# GOSFORD HIGH SCHOOL

**2006**
**YEAR 12 HALF YEARLY HIGHER SCHOOL CERTIFICATE**

# MATHEMATICS EXTENSION 2

**General Instructions:**

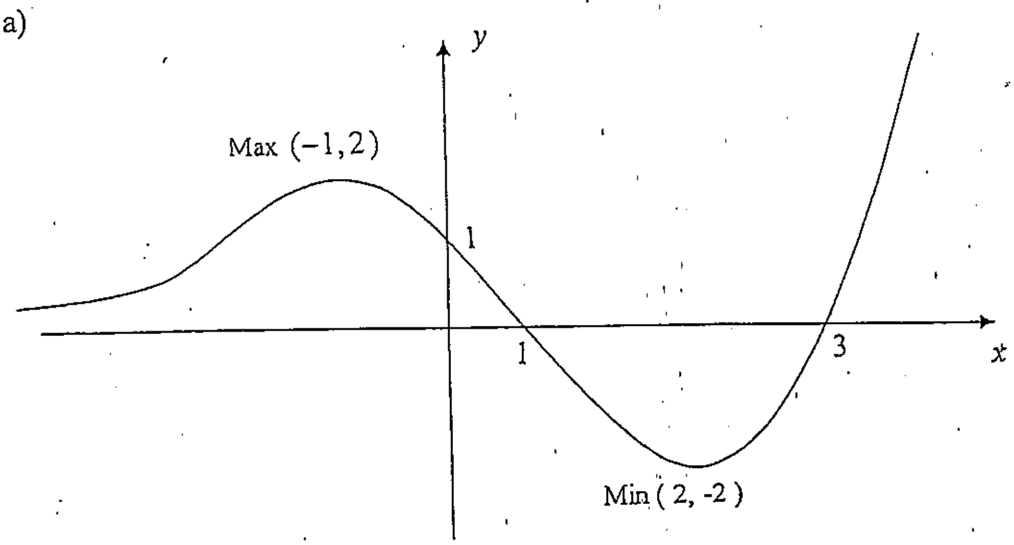
- Reading time – 5minutes
- Working time – 3 hours
- Write using black or blue pen.
- Board-approved calculators may be used
- A table of standard integrals is provided at the back of this paper
- All necessary working should be shown in every question.

**Total marks: - 105**

- Attempt Questions 1 -7
- All questions are of equal value.

**Question 1.**

a)




The graph above shows the curve $y = f(x)$. Without the use of calculus sketch the following curves on the answer sheet provided. Show any intercepts, asymptotes, end points and turning points.

i) $y = |f(x)|$ 2.

ii) $y = \dfrac{1}{f(x)}$ 2.

iii) $y = \sqrt{f(x)}$ 2.

iv) $y = \ln f(x)$ 2.

v) $y = f(2x)$ 2.

vi) $|y| = f(x)$ 2.

b) By using implicit differentiation show that the equation of the normal to the curve $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ ( $a$ and $b$ constants) at the point $(a\cos\theta, b\sin\theta)$ is given by $\dfrac{ax}{\cos\theta} - \dfrac{by}{\sin\theta} = a^2 - b^2$. 3.

**Question 2.**

a) i) Find $\int \frac{5}{(x+3)(2x+1)}dx.$ 2.

ii) Find $\int x\cos x \; dx.$ 2.

b) Use the substitution $t = \tan\frac{x}{2}$ to evaluate $\int_0^{\frac{\pi}{2}} \frac{2}{5+3\cos x}dx.$ 2.

c) i) Use the substitution $u = \frac{\pi}{4} - x$ to show that

$$\int_0^{\frac{\pi}{4}} \ln(1+\tan x)dx. = \int_0^{\frac{\pi}{4}} \ln\left[\frac{2}{1+\tan x}\right]dx$$ 3.

ii) Hence find the exact value of $\int_0^{\frac{\pi}{4}} \ln(1+\tan x)dx$ 2.

d) i) If $I_n = \int_1^e x^3(\ln x)^n\,dx$ for $n = 0,1,2\ldots..$ 2.

show that $I_n = \frac{e^4}{4} - \frac{n}{4}I_{n-1}$ for n = 1,2,3,....

ii) Hence find the value of $\int_1^e x^3(\ln x)^2.dx$ 2.

**Question 3.**

a) If $Z = \dfrac{4+3i}{1-2i}$ determine

i) $|Z|$ ii) Re(Z) iii) $Z.\overline{Z}$ iv) arg $Z$

b) i)Solve $z^5 - 1 = 0$.
Show that the roots form the vertices of a regular pentagon in the argand plane.

ii) Hence prove that $\cos\dfrac{2\pi}{5} + \cos\dfrac{4\pi}{5} + \dfrac{1}{2} = 0$

c) i) Express $1+i$ in modulus argument form. 1.

ii) Given that $(1+i)^n = x+iy$ where $x$ and $y$ are real and $n$ is an integer, prove that $x^2 + y^2 = 2^n$. 2.

d) Represent the locus on an Argand diagram for which the inequality $2|z-i| \le |z-\overline{z}|$ is satisfied. 3.

**Question 4.**

a) Factorise $x^4 + 3x^2 + 2$ over the complex field. 2.

b) The roots of the polynomial equation $2x^3 - 8x^2 + 3x + 5 = 0$ are $\alpha, \beta, \gamma$.

i) Find the polynomial equation with roots $\alpha^2, \beta^2, \gamma^2$. 2.

ii) Find the value of $\alpha^2 + \beta^2 + \gamma^2$ 2.

c) i) If $\alpha$ is a multiple root of the polynomial equation $P(x) = 0$, prove that $P'(\alpha) = 0$. 2.

ii) The polynomial equation $Q(x) = x^4 + 2x^3 - ax^2 + bx + 12$ has a double root at $x = -2$. Find the values of $a$ and $b$. 2.

d) i)Show that if $z = (\cos\theta + i\sin\theta)$ then $z^n + z^{-n} = 2\cos n\theta$. 2.

ii) Hence or otherwise solve $z^4 + 2z^3 + 3z^2 + 2z + 1 = 0$. 3.

**Question 5.**

a) By taking slices perpendicular to the axis of rotation use the method of slicing to find the volume of the solid obtained by rotating the region bounded by the curve $y = x^2$, the $x$ axis and the line $x = 1$ through one complete revolution about the line $y = 1$. 3.

b) i) Sketch the curve $y = \sin^{-1} x$, for $-1 \le x \le 1$ 1.

ii) By taking slices perpendicular to the axis of rotation use the method of slicing to find the volume of the solid generated by rotating the region bounded by the curve $y = \sin^{-1} x$, the '$x$' axis and the ordinate $x = 1$ about the '$y$' axis. 3.

c) Find $\displaystyle\int \frac{dx}{\sqrt{9 + 16x - 4x^2}}$. 3.

d) i) Show that $\displaystyle\int_{-a}^{a} f(x).dx = \int_{0}^{a} \{f(x) + f(-x)\} dx$ . 2.

ii) Hence deduce that $\displaystyle\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{e^x}{1+e^x} \sin^2 x dx = \frac{\pi}{4}$ 3.

**Question 6.**

a)If $y = \frac{1}{2}(e^x - e^{-x})$

i) show that $x = \log_e y + \sqrt{y^2 + 1}$ 3.

ii) Show that $\left(\frac{dy}{dx}\right)^2 - y^2 = 1$ 3.

b) Let $z = \cos\theta + i\sin\theta$

(i) Show that $z^n - z^{-n} = 2i\sin n\theta$ 2

(ii) Expand $(z - z^{-1})^3$ 2

(iii) Hence show that $\sin^3\theta = \frac{1}{4}(3\sin\theta - \sin 3\theta)$ 3

(iv) Evaluate $\int_0^{\frac{\pi}{2}} \sin^3\theta . d\theta$ 2

**Question 7.**

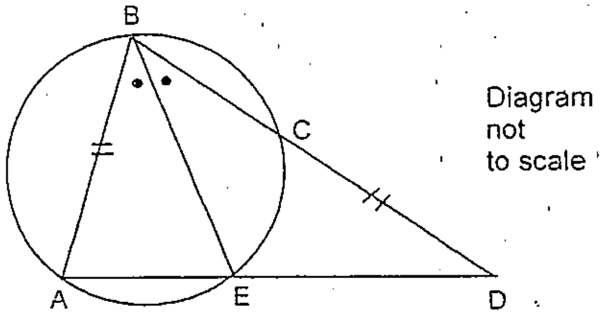
a) Prove by mathematical induction that for every positive integer $n$

$\frac{n^5}{5}+\frac{n^3}{3}+\frac{7n}{15}$ is an integer. (You may use the expansion

$(A+B)^5 = A^5 + 5A^4B + 10A^3B^2 + 10A^2B^3 + 5AB^4 + B^5$) 4

a) Sketch (without using calculus) the curve $y=\frac{x+1}{x^2+2x}$ showing all asymptotes. 3

c)




In the diagram BE bisects $\angle ABD$, and CD = AB

i) Prove that triangle CED is congruent to triangle BAE. 3.

ii) Prove that $\angle BEA = 2\times \angle EDB$ 2.

d) If $\omega$ is one of the non-real cube roots of unity find the value of $(2-\omega)(2-\omega^2)(2-\omega^4)(2-\omega^5)$. 3.