2\pi}{3}, \frac{4\pi}{3}$

$\therefore \alpha = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}, \frac{2\pi}{3}, \frac{4\pi}{3}$

(4)

Question 7 (19 marks)

a) $P(x) = x^4 + 7x^3 + 9x^2 - 27x + c$

Triple root $\Rightarrow P''(x) = 0$

i) $P'(x) = 4x^3 + 21x^2 + 18x - 27$

$P''(x) = 12x^2 + 42x + 18$

$6(2x^2 + 7x + 3) = 0$

$(2x+1)(x+3) = 0$

$x = -\frac{1}{2}$ or $x = -3$

$P'(-3) = 4(-3)^3 + 21(-3)^2 + 18(-3) - 27$

$= 0$ (3)

$\therefore$ Triple root at $x = -3$

ii) $P(-3) = 0$

$0 = (-3)^4 + 7(-3)^3 + 9(-3)^2 - 27(-3) + c$

$c = -54$ (1)

iii) $P(x) = (x+3)^3 (x+a)$

$= x^4 + 7x^3 + 9x^2 - 27x - 54$

$3^3 \times a = -54$

$\therefore a = -2$ (1)

$\therefore P(x) = (x+3)^3 (x-2)$

b. $x^3 - x^2 + 3 = 0$; roots α, β, γ

i) $y = x^2$

$\therefore x = \sqrt{y}$

$\therefore (\sqrt{y})^3 - (\sqrt{y})^2 + 3 = 0$

$y^{3/2} - y + 3 = 0$

$y^{3/2} = y - 3$

Squaring both sides,

$y^3 = y^2 - 6y + 9$

$y^3 - y^2 + 6y - 9 = 0$

$\Rightarrow x^3 - x^2 + 6x - 9 = 0$ (2)

ii) $\alpha^4 + \beta^4 + \gamma^4$

$= (\alpha^2 + \beta^2 + \gamma^2)^2 - 2\sum\alpha^2\beta^2$

$= \left(\frac{-b}{a}\right)^2 - 2\left(\frac{c}{a}\right)$

$= 1 - 2(6)$ (2)

$= -11$

c. $f(x) = x^4 + px^3 + qx^2 + rx + s$

zeros : $\alpha, \beta, \gamma, \delta$

$C = \alpha + \beta = \gamma + \delta$; $P = \alpha\beta$; $Q = \gamma\delta$

i) sum of roots $= \alpha + \beta + \gamma + \delta$

$-p = 2C$

$p = -2C \Rightarrow C = \frac{-p}{2}$

$\sum\alpha\beta = \alpha\beta + \alpha\gamma + \alpha\delta + \beta\gamma + \beta\delta + \gamma\delta$

$= P + Q + \alpha(\gamma+\delta) + \beta(\gamma+\delta)$

$= P + Q + (\alpha+\beta)(\gamma+\delta)$

$\therefore q = P + Q + C^2$

$\sum\alpha\beta\gamma = \alpha\beta\gamma + \alpha\beta\delta + \alpha\gamma\delta + \beta\gamma\delta$

$= P\gamma + \alpha Q + \beta Q + \delta P$

$= P(\gamma+\delta) + Q(\alpha+\beta)$

$-r = PC + QC$

$\therefore r = -PC - QC$

$\sum\alpha\beta\gamma\delta = PQ = s$

$\therefore s = PQ$ (4)

7)

c. ii) $p^3 + 8r = 4pq$

$LHS = p^3 + 8r$

$= (-2C)^3 + 8(-PC - QC)$

$= -8C^3 - 8PC - 8QC$

$= -8C(C^2 + P + Q)$

$= -8Cq$

$= -8\left(-\frac{p}{2}\right)q$ from (i) $\left[C = \frac{-p}{2}\right]$

$= 4pq$

$= RHS$ (2)

iii) $g(x) = x^4 - 18x^3 + 79x^2 + 18x - 440$

$\alpha + \beta = \gamma + \delta$

$-2C = -18$

$C = 9$

$P + Q + C^2 = 79$

$\therefore P + Q = -2$; $PQ = -440$

$P(-2-P) = -440$

$P^2 + 2P - 440 = 0$

$(P+22)(P-20) = 0$

$P = -22, 20$

$\Sigma\alpha = 18 \Rightarrow \alpha + \beta = \gamma + \delta = 9$

$\alpha\beta\gamma\delta = -440$

$\alpha\beta = 20$; $\gamma\delta = -22$

$\alpha + \beta = 9$; $\gamma + \delta = 9$

5, 4 ; 11, −2

$\therefore$ Roots are $-2, 4, 5, 11$ (4)

Question 8 (10 marks)

i) $\frac{x^2}{81} + \frac{y^2}{49} = 1$

$a = 9$; $b = 7$

$b^2 = a^2(1 - e^2)$

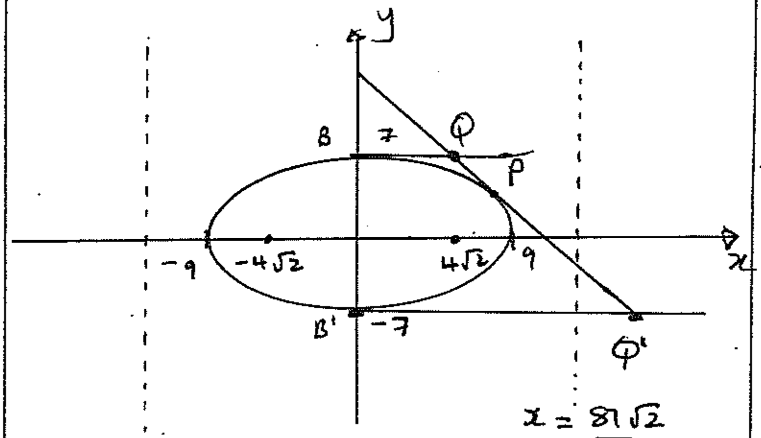
$1 - e^2 = \frac{49}{81}$

$e = \frac{\sqrt{32}}{9} = \frac{4\sqrt{2}}{9}$

Foci : $(\pm ae, 0) = (\pm 4\sqrt{2}, 0)$

Directrices : $x = \pm\frac{a}{e} = \pm\frac{81\sqrt{2}}{8}$

Centre $(0,0)$



(4)

ii) $\frac{x^2}{81} + \frac{y^2}{49} = 1$

$\frac{2x}{81} + \frac{2y}{49}\frac{dy}{dx} = 0$

$\frac{dy}{dx} = -\frac{49}{81}\frac{x}{y}$

At $P(9\cos\theta, 7\sin\theta)$,

$m_{tangent} = \frac{-7\cos\theta}{9\sin\theta}$

$E_{tangent}$: $y - 7\sin\theta = \frac{-7\cos\theta}{9\sin\theta}(x - 9\cos\theta)$

$9y\sin\theta - 63\sin^2\theta = -7x\cos\theta + 63\cos^2\theta$

$9y\sin\theta + 7x\cos\theta = 63$ $\div 63$

$\frac{y\sin\theta}{7} + \frac{x\cos\theta}{9} = 1$ (3)

8.iii) At Q, $y = 7$

Q lies on tangent at P

$\therefore \dfrac{x\cos\theta}{9} + \sin\theta = 1$

$x = \dfrac{9(1-\sin\theta)}{\cos\theta}$

$\therefore Q = \left(\dfrac{9(1-\sin\theta)}{\cos\theta}, 7\right)$

At Q', $y = -7$

Q' lies on tangent at P

$\therefore \dfrac{x\cos\theta}{9} - \sin\theta = 1$

$x = \dfrac{9(1+\sin\theta)}{\cos\theta}$

$Q' = \left(\dfrac{9(1+\sin\theta)}{\cos\theta}, -7\right)$

$BQ \cdot B'Q$

$= \dfrac{9(1-\sin\theta)}{\cos\theta} \cdot \dfrac{9(1+\sin\theta)}{\cos\theta}$

$= \dfrac{81(1-\sin^2\theta)}{\cos^2\theta}$

$= \dfrac{81\cos^2\theta}{\cos^2\theta}$ (3)

$= 81$

Question 9 (8 marks)

a) $5x^2 - 4y^2 = 20$ $(\div 20)$

i) $\dfrac{x^2}{4} - \dfrac{y^2}{5} = 1$

$a^2 = 4,\ b^2 = 5$

$b^2 = a^2(e^2 - 1)$

$5 = 4(e^2 - 1)$

$e^2 = \dfrac{9}{4}$

$e = \dfrac{3}{2}$ (2)

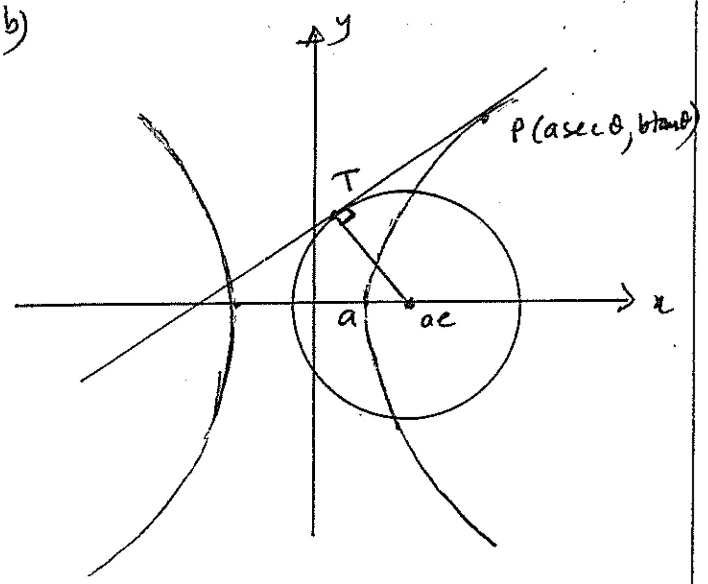
Foci $(\pm ae, 0)$

$= (\pm 3, 0)$ (1)

ii) Asymptotes: $y = \pm \dfrac{b}{a} x$

$y = \pm \dfrac{\sqrt{5}}{2} x$ (1)

9 b)



$P(a\sec\theta, b\tan\theta)$ $\left[\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1\right]$

$E_{tangent}: \frac{x\sec\theta}{a} - \frac{y\tan\theta}{b} - 1 = 0$

If tangent at P also tangent to circle, then the perpendicular distance from the centre of the circle to the tangent is equal to the radius of the circle. $(x_1, y_1) = (ae, 0)$

$$\therefore d = \left|\frac{ax_1 + by_1 + c}{\sqrt{a^2+b^2}}\right|$$

$$= \left|\frac{e\sec - 1}{\sqrt{\frac{\sec^2\theta}{a^2} + \frac{\tan^2\theta}{b^2}}}\right|$$

$$= \left|\frac{e\sec\theta - 1}{\sqrt{\frac{\sec^2\theta}{a^2} + \frac{\tan^2\theta}{a^2(e^2-1)}}}\right|$$

$$= \left|\frac{e\sec\theta - 1}{\sqrt{\frac{(e^2-1)\sec^2\theta + \tan^2\theta}{a^2(e^2-1)}}}\right|$$

$$= \left|\frac{e\sec\theta - 1}{\sqrt{\frac{e^2\sec^2\theta - \sec^2\theta + \sec^2\theta - 1}{a^2(e^2-1)}}}\right|$$

$$= \left|\frac{e\sec\theta - 1}{\sqrt{\frac{e^2\sec^2\theta - 1}{a^2(e^2-1)}}}\right|$$

$$= \left|\frac{a\sqrt{e^2-1}\,(e\sec\theta - 1)}{\sqrt{e^2\sec^2\theta - 1}}\right|$$

$$= \left|\frac{(e\sec\theta - 1)\,a\sqrt{e^2-1}}{\sqrt{(e\sec\theta - 1)(e\sec\theta + 1)}}\right|$$

$$= \frac{\sqrt{e\sec\theta - 1}}{\sqrt{e\sec\theta + 1}} \cdot a\sqrt{e^2-1}$$

$$d = r = a\sqrt{e^2+1}$$

$$\therefore \frac{\sqrt{e\sec\theta - 1}}{\sqrt{e\sec\theta + 1}} = \frac{\sqrt{e^2+1}}{\sqrt{e^2-1}}$$

squaring both sides.

$$\frac{e\sec\theta - 1}{e\sec\theta + 1} = \frac{e^2+1}{e^2-1}$$

$$e^3\sec\theta + e^2 - e\sec\theta + 1 = e^3\sec\theta - e^2 - e\sec\theta + 1$$

$$\therefore 2e^2 = -2e\sec\theta$$

$$\therefore \sec\theta = -e$$

(4)

Question 10 (10 marks)

a) i) $P\left(pt, \frac{t}{p}\right)$; $Q\left(qt, \frac{t}{q}\right)$

$xy = t^2$

$m_{PQ} = \dfrac{\frac{t}{q} - \frac{t}{p}}{qt - pt}$

$= \dfrac{pt - qt}{pq} \times \dfrac{1}{-(pt - qt)}$

$= \dfrac{-1}{pq}$ (1)

ii) $y = \dfrac{t^2}{x}$

$\dfrac{dy}{dx} = \dfrac{-t^2}{x^2}$

$m_{at\ P} = \dfrac{-t^2}{p^2t^2} = -\dfrac{1}{p^2}$ (1)

iii) If PQ is a normal, then

$m_{PQ} \cdot m_{tangent} = -1$

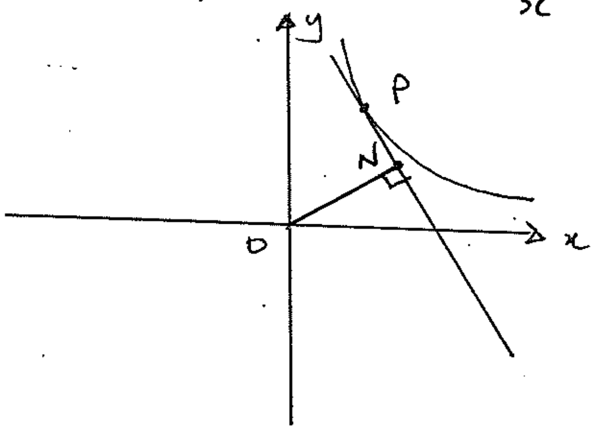
$-\dfrac{1}{pq} \cdot -\dfrac{1}{p^2} = -1$

$\dfrac{1}{p^3q} = -1$

$q = \dfrac{-1}{p^3}$ (2)

b) $P\left(6p, \frac{6}{p}\right)$; $y = \dfrac{36}{x}$

i)



$E_{tangent\ at\ P}: x + p^2y - 12p = 0$

$y = \dfrac{-x + 12p}{p^2}$

i) $m_{tangent\ at\ P} = -\dfrac{1}{p^2}$

$\therefore m_{ON} = p^2$

$\therefore E_{ON} \Rightarrow y = p^2x$

Point of Intersection of PN and ON

$\Rightarrow \dfrac{-x + 12p}{p^2} = p^2x$

$-x + 12p = p^4x$

$p^4x + x = 12p$

$x(p^4 + 1) = 12p$

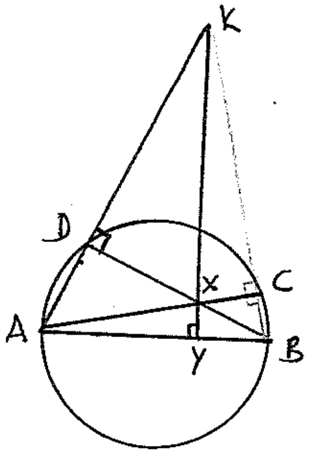
$x = \dfrac{12p}{1 + p^4}$

$y_N = p^2\left(\dfrac{12p}{1+p^4}\right)$

$= \dfrac{12p^3}{1+p^4}$

$\therefore N = \left(\dfrac{12p}{1+p^4}, \dfrac{12p^3}{1+p^4}\right)$ (3)

Question 11 (8 marks)



i) Show $\angle AKY = \angle ABD$

$\angle ADB = 90°$ ($\angle$ in a semi-circle)

$\angle ABD = 90° - \angle BAD$ ($\angle$ sum of ΔADB)

$\angle AKY = 90° - \angle BAD$ ($\angle$ sum of ΔAKY)

$\therefore \angle AKY = \angle ABD$ (both $= 90 - \angle BAD$)

[Alternatively, use similar triangles] (2)

ii) Show CKDX is a cyclic quadrilateral

$\angle AKY = \angle ABD$ (from (i))

$\angle ABD = \angle ACD$ ($\angle$ subtended by same arc AD)

$\therefore \angle AKX = \angle ACD$ (both equal to $\angle ABD$)

$\therefore$ CKDX is cyclic ($\angle$s subtended by the same chord are equal)

(3)

iii) Show B, C & K are collinear.

$\angle ACK = 180 - \angle XDK$ (opposite $\angle$s of cyclic quadrilateral CKDX)

$\angle XDK = 90°$ (adjacent supplementary $\angle$ to $\angle ADB$)

$\therefore \angle ACK = 90°$

$\angle ACB = 90°$ ($\angle$ in a semi-circle)

$\therefore$ B, C & K are collinear. (3)

10. b ii)

N: $x = \dfrac{12p}{1+p^4}$; $y = \dfrac{12p^3}{1+p^4}$

$$\frac{y}{x} = \frac{12p^3}{1+p^4} \times \frac{1+p^4}{12p}$$

$$\Rightarrow p^2 = \frac{y}{x}$$

$$\therefore x = \frac{12\sqrt{\frac{y}{x}}}{1 + \frac{y^2}{x^2}}$$

$$x + \frac{y^2}{x} = 12\sqrt{\frac{y}{x}}$$

squaring both sides

$$x^2 + 2y^2 + \frac{y^4}{x^2} = 144\frac{y}{x}$$

$$x^4 + 2x^2y^2 + y^4 = 144xy$$

$$144xy = (x^2+y^2)^2$$

(3)