GIRRAWEEN HIGH SCHOOL

HALF YEARLY EXAMINATIONS

2018

MATHEMATICS

EXTENSION 2

Time Allowed: Two hours

(Plus 5 minutes reading time)

Total Marks: 95

Instructions:

- There are 11 questions in this paper. All questions are compulsory.
- Use blue or black pen.
- Write all your answers in the Answer Booklets provided.
- For Questions 1 – 5, fill in the circle corresponding to the correct answer in your answer booklet.
- For Questions 6 – 11, start each question on a new page.
- Write on both sides of the paper.
- Show all necessary working.
- Board-approved calculators may be used.
- Mathematics reference sheets are provided.
- Marks may be deducted for careless or badly arranged work.
- Write ‘End of Solutions’ at the conclusion of your solutions to the task.

Questions 1 - 5 (5 marks)

Fill in the circle corresponding to the correct answer in your answer booklet.

1 What is the value of $\frac{6}{iz}$ if $z = -1 + i$?

(A) $-3-3i$

(B) $-3+3i$

(C) $3-3i$

(D) $3+3i$

2 If ω is an imaginary cube root of unity, then $(1+\omega-\omega^2)^{2017}$ is equal to:

(A) $-2^{2017}\omega$

(B) $2^{2017}\omega$

(C) $-2^{2017}\omega^2$

(D) $2^{2017}\omega^2$

3 The equation $24x^3 - 12x^2 - 6x + 1 = 0$ has roots α, β and γ.
What is the value of α if $\alpha = \beta + \gamma$.

(A) $-\frac{1}{2}$

(B) $\frac{1}{4}$

(C) $\frac{1}{2}$

(D) 1

4 The equations of the asymptotes of the hyperbola $9x^2 - 4y^2 = 36$ are:

(A) $x = \pm\frac{3}{2}y$

(B) $y = \pm\frac{3}{2}x$

(C) $x = \pm\frac{9}{4}y$

(D) $y = \pm\frac{9}{4}x$

5 The foci of the ellipse $\frac{x^2}{9} + \frac{y^2}{16} = 1$ are:

(A) $(0, \pm\sqrt{7})$

(B) $(\pm\sqrt{7}), 0)$

(C) $\left(0, \pm\frac{\sqrt{7}}{4}\right)$

(D) $\left(\pm\frac{\sqrt{7}}{4}, 0\right)$

Question 6 (28 marks)

a. Let $\omega_1 = 8 - 2i$ and $\omega_2 = -5 + 3i$. Find the value of:

[2]

(i) $\omega_1\omega_2$

(ii) $\omega_1 + \overline{\omega_2}$ [2]

b. Reduce the complex expression $\dfrac{(2 - i)(8 + 3i)}{3 + i}$ to the form $a + ib$ where a and b are real numbers. [3]

c. (i) Express $\sqrt{3} - i$ in modulus-argument form. [2]

(ii) Hence evaluate $(\sqrt{3} - i)^6$. [3]

d. Sketch the region defined by:

$|z - (3 + i)| \leq 3$ and $\dfrac{\pi}{4} < \arg[z - (1 + i)] \leq \dfrac{\pi}{2}$ are satisfied. [4]

e. If $z = \cos\theta + i\sin\theta$

(i) Show that $z^n + \dfrac{1}{z^n} = 2\cos n\theta$. [2]

(ii) Hence show that $\cos^4\theta = \dfrac{1}{8}(\cos 4\theta + 4\cos 2\theta + 3)$. [3]

f. (i) Show that $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$. [2]

(ii) Using the equation $8x^3 - 6x - 1 = 0$ and letting $x = \cos\theta$, deduce that $\cos 3\theta = \dfrac{1}{2}$. [1]

(iii) Find the roots of $8x^3 - 6x - 1 = 0$ in terms of $\cos\theta$. [2]

(iv) Hence evaluate $\cos\dfrac{\pi}{9}\cos\dfrac{2\pi}{9}\cos\dfrac{4\pi}{9}$. [2]

Question 7 (20 marks)

a. The equation $x^3 - 4x^2 + 5x + 2 = 0$ has roots α, β and γ.

Find (i) $\alpha + \beta + \gamma$ [1]

(ii) $\alpha\beta + \beta\gamma + \gamma\alpha$ [1]

(iii) $\alpha^2 + \beta^2 + \gamma^2$ [2]

(iv) $(2 + \alpha)(2 + \beta)(2 + \gamma)$ [2]

b. (i) Find a and b such that $x = 2$ is a double root of $f(x) = x^4 + ax^3 + x^2 + b$. [2]

(ii) For the values of *a* and *b* above, factorise $f(x)$ over the real numbers. [2]

c. Let α, β and γ be the roots of the equation $x^3 - 5x^2 + 5 = 0$.

(i) Find a polynomial equation with integer coefficients whose roots are $\alpha - 1, \beta - 1$ and $\gamma - 1$. [2]

(ii) Find a polynomial equation with integer coefficients whose roots are α^2, β^2 and γ^2. [2]

(iii) Find the value of $\alpha^3 + \beta^3 + \gamma^3$. [2]

d. Given that $2 + 3i$ is a solution to $x^4 - 6x^3 + 26x^2 - 46x + 65 = 0$

(i) State why $2 - 3i$ is also a solution. [1]

(ii) Hence, completely solve $x^4 - 6x^3 + 26x^2 - 46x + 65 = 0$ [3]

Question 8 (12 marks)

a. Point $P\,(x_0, y_0)$ is on the ellipse $\frac{x^2}{25} + \frac{y^2}{9} = 1$.

(i) Find the coordinates of the foci and the equations of the directrices. [2]

(ii) Sketch the ellipse, clearly showing all important features. [2]

(iii) Show that the equation of the tangent at *P* is $\frac{x\,x_0}{25} + \frac{y\,y_0}{9} = 1$. [2]

(iv) Let the tangent at P meet a directrix at a point *Q*.

Show that $\angle PSQ$ is a right angle, where S is the corresponding focus. [4]

b. Describe the locus of *z*, stating its equation, if $|z + 3| + |z - 3| = 10$. [2]

Question 9 (7 marks)

Consider the hyperbola $\frac{x^2}{4} - \frac{y^2}{12} = 1$.

(i) Sketch the graph of the hyperbola, clearly showing:
- any intercepts made with the axes
- the coordinates of the foci
- the equations of the directrices
- the equations of the asymptotes. [4]

(ii) $T(x_1, y_1)$ is a point on the hyperbola.

The equation of the tangent to the hyperbola at *T* is

$\frac{x\,x_1}{4} - \frac{y\,y_1}{12} = 1$ [Do not prove this].

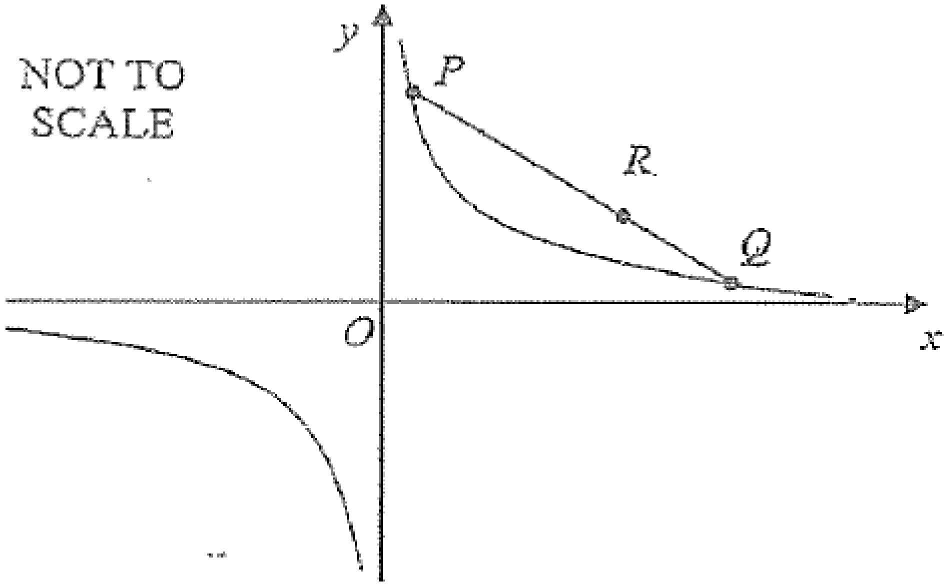
$P(4, 6)$ and $Q(14, 24)$ are two points on the hyperbola.
M is the midpoint of *PQ* and *O* is the origin.
The tangents to the hyperbola at *P* and *Q* intersect at the point *R*.
Show that the points *R*, *O* and *M* are collinear. [3]

Question 10 (15 marks)

a. $P\left(2p,\dfrac{2}{p}\right)$ and $Q\left(2q,\dfrac{2}{q}\right)$ are points on the rectangular hyperbola $xy = 4$.

M is the midpoint of the chord *PQ*. *P* and *Q* move on the hyperbola so

that the chord *PQ* always passes through the point *R(4, 2).*



(i) Show that the equation of the chord *PQ* is $x + pqy = 2(p + q)$. [3]

(ii) Show that $pq = p + q - 2$. [1]

(iii) Hence find the equation of the locus of *M*, as *P* and *Q* move on the curve $xy = 4$. [3]

b. $P\left(cp,\dfrac{c}{p}\right)$ and $Q\left(cq,\dfrac{c}{q}\right)$ are points on the rectangular hyperbola $x\,y = c^2$.

Tangents to the hyperbola at *P* and *Q* intersect at the point *R* .

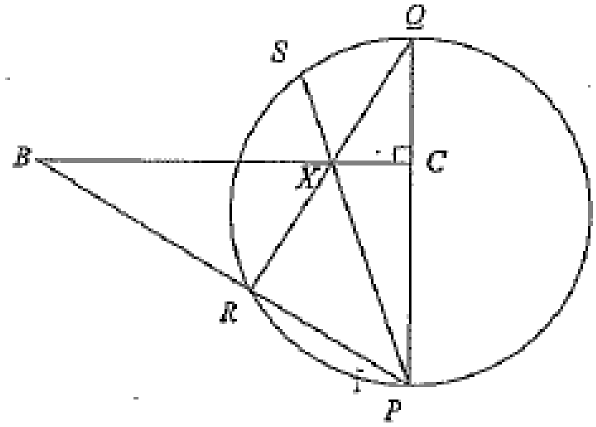
(i) Show that the tangent to the rectangular hyperbola at the point $\left(ct,\dfrac{c}{t}\right)$ has the equation $x + t^2y = 2ct$. [2]

(ii) Show that $R = \left(\dfrac{2cpq}{p+q}\ ,\ \dfrac{2c}{p+q}\right)$ [3]

(iii) If *P* and *Q* are variable points on the rectangular hyperbola which move so that $p^2 + q^2 = 2$, find the equation of the locus of *R*. [3]

Question 11 (8 marks)

In the diagram, PQ is a diameter of the circle. The chords QR and PS intersect at X. The point C lies on PQ such that XC is perpendicular to PQ. The point B is the intersection of PR produced and CX produced.



(i) Trace or copy the diagram into your booklet and show that $\angle PBC = \angle PQR$. [2]

(ii) Show that *SBRX* is a cyclic quadrilateral. [3]

(iii) Show that the points B, S and Q are collinear. [3]

END OF EXAMINATION

EXTENSION 2

HY 2018 SOLUTIONS

MC

1. $\frac{6}{iz} = \frac{6}{i(-1+i)} = \frac{6}{-1-i} \times \frac{-1+i}{-1+i}$

$= \frac{6(-1+i)}{2}$

$= -3+3i$ $\boxed{B}$

2. $(1+\omega-\omega^2)^{2017}$

$= (1+\omega+\omega^2-2\omega^2)^{2017}$

$= (-2\omega^2)^{2017}$ $[1+\omega+\omega^2=0]$

$= (-2)^{2017} \cdot \omega^{4034}$

$= (-2)^{2017} \cdot \omega^2 \cdot \omega^{4032}$

$= (-2)^{2017} \cdot \omega^2 \cdot (\omega^3)^{1344}$ $[\omega^3=1]$

$= -2^{2017}\omega^2$

$\boxed{C}$

3. Sum of roots : $\alpha+\beta+\gamma = \frac{-(-12)}{24} = \frac{1}{2}$

$\alpha+\alpha = \frac{1}{2}$

$2\alpha = \frac{1}{2}$

$\alpha = \frac{1}{4}$

$\boxed{B}$

4. $9x^2-4y^2=36$

$\frac{x^2}{4}-\frac{y^2}{9}=1$

$a=2,\ b=3$

Asymptotes : $y = \pm\frac{b}{a}x$

$y = \pm\frac{3}{2}x$ $\boxed{B}$

5. $\frac{x^2}{9}+\frac{y^2}{16}=1$ $a=3,\ b=4$

major axis along y-axis

Foci $(0, \pm be)$

$= (0, \pm\sqrt{7})$

$e^2 = 1-\frac{a^2}{b^2}$

$= \frac{7}{16}$

$e = \frac{\sqrt{7}}{4}$

$\boxed{A}$

Question 6

a) $w_1 = 8 - 2i$; $w_2 = -5 + 3i$

(i) $w_1 w_2$

$= (8-2i)(-5+3i)$

$= -40 + 24i + 10i + 6$

$= -34 + 34i$

(ii) $w_1 + \overline{w_2}$

$= 8 - 2i - 5 - 3i$

$= 3 - 5i$

b) $\dfrac{(2-i)(8+3i)}{3+i}$

$= \dfrac{16 + 6i - 8i + 3}{3+i}$

$= \dfrac{19 - 2i}{3+i} \times \dfrac{3-i}{3-i}$

$= \dfrac{55 - 25i}{10}$

$= \dfrac{11}{2} - \dfrac{5i}{2}$

c) (i) $\sqrt{3} - i$

$|\sqrt{3} - i| = \sqrt{3+1} = 2$

$\arg(\sqrt{3} - i) = \tan^{-1}\left(\frac{-1}{\sqrt{3}}\right)$

$= \frac{-\pi}{6}$

$\sqrt{3} - i = 2\,\text{cis}\left(-\frac{\pi}{6}\right)$

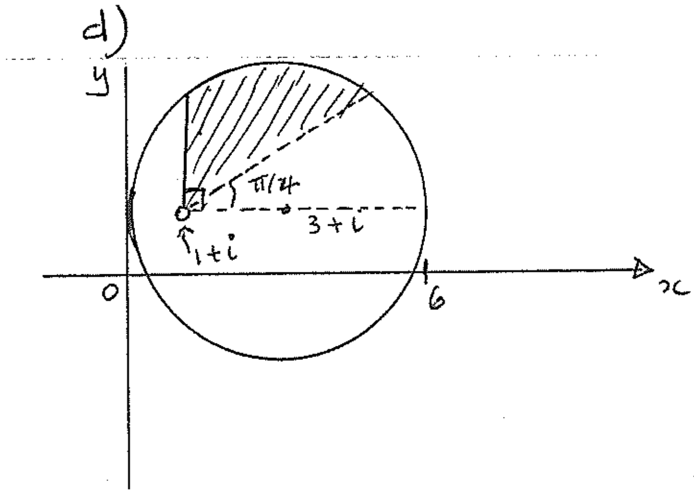
(ii) $(\sqrt{3} - i)^6 = \left(2\,\text{cis}\,\frac{-\pi}{6}\right)^6$

$= 64\,\text{cis}(-\pi)$

$= 64(\cos(-\pi) + i\sin(-\pi))$

$= -64$

d)



e) (i) $z = \cos\theta + i\sin\theta$

$z^n = \cos n\theta + i\sin n\theta$ (De Moivre's thm)

$z^{-n} = \cos(-n\theta) + i\sin(-n\theta)$

$= \cos n\theta - i\sin n\theta$ (sin odd, cos even)

$\therefore z^n + \frac{1}{z^n} = 2\cos n\theta$

(ii) $\left(z + \frac{1}{z}\right)^4 = z^4 + 4z^2 + 6 + \frac{4}{z^2} + \frac{1}{z^4}$

$(2\cos\theta)^4 = \left(z^4 + \frac{1}{z^4}\right) + 4\left(z^2 + \frac{1}{z^2}\right) + 6$

$16\cos^4\theta = 2\cos 4\theta + 8\cos 2\theta + 6$

$\cos^4\theta = \frac{1}{8}(\cos 4\theta + 4\cos 2\theta + 3)$

f) i) $\cos 3\theta + i\sin 3\theta$

$= (\cos\theta + i\sin\theta)^3$

$= \cos^3\theta + 3i\sin\theta\cos^2\theta - 3\cos\theta\sin^2\theta - i\sin^3\theta$

Equating real parts,

$\cos 3\theta = \cos^3\theta - 3\cos\theta\sin^2\theta$

$= \cos^3\theta - 3\cos\theta(1 - \cos^2\theta)$

$= \cos^3\theta - 3\cos\theta + 3\cos^3\theta$

$= 4\cos^3\theta - 3\cos\theta$

f (ii) $8x^3 - 6x - 1 = 0$

$\Rightarrow \quad 4x^3 - 3x = \frac{1}{2}$

let $x = \cos\theta$

$4\cos^3\theta - 3\cos\theta = \frac{1}{2}$

$\therefore \cos 3\theta = \frac{1}{2}$

(iii) $\cos 3\theta = \frac{1}{2}$

$3\theta = \frac{\pi}{3}, \frac{5\pi}{3}, \frac{7\pi}{3}$

$\theta = \frac{\pi}{9}, \frac{5\pi}{9}, \frac{7\pi}{9}$

$\therefore x = \cos\frac{\pi}{9}, \cos\frac{5\pi}{9}, \cos\frac{7\pi}{9}$

$= \cos\frac{\pi}{9}, -\cos\frac{4\pi}{9}, -\cos\frac{2\pi}{9}$

iv) Product of roots :

$\left(\cos\frac{\pi}{9}\right)\left(-\cos\frac{4\pi}{9}\right)\left(-\cos\frac{2\pi}{9}\right) = \frac{1}{8}$

$\therefore \cos\frac{\pi}{9}\cos\frac{2\pi}{9}\cos\frac{4\pi}{9} = \frac{1}{8}$

Question 7

a. $x^3 - 4x^2 + 5x + 2 = 0$

i) $\alpha + \beta + \gamma = 4$

ii) $\alpha\beta + \beta\gamma + \gamma\alpha = 5$

iii) $\alpha^2 + \beta^2 + \gamma^2 = (\alpha+\beta+\gamma)^2 - 2(\alpha\beta + \beta\gamma + \gamma\alpha)$

$= 4^2 - 2(5)$

$= 6$

iv) $(2+\alpha)(2+\beta)(2+\gamma)$

$= 8 + 2(\alpha\beta + \alpha\gamma + \beta\gamma) + 4(\alpha+\beta+\gamma) + \alpha\beta\gamma$

$= 8 + 2(5) + 4(4) + (-2)$

$= 32$

Q7 (con't)

b. (i) $f(x) = x^4 + ax^3 + x^2 + b$

$f'(x) = 4x^3 + 3ax^2 + 2x$

$x = 2$ is a double root

$\therefore f(2) = 0$ & $f'(2) = 0$

ie $16 + 8a + 4 + b = 0$ & $32 + 12a + 4 = 0$

$20 + 8a + b = 0$ $\qquad 12a = -36$

$20 - 24 + b = 0$ $\qquad a = -3$

$b = 4$

$\therefore a = -3,\ b = 4$

ii) $f(x) = x^4 - 3x^3 + x^2 + 4$

$= (x-2)^2 \cdot Q(x)$

$= (x-2)^2 (x^2 + x + 1)$

c. $x^3 - 5x^2 + 5 = 0$

i) let $y = x - 1 \Rightarrow x = y + 1$

E: $(y+1)^3 - 5(y+1)^2 + 5 = 0$

$y^3 - 2y^2 - 7y + 1 = 0$

(OR) $x^3 - 2x^2 - 7x + 1 = 0$

ii) Let $y = x^2 \Rightarrow x = \sqrt{y}$

E: $(\sqrt{y})^3 - 5(\sqrt{y})^2 + 5 = 0$

$y\sqrt{y} - 5y + 5 = 0$

$y\sqrt{y} = 5y - 5$ squaring Both sides

$y^3 = 25y^2 - 50y + 25$

$y^3 - 25y^2 + 50y - 25 = 0$

(OR) $x^3 - 25x^2 + 50x - 25 = 0$

iii) If α, β & γ are roots of $x^3 - 5x^2 + 5 = 0$, then

$\alpha^3 - 5\alpha^2 + 5 = 0$ —①

$\beta^3 - 5\beta^2 + 5 = 0$ —②

$\gamma^3 - 5\gamma^2 + 5 = 0$ —③

Adding ①, ② & ③ $\quad \alpha^3 + \beta^3 + \gamma^3 - 5(\alpha^2 + \beta^2 + \gamma^2) + 15 = 0$

$\alpha^3 + \beta^3 + \gamma^3 = 5(\alpha^2 + \beta^2 + \gamma^2) - 15$ $\quad$ [sum of roots from (ii) $\alpha^2 + \beta^2 + \gamma^2 = 25$

$= 5(25) - 15$

$= 110$

Q7 (cont)

d. $x^4 - 6x^3 + 26x^2 - 46x + 65 = 0$

i) Polynomials with real coefficients have complex roots in conjugate pairs

$\therefore$ $2-3i$ is also a solution.

ii) $2+3i$, $2-3i$

let $z = 2+3i$

$\bar{z} = 2-3i$

$z + \bar{z} = 4$; $z\bar{z} = 13$

$$\therefore (x-z)(x-\bar{z}) = x^2 - (z+\bar{z})x + z\bar{z}$$
$$= x^2 - 4x + 13$$

$$\begin{array}{r} x^2 - 2x + 5 \\ x^2 - 4x + 13 \overline{\big) x^4 - 6x^3 + 26x^2 - 46x + 65} \\ - \underline{x^4 - 4x^3 + 13x^2} \\ -2x^3 + 13x^2 - 46x \\ - \underline{-2x^3 + 8x^2 - 26x} \\ 5x^2 - 20x + 65 \\ - \underline{5x^2 - 20x + 65} \end{array}$$

$x^2 - 2x + 5 = 0$

$x^2 - 2x^{+1} = -5^{+1}$

$(x-1)^2 = -4$

$x - 1 = \pm\sqrt{4i^2}$

$x - 1 = \pm 2i$

$x = 1 + 2i$

$\therefore$ Roots : $2+3i$, $2-3i$, $1+2i$, $1-2i$

Question 8

a. i) $P(x_0, y_0)$; $\frac{x^2}{25} + \frac{y^2}{9} = 1$

$a = 5,\ b = 3$

$b^2 = a^2(1 - e^2)$

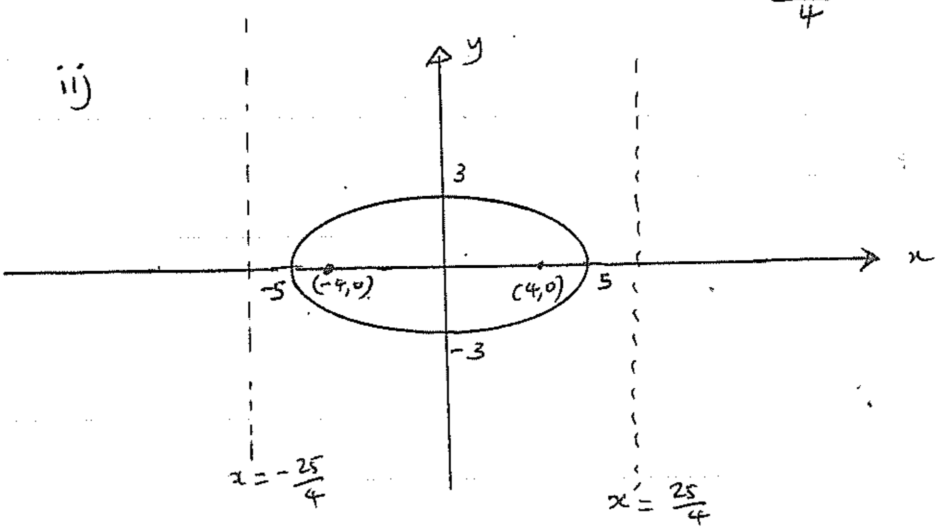
$9 = 25(1 - e^2)$

$e = \frac{4}{5}$

Foci: $(\pm ae, 0)$
$= (\pm 4, 0)$

Directrices:
$x = \pm \frac{a}{e}$
$x = \pm \frac{25}{4}$

ii)



iii) $\frac{x^2}{25} + \frac{y^2}{9} = 1$

By implicit differentiation,

$$\frac{2x}{25} + \frac{2y}{9}\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = \frac{-9x}{25y}$$

At $P(x_0, y_0)$, $m_{tangent} = \frac{-9x_0}{25y_0}$

$E_{tangent}$: $y - y_0 = \frac{-9x_0}{25y_0}(x - x_0)$

$$25y_0 y - 25y_0^2 = -9x_0 x + 9x_0^2$$

$$9x_0 x + 25y_0 y = 9x_0^2 + 25y_0^2 \qquad (\div 225)$$

$$\frac{xx_0}{25} + \frac{yy_0}{9} = \frac{x_0^2}{25} + \frac{y_0^2}{9}$$

$\therefore E_{tangent}: \frac{xx_0}{25} + \frac{yy_0}{9} = 1$ (since (x_0, y_0) lies on the ellipse $\frac{x_0^2}{25} + \frac{y_0^2}{9} = 1$

Q8 cont

iv) Q on directrix $\therefore x = \frac{25}{4}$

Q lies on tangent

$$\therefore \frac{\frac{25}{4}x_0}{25} + \frac{yy_0}{9} = 1$$

$$\frac{yy_0}{9} = 1 - \frac{x_0}{4}$$

$$\frac{yy_0}{9} = \frac{4-x_0}{4}$$

$$y = \frac{9(4-x_0)}{4y_0}$$

$$\therefore Q\left(\frac{25}{4}, \frac{9(4-x_0)}{4y_0}\right)$$

$$m_{QS} = \frac{\frac{9(4-x_0)}{4y_0} - 0}{\frac{25}{4} - 4} \qquad m_{PS} = \frac{y_0 - 0}{x_0 - 4}$$

$$= \frac{4-x_0}{y_0} \qquad\qquad = \frac{y_0}{x_0-4}$$

$$m_{QS} \cdot m_{PS} = -1$$

$\therefore \angle PSQ$ is a right angle.

b. i) The equation $|z+3| + |z-3| = 10$ describes a point z moving in the complex plane such that the sum of its distances to the points $(3,0)$ & $(-3,0)$ is a constant (10) ie. an ellipse

$$2a = 10 \Rightarrow a = 5$$

Foci $S(3,0)$, $S'(-3,0)$

$$ae = 3 \therefore e = \frac{3}{5} \quad ; \quad b^2 = 5^2\left(1 - \left(\frac{3}{5}\right)^2\right)$$

$$b = 4$$

Equation $\frac{x^2}{25} + \frac{y^2}{16} = 1$

Question 9

$$\frac{x^2}{4} - \frac{y^2}{12} = 1$$

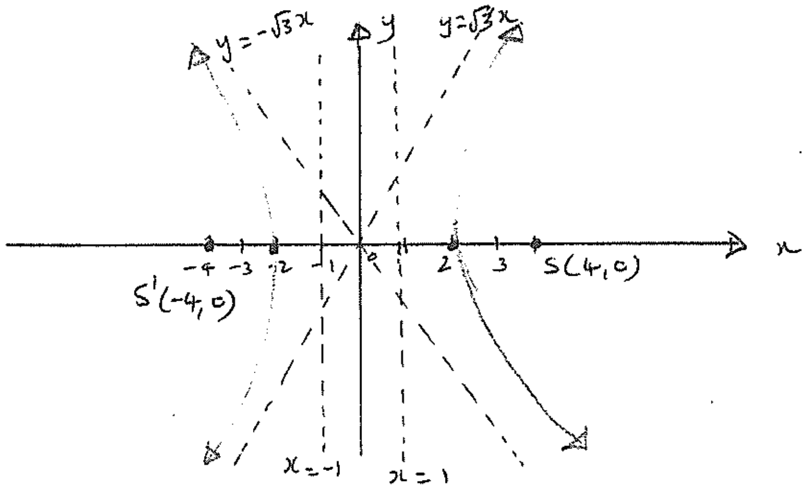
(i) $a^2 = 4$, $b^2 = 12$

$$b^2 = a^2(e^2 - 1)$$
$$e^2 = 4$$
$$e = 2$$

Foci : $(\pm ae, 0)$
$= (\pm 4, 0)$

Directrices : $x = \pm \frac{a}{e}$
$x = \pm 1$

Asymptotes : $y = \pm \frac{b}{a} x$
$= \pm \frac{\sqrt{12}}{\sqrt{4}} x$
$y = \pm \sqrt{3}\, x$



(ii) $P(4,6)$; $Q(14,24)$

$\therefore M_{PQ} = (9, 15)$

Tangent at P : $\frac{4x}{4} - \frac{6y}{12} = 1$
$x - \frac{y}{2} = 1$
$2x - y = 2$

Tangent at Q : $\frac{14x}{4} - \frac{24y}{12} = 1$
$\frac{7x}{2} - 2y = 1$
$7x - 4y = 2$

Tangents intersect at R
Solving ① & ② simultaneously,

$2x - y = 2$ — ① $\boxed{\times 4}$
$7x - 4y = 2$ — ②
$8x - 4y = 8$ — ③

③ − ① $x = 6$
$2(6) - y = 2$
$y = 10$

$\therefore R(6, 10)$

Gradient of OM $= \frac{15}{9} = \frac{5}{3}$
Gradient of OR $= \frac{10}{6} = \frac{5}{3}$

$\therefore$ R, O and M are collinear.

Question 10

a. $P\left(2p, \frac{2}{p}\right)$; $Q\left(2q, \frac{2}{q}\right)$; $xy = 4$

i) $m_{PQ} = \frac{y_2 - y_1}{x_2 - x_1}$

$= \frac{\frac{2}{q} - \frac{2}{p}}{2q - 2p}$

$= \frac{\frac{2p - 2q}{pq}}{-2(p-q)}$

$= -\frac{1}{pq}$

E_{PQ} : $y - \frac{2}{p} = \frac{-1}{pq}(x - 2p)$

$pqy - 2q = -x + 2p$

$x + pqy = 2(p+q)$

ii) PQ passes through $R(4,2)$

$\therefore 4 + pq(2) = 2(p+q)$

$2(2 + pq) = 2(p+q)$

$2 + pq = p+q$

$pq = p+q-2$

iii) M is the midpoint of PQ

$\therefore M = \left(p+q, \frac{p+q}{pq}\right)$

$x = p+q$; $y = \frac{p+q}{pq}$

$\therefore y = \frac{x}{pq}$

$= \frac{x}{p+q-2}$ (from ii)) ($x = p+q$)

$y = \frac{x}{x-2}$ is the locus of M.

Q10 (con't)

b. $P\left(cp, \frac{c}{p}\right)$; $Q\left(cq, \frac{c}{q}\right)$; $xy = c^2$

i) At $\left(ct, \frac{c}{t}\right)$

$$y = \frac{c^2}{x} = c^2x^{-1}$$

$$\frac{dy}{dx} = \frac{-c^2}{x^2}$$

$$m_{tangent} = \frac{-c^2}{c^2t^2} = \frac{-1}{t^2}$$

$$E_{tangent} : y - \frac{c}{t} = \frac{-1}{t^2}(x - ct)$$

$$yt^2 - ct = -x + ct$$

$$\therefore x + t^2y = 2ct$$

ii) $E_{tangent\ at\ P} : \quad x + p^2y = 2cp \quad — ①$

$E_{tangent\ at\ Q} : \quad x + q^2y = 2cq \quad — ②$

Tangents at P and Q intersect where

$$(p^2 - q^2)y = 2c(p-q)$$

$$(p-q)(p+q)y = 2c(p-q)$$

$$y = \frac{2c}{p+q}$$

substituting into ①

$$x + p^2\left(\frac{2c}{p+q}\right) = 2cp$$

$$x(p+q) + 2cp^2 = 2cp(p+q)$$

$$x(p+q) + 2cp^2 = 2cp^2 + 2cpq$$

$$x(p+q) = 2cpq$$

$$x = \frac{2cpq}{p+q}$$

$$\therefore X = \frac{2cpq}{p+q} ; \quad Y = \frac{2c}{p+q} \qquad (p \neq q)$$

Q10 con't

iii) $p^2+q^2 = (p+q)^2 - 2pq$
$(p^2+q^2 = 2)$

$\therefore (p+q)^2 = 2 + 2pq$
$(p+q)^2 = 2(1+pq)$

At R

$X = \frac{2cpq}{p+q}$; $Y = \frac{2c}{p+q}$

$pq = \frac{X(p+q)}{2c}$

$pq = \frac{X}{Y}$ $\qquad p+q = \frac{2c}{Y}$

$\therefore$ the locus of R has equation

$$\left(\frac{2c}{y}\right)^2 = 2\left(1+\frac{x}{y}\right)$$

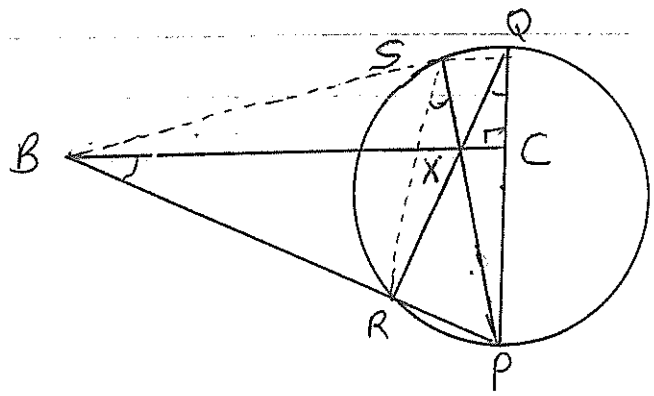
$$\frac{4c^2}{y^2} = 2\left(1+\frac{x}{y}\right)$$

$$2c^2 = y^2\left(1+\frac{x}{y}\right)$$

$$2c^2 = y^2 + xy$$

or $y^2 + xy = 2c^2$

Question 11 (8 marks)



i) In Δs BCP and PRQ,

$\angle PRQ = 90°$ ($\angle$ in a semi-circle)

$\angle BPC$ is common

$\therefore \Delta BCP \parallel\!| \Delta QRP$ (equiangular)

$\therefore \angle PBC = \angle PQR$ (matching $\angle$s in $\parallel\!|\Delta$s)

ii) Join SR

$\angle RSP = \angle RQP$ ($\angle$s at the circumference standing on same arc RP)

$\angle RBC = \angle RSP$ (both equal to $\angle RQP$)

$\therefore$ SBRX is a cyclic quadrilateral

(Interval RX subtends equal $\angle$s on same side)

iii) Join BS and SQ

$\angle PSQ = 90°$ ($\angle$ in a semi-circle)

$\angle PRX = \angle BSX$ (exterior $\angle$ of cyclic quadrilateral = interior opposite $\angle$)

Since $\angle PRX = 90°$, $\angle BSX = 90°$

$\angle BSQ = \angle BSX + \angle XSQ$

$= 90° + 90°$

$= 180°$

$\therefore \angle BSQ$ is a straight $\angle$ and

B, S and Q are collinear