GOSFORD HIGH SCHOOL

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2008
YEAR 12 HALF YEARLY HIGHER SCHOOL CERTIFICATE

MATHEMATICS EXTENSION 2

General Instructions:

- Reading time – 5minutes
- Working time – 2 hours
- Write using black or blue pen.
- Board-approved calculators may be used
- A table of standard integrals is provided at the back of this paper
- All necessary working should be shown in every question.

Total marks: - 85

- Attempt Questions 1 -4

Question 1.

a) Write $\sqrt{3}+i$ in modulus argument form and hence evaluate $(\sqrt{3}+i)^6$. **3**

b) Write down the conjugate of $a+ib$ and hence show that if $z = x + iy$ then $\dfrac{z+\bar{z}}{z\bar{z}}$ is real. **2**

c) i) If $\left|\dfrac{z-1}{z+1}\right| = 2$ where $z = x+iy$, show that the equation of the locus of z is

$$\left(x+\frac{5}{3}\right)^2 + y^2 = \frac{16}{9}$$ **2**

ii) Represent this locus on an Argand diagram and shade the region for which the inequalities $\left|\dfrac{z-1}{z+1}\right| \le 2$ and $0 \le \arg z \le \dfrac{3\pi}{4}$ are both satisfied. **3**

d) What is the maximum value of $|z|$ for $|z-1-i| \le 2$ **2**

e) i) Find the stationary points and the asymptote(s) of the function

$$f(x) = \frac{(x+1)^4}{x^4+1}$$ **4**

ii) Sketch this function labelling all essential features. **2**

iii) Use the graph to find the set of values of k for which $(x+1)^4 = k(x^4+1)$ has two distinct real roots. **2**

iv) Redraw your sketch of $f(x)$ from part (ii) and use it to do a neat sketch of

$$y = \frac{1}{f(x)}$$ **2**

Question 2.

a) By using the substitution $x = u^2$ find $\int_0^{\frac{1}{4}} \frac{dx}{\sqrt{x} \cdot \sqrt{1-x}}$ **3**

b) Using the substitution $x = 5\sec\theta$ find $\int \frac{\sqrt{x^2 - 25}}{x} dx$ where $5 < x$. **3**

c) By using partial fractions find $\int \frac{9x-2}{2x^2 - 7x + 3} dx$ **3**

d) i) Show that the function $y = \frac{x}{\sqrt{x^2+16}}$ is increasing for all values of x **2**

ii) The region R is bounded by $y = \frac{x}{\sqrt{x^2+16}}$, the x axis and the line $x = 4$. Show by using the substitution $y = \sin\theta$ and the result $\int \sec x dx = \ln(\sec x + \tan x)$, that the volume generated by rotating R about the y axis is $16\pi(\sqrt{2} - \ln(\sqrt{2}+1))units^3$ **4**

Question 3

a) For the ellipse $\frac{x^2}{25} + \frac{y^2}{9} = 1$, find : **5**

i) The length of the axes.
ii) Its eccentricity.
iii) The coordinates of the foci.
iv) The equations of the directrices.
v) Sketch the ellipse showing the above features.

b) **i)** Show that the equation of the normal to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ at any point θ is $\frac{ax}{\sec\theta} + \frac{by}{\tan\theta} = a^2 + b^2$. **3**

ii) The normal at any point P on the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ meets the x-axis and y-axis at A and B respectively and $OAQB$ is a rectangle, O the origin. Find the coordinates of Q in terms of θ. **2**

c) P is a point on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ with centre 'O'. A line drawn from O, parallel to the tangent to the ellipse at P, meets the ellipse at Q. Prove that the area of the triangle OPQ is independent of the position of P. **5**

d) The point P $(a\sec\theta, b\tan\theta)$ on the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ with focus S, is such that the tangent at P, the latus rectum through S, and one asymptote are concurrent. Prove that SP is parallel to the other asymptote. **4**

(you may assume the equation of the tangent at P is $\frac{x\sec\theta}{a}-\frac{y\tan\theta}{b}=1$)

Question 4.

a) If $2+i$ is a root of $x^3-2x^2-3x+10$, find the other two roots. **2**

b) If $P(x)=x^4+2x^3-12x^2+14x-5$ has a triple root, find the roots of $P(x)$. **3**

c) If α,β,γ are the roots of the equation $2x^3-4x^2-3x-1=0$, find

i) $\sum\alpha$ **ii)** $\sum\alpha\beta$ **6**

iii) $\alpha^2+\beta^2+\gamma^2$ **iv)** $\alpha^3+\beta^3+\gamma^3$

d) If α,β,γ are the roots of the equation $x^3-x-1=0$, find the cubic equation with roots $\frac{1}{\alpha},\frac{1}{\beta},\frac{1}{\gamma}$ **2**

e) Find all values of z such that $z^4+1=0$. **2**

f) Show that the zeros of $P(x)=x^4+x^2+1$ are included in the zeros of x^6-1. Hence factorise $P(x)$ over the real numbers. **3**

g) Solve $2z^4+3z^3+5z^2+3z+2=0$. **3**

h) Find $\cos 4\theta$ in terms of:

i) $\sin\theta$ and $\cos\theta$. **3**

ii) $\cos\theta$ alone. **2**

iii) Hence solve $8\cos^4 x-8\cos^2 x+1=0$ for $0\le x\le\pi$. **3**

STANDARD INTEGRALS

$$\int x^n \, dx = \frac{1}{n+1} x^{n+1}, \; x \neq 0 \text{ if } n < 0$$

$$\int \frac{1}{x} \, dx = \ln x, \; x > 0$$

$$\int e^{ax} \, dx = \frac{1}{a} e^{ax}, \; a \neq 0$$

$$\int \cos ax \, dx = \frac{1}{a} \sin ax, \; a \neq 0$$

$$\int \sin ax \, dx = -\frac{1}{a} \cos ax, \; a \neq 0$$

$$\int \sec^2 ax \, dx = \frac{1}{a} \tan ax, \; a \neq 0$$

$$\int \sec ax \tan ax \, dx = \frac{1}{a} \sec ax, \; a \neq 0$$

$$\int \frac{1}{a^2 + x^2} \, dx = \frac{1}{a} \tan^{-1} \frac{x}{a}, \; a \neq 0$$

$$\int \frac{1}{\sqrt{a^2 - x^2}} \, dx = \sin^{-1} \frac{x}{a}, \; a > 0, \; -a < x < a$$

$$\int \frac{1}{\sqrt{x^2 - a^2}} \, dx = \ln\left(x + \sqrt{x^2 - a^2}\right), \; x > a > 0$$

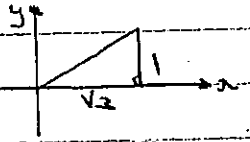
$$\int \frac{1}{\sqrt{x^2 + a^2}} \, dx = \ln\left(x + \sqrt{x^2 + a^2}\right)$$

NOTE: $\ln x = \log_e x, \; x > 0$

Ext2 half yearly 2008 Solutions

Q1)

a) $\sqrt{3}+i$



$|\sqrt{3}+i| = \sqrt{3+1}$

$= 2.$

$\arg(\sqrt{3}+i) = \tan^{-1}\frac{1}{\sqrt{3}}$

$= \frac{\pi}{6}$

$\therefore \sqrt{3}+i = 2\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)$

$(\sqrt{3}+i)^6 = \left(2\left(\cos\frac{\pi}{6}+i\sin\frac{\pi}{6}\right)\right)^6$

$= 2^6(\cos\pi + i\sin\pi)$

$= -2^6$

$= -64.$

b) conjugate $= a - ib$

$\frac{z+\bar{z}}{z\bar{z}} = \frac{x+iy+x-iy}{(x+iy)(x-iy)}$

$= \frac{2x}{x^2+y^2}$

which is real.

c) i) $\left|\frac{z-1}{z+1}\right| = 2$

$\frac{|z-1|}{|z+1|} = 2$

$\frac{\sqrt{(x-1)^2+y^2}}{\sqrt{(x+1)^2+y^2}} = 2$

$\sqrt{(x-1)^2+y^2} = 2\sqrt{(x+1)^2+y^2}$

$x^2-2x+1+y^2 = 4(x^2+2x+1+y^2)$

$x^2-2x+1+y^2 = 4x^2+8x+4+4y^2$

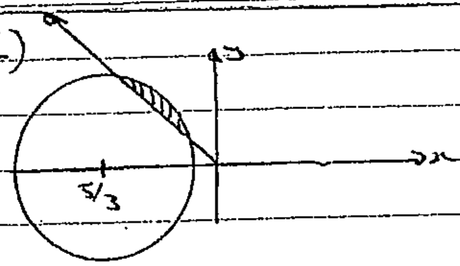
$3x^2+10x+3y^2 = -3$

$x^2+\frac{10x}{3}+y^2 = -1$

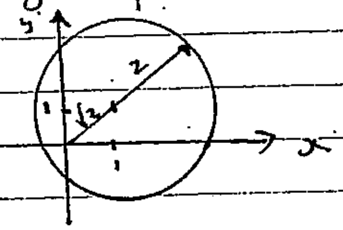
$x^2+\frac{10}{3}x+\frac{25}{9}+y^2 = -1+\frac{25}{9}$

$(x+\frac{5}{3})^2+y^2 = \frac{16}{9}$

ii)



d) $|z-1-i| \le 2.$



(circle centre $(1,1)$ rad $= 2$)

from diagram max value $= 2+\sqrt{2}$.

e) $f(x) = \frac{(x+1)^4}{x^4+1}$

$f'(x) = \frac{4(x^4+1)(x+1)^3 - 4x^3(x+1)^4}{(x^4+1)^2}$

For s.p. $f'(x) = 0$

$\Rightarrow 4(x^4+1)(x+1)^3 - 4x^3(x+1)^4 = 0$

$4(x+1)^3(x^4+1-x^3(x+1)) = 0$

$4(x+1)^3(x^4+1-x^4-x^3) = 0$

$4(x+1)^3(1-x^3) = 0$

$\left.\begin{matrix} x = -1 \\ y = 0 \end{matrix}\right\}$ or $\left.\begin{matrix} 1 \\ 8 \end{matrix}\right\}$

$f'(-2) < 0$

$f'(0) > 0$

$f'(2) < 0$

$\therefore (-1,0)$ min

$(1,8)$ max.

asymptote

$\lim_{x\to\infty} \frac{(x+1)^4}{x^4+1}$

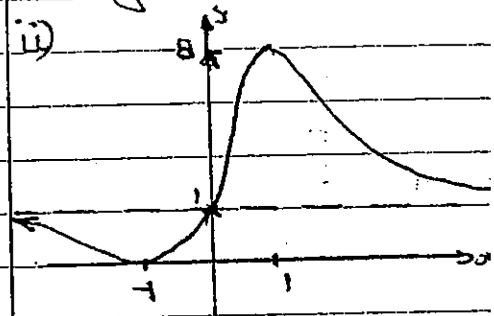
$= \lim_{x\to\infty} \frac{x^4+4x^3+6x^2+4x+1}{x^4+1}$

$\lim_{x\to\infty} \frac{\frac{x^4}{x^4}+\frac{4}{x}+\frac{6}{x^2}+\frac{4}{x^3}+}{\frac{x^4}{x^4}+\frac{1}{x^4}}$

$= 1.$

$\therefore$ horizontal asymptote at $y = 1$.

ii)



iii) $(x+1)^4 = k(x^4+1)$

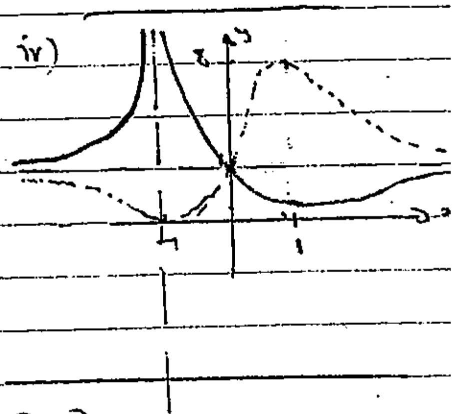
$\frac{(x+1)^4}{x^4+1} = k$

2 distinct real roots $\Rightarrow$ horizontal line hi

the curve twice.

i.e. $0<k<1$ or $1<k<$

iv)



Q2)

a) $\int_0^{1/4} \frac{dx}{\sqrt{x}.\sqrt{1-x}}$

$x = u^2$ $\quad x = 0 :$

$\frac{dx}{du} = 2u$ $\quad x = \frac{1}{4}$

$dx = 2u\,du.$

$\int_0^{1/2} \frac{2u\,du}{u\sqrt{1-u^2}}$

$= \int_0^{1/2} \frac{2\,du}{\sqrt{1-u^2}}$

$2\left[\sin^{-1} u\right]_0^{1/2}$

$2\left[\sin^{-1}\tfrac{1}{2} - \sin^{-1} 0\right]$

$2\left(\pi/6 - 0\right)$

$= \pi/3$

b) $\int \frac{\sqrt{x^2-25}}{x}\,dx$

$x = 5\sec\theta$

$\frac{dx}{d\theta} = \frac{5\sin\theta}{\cos^2\theta}$

$\int \frac{\sqrt{25\sec^2\theta - 25}}{5\sec\theta}\times\frac{5\sin\theta}{\cos^2\theta}$

$= \int \frac{\sqrt{25\tan^2\theta}}{5\sec\theta}\times \frac{5\sin\theta}{\cos^2\theta}$

$= \int \frac{\tan\theta}{\sec\theta}\times\frac{5\sin\theta}{\cos^2\theta}$

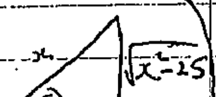
$= 5\int \frac{\sin^2\theta}{\cos^2\theta}\,d\theta$

$= 5\int \tan^2\theta\,d\theta$

$= 5\int \sec^2\theta - 1\,d\theta$

$= 5\left(\tan\theta - \theta\right)$

$\left(x = 5\sec\theta \quad \sec\theta = \frac{x}{5}\right)$



$= 5\frac{\sqrt{x^2-25}}{5} - \sec^{-1}\frac{x}{5} + c$

$= -\sec^{-1}\frac{x}{5} + \sqrt{x^2-25} + c$

c) $\int \frac{9x-2}{2x^2-7x+3}\,dx$

$\int \frac{9x-2}{(2x-1)(x-3)}\,dx$

let $\frac{9x-2}{(2x-1)(x-3)} = \frac{a}{2x-1} + \frac{b}{x-3}$

$9x-2 = a(x-3) + b(2x-1)$

let $x=3$: $25 = 5b$

$5 = b$

let $x = \frac{1}{2}$: $2\frac{1}{2} = -2\frac{1}{2}a$

$-1 = a$

$\therefore \int \frac{9x-2}{2x^2-7x+3}\,dx = \int \frac{5}{x-3} - \frac{1}{2x+1}\,dx$

$= 5\ln(x-3) - \frac{1}{2}\ln(2x+1)$

$= \ln\left(\frac{(x-3)^5}{\sqrt{2x+1}}\right) + c$

d) i) $f(x) = \frac{x}{\sqrt{x^2+16}}$

$f'(x) = \frac{\sqrt{x^2+16} - \frac{x^2}{\sqrt{x^2+16}}}{x^2+16}$

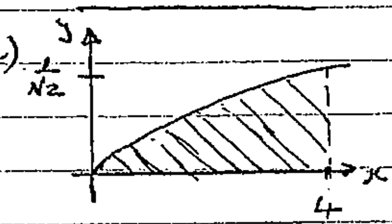
$= \frac{x^2+16-x^2}{\sqrt{(x^2+16)^3}}$

$= \frac{16}{\sqrt{(x^2+16)^3}}$

which is positive for all values of x

$\therefore$ the curve is increasing for all values of x

ii)



V = volume of cylinder $- \pi\int_c^d x^2\,dy$

$= \pi r^2 h - \pi\int_0^{1/\sqrt{2}} \frac{16y^2}{1-y^2}\,dy$

$= \pi\times 4^2\times\frac{1}{\sqrt{2}} -$ $\quad y = \sin\theta$

$\frac{dy}{d\theta} = \cos\theta$

$dy = \cos\theta\,d\theta$

$= 8\pi\sqrt{2} - \pi\int_0^{\pi/4} \frac{16\sin^2\theta}{1-\sin^2\theta}\times\cos\theta\,d\theta$

$= 8\pi\sqrt{2} - \pi\int_0^{\pi/4} \frac{16(1-\cos^2\theta)}{\cos^2\theta}\cos\theta\,d\theta$

$= 8\pi\sqrt{2} - 16\pi\int_0^{\pi/4} \frac{1-\cos^2\theta}{\cos\theta}\,d\theta$

$= 8\pi\sqrt{2} - 16\pi\int_0^{\pi/4} \sec\theta - \cos\theta\,d\theta$

$= 8\pi\sqrt{2} - 16\pi\left[\ln(\sec\theta+\tan\theta) - \sin\theta\right]$

$= 8\pi\sqrt{2} - 16\pi\left[\ln(\sqrt{2}+1) - \frac{1}{\sqrt{2}} - (\ln 1)\right]$

$= 8\pi\left(\sqrt{2} - 2\ln(\sqrt{2}+1) + \sqrt{2}\right]$

$= 8\pi\left(2\sqrt{2} - 2\ln(\sqrt{2}+1)\right)$

$= 16\pi\left(\sqrt{2} - \ln(\sqrt{2}+1)\right)$

Q3) $\frac{x^2}{25} + \frac{y^2}{9} = 1$

i) major axis $= 5$

minor axis $= 3$

ii) $b^2 = a^2(1-e^2)$

$9 = 25(1-e^2)$

$\frac{9}{25} = 1 - e^2$

$e^2 = \frac{16}{25}$

$e = \frac{4}{5}$

iii) foci $(\pm ae, 0)$

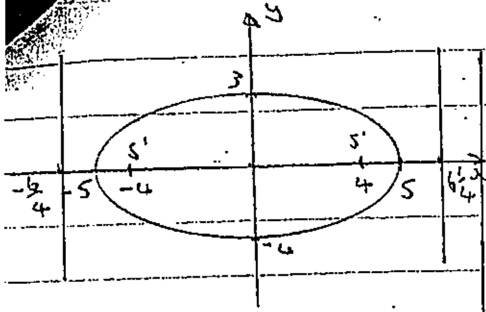
$= (4, 0) \quad (-4, 0)$

iv) $x = \pm\frac{a}{e}$

$= \pm\frac{5}{\frac{4}{5}}$

$x = \pm\frac{25}{4}$

$x = \frac{25}{4}, \quad x = -\frac{25}{4}$



b) i) $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

$\frac{2x}{a^2} - \frac{2y}{b^2}\frac{dy}{dx} = 0$

$\frac{dy}{dx} = \frac{b^2x}{a^2y}$

∴ Slope normal = $\frac{-a^2y}{b^2x}$.

at $(a\sec\theta, b\tan\theta)$

∴ $m = \frac{-a\tan\theta}{b\sec\theta}$

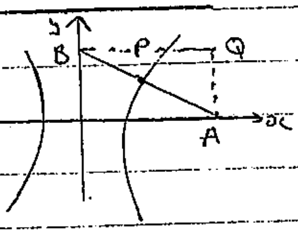
∴ eqn normal

$y - b\tan\theta = \frac{-a\tan\theta}{b\sec\theta}(x - a\sec\theta)$

$by\sec\theta - b^2\tan\theta\sec\theta = -ax\tan\theta + a^2\sec\theta\tan\theta$

$ax\tan\theta + by\sec\theta = \sec\theta\tan\theta(a^2+b^2)$

$\frac{ax}{\sec\theta} + \frac{by}{\tan\theta} = a^2+b^2$

ii) 

for A $(y=0)$

$\frac{ax}{\sec\theta} = a^2+b^2$

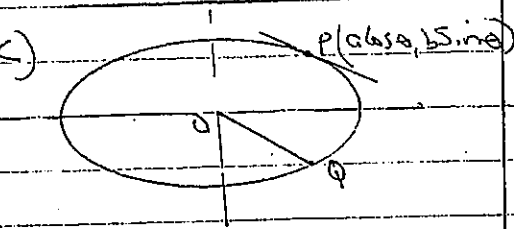
$x = \frac{\sec\theta}{a}(a^2+b^2)$

For B $(x=0)$

$\frac{by}{\tan\theta} = a^2+b^2$

$y = \frac{\tan\theta}{b}(a^2+b^2)$

∴ Q $\left(\frac{\sec\theta}{a}(a^2+b^2), \frac{\tan\theta}{b}(a^2+b^2)\right)$

c) 

$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ ---(1)

$\frac{2x}{a^2}+\frac{2y}{b^2}\frac{dy}{dx}=0$

$\frac{dy}{dx} = \frac{-b^2x}{a^2y}$.

at $(a\cos\theta, b\sin\theta)$

$\frac{dy}{dx} = \frac{-b\cos\theta}{a\sin\theta}$

∴ eqn. OQ

$y = -\frac{b\cos\theta x}{a\sin\theta}$

Sub into (1)

$\frac{x^2}{a^2} + \frac{b^2\cos^2\theta x^2}{b^2a^2\sin^2\theta} = 1$

$\frac{x^2}{a^2} + \frac{x^2\cos^2\theta}{a^2\sin^2\theta} = 1$

$\frac{x^2\sin^2\theta + x^2\cos^2\theta}{a^2\sin^2\theta} = 1$

$x^2(\sin^2\theta+\cos^2\theta) = a^2\sin^2\theta$

$x^2 = a^2\sin^2\theta$

$x = a\sin\theta$

∴ $y = -\frac{b\cos\theta}{a\sin\theta} \times a\sin\theta$

$= -b\cos\theta$

∴ Q $(a\sin\theta, -b\cos\theta)$

∴ OQ $= \sqrt{a^2\sin^2\theta + b^2\cos^2\theta}$

eqn OQ: $b\cos\theta x + a\sin\theta y = 0$

Now perpendicular distance P to OQ

$= \left|\frac{b\cos\theta . a\cos\theta + a\sin\theta b\sin\theta}{\sqrt{b^2\cos^2\theta + a^2\sin^2\theta}}\right|$

$= \frac{ab\cos^2\theta + ab\sin^2\theta}{\sqrt{a^2\sin^2\theta + b^2\cos^2\theta}}$

$= \frac{ab(\cos^2\theta+\sin^2\theta)}{\sqrt{a^2\sin^2\theta+b^2\cos^2\theta}}$

$= \frac{ab}{\sqrt{a^2\sin^2\theta+b^2\cos^2\theta}}$

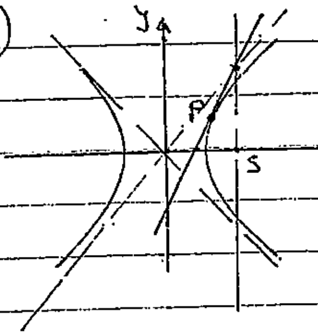
Now area of ΔOPQ

$= \frac{1}{2}bh$

$= \frac{1}{2}\sqrt{a^2\sin^2\theta+b^2\cos^2\theta} \times \frac{ab}{\sqrt{a^2\sin^2\theta+b^2\cos^2\theta}}$

$= \frac{ab}{2}$

∴ Area independent of P

d) 

equation latus rectum

$x = ae$ ---(1)

equation of asymptote

$y = \frac{bx}{a}$ ---(2)

Sub (1) into (2)

$y = \frac{bae}{a}$

$y = be$

∴ pt. of intersection (ae, be)

Now this point lies on the tangent

$\frac{x\sec\theta}{a} - \frac{y\tan\theta}{b} = 1$

$\frac{ae\sec\theta}{a} - \frac{be\tan\theta}{b} = 1$

$e\sec\theta - e\tan\theta = 1$

$e(\sec\theta - \tan\theta) = 1$ ...(A)

Now gradient SP

$m = \frac{b\tan\theta}{a\sec\theta - ae}$

$= \frac{b\tan\theta}{a(\sec\theta - e)}$

from (A) $e = \frac{1}{\sec\theta - \tan\theta}$

$\therefore m = \frac{b\tan\theta}{a\left(\sec\theta - \frac{1}{\sec\theta-\tan\theta}\right)}$

$= \frac{b\tan\theta}{a\left(\frac{\sec^2\theta - \sec\theta\tan\theta - 1}{\sec\theta - \tan\theta}\right)}$

$= \frac{b\tan\theta}{a\left(\frac{\tan^2\theta - \sec\theta\tan\theta}{\sec\theta - \tan\theta}\right)}$

$= \frac{b\tan\theta}{a\left(\frac{\tan\theta(\tan\theta - \sec\theta)}{\sec\theta - \tan\theta}\right)}$

$= \frac{b\tan\theta}{a(-\tan\theta)}$

$= -\frac{b}{a}$

$\therefore$ SP parellel to the other asymptote.

Q4)

a) $x^3 - 2x^2 - 3x + 10$

IF $2+i$ is a root then $2-i$ is a root (real coefficients)

Let the other root be α

$\therefore \alpha + 2 + i + 2 - i = -\frac{b}{a}$

$= 2$

$4 + \alpha = 2$

$\alpha = -2$

$\therefore$ other roots; $2-i$, -2.

b) $P(x) = x^4 + 2x^3 - 12x^2 + 14x - 5$

$P'(x) = 4x^3 + 6x^2 - 24x + 14$

$P''(x) = 12x^2 + 12x - 24$

triple root $\Rightarrow$ root of $P(x)$, $P'(x)$ and $P''(x)$

$P''(x) = 0$

$12x^2 + 12x - 24 = 0$

$x^2 + x - 2 = 0$

$(x+2)(x-1) = 0$

$x = 1, -2.$

Now $P'(1) = 4 + 6 - 24 + 14 = 0$

$\therefore$ 1 root of $P'(x)$

$P(1) = 1 + 2 - 12 + 14 - 5$

$= 0$

$\therefore$ 1 a root of $P(x)$

$\therefore P(x) = (x-1)^3(x+a)$

$\therefore$ roots are 1, 5.

c) $2x^3 - 4x^2 - 3x - 1 = 0$

i) $\Sigma\alpha = -\frac{b}{a}$

$= 2.$

ii) $\Sigma\alpha\beta = \frac{c}{a}$

$= -\frac{3}{2}.$

iii) $\alpha^2 + \beta^2 + \gamma^2 = (\alpha+\beta+\gamma)^2 - 2(\alpha\beta + \alpha\gamma + \beta\gamma)$

$= 2^2 - 2 \times -\frac{3}{2}$

$= 4 + 3$

$= 7$

iv) $\alpha^3 + \beta^3 + \gamma^3$

if α a root then

$2\alpha^3 - 4\alpha^2 - 3\alpha - 1 = 0$ --(1)

if β a root then

$2\beta^3 - 4\beta^2 - 3\beta - 1 = 0$ --(2)

if γ a root then

$2\gamma^3 - 4\gamma^2 - 3\gamma - 1 = 0$ --(3)

(1) + (2) + (3)

$2(\alpha^3+\beta^3+\gamma^3) - 4(\alpha^2+\beta^2+\gamma^2) - 3(\alpha+\beta+\gamma) - 3 = 0$

$2(\alpha^3+\beta^3+\gamma^3) - 4\times 7 - 3\times 2 - 3 = 0$

$2(\alpha^3+\beta^3+\gamma^3) - 28 - 6 - 3 = 0$

$2(\alpha^3+\beta^3+\gamma^3) = 37$

$\alpha^3+\beta^3+\gamma^3 = 18\frac{1}{2}.$

d) $x^3 - x - 1 = 0$

Let root be X s.t $X = \frac{1}{\alpha}$ $\therefore \alpha =$

Now α is a solution of $x^3 - x - 1 =$

ie $\alpha^3 - \alpha - 1 = 0$

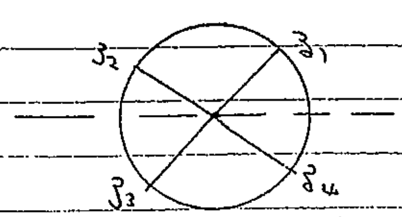
$\left(\frac{1}{X}\right)^3 - \frac{1}{X} - 1 = 0$

$1 - X^2 - X^3 = 0$

e) $z^4 + 1 = 0$

$z^4 = -1$

roots will be equally spaced around the unit circle starting at $\frac{\pi}{4}$



$z_1 = \cos\frac{\pi}{4} + i\sin\frac{\pi}{4} = \frac{1}{\sqrt{2}}(1+i)$

$z_2 = \cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4} = \frac{1}{\sqrt{2}}(-1+i)$

$z_3 = \cos\frac{5\pi}{4} + i\sin\frac{5\pi}{4} = \frac{1}{\sqrt{2}}(-1-i)$

$z_4 = \cos\frac{7\pi}{4} + i\sin\frac{7\pi}{4} = \frac{1}{\sqrt{2}}(1-i$

f) $P(x) = x^4 + x^2 + 1$

$x^6 - 1$

$= (x^2)^3 - 1$

$= (x^2-1)(x^4+x^2+1)$

$= (x+1)(x-1)(x^4+x^2+1)$ --(A)

$\therefore$ zero of $P(x)$ are included in zeros of $x^6 - 1$

Now

$x^6 - 1$

$= (x^3-1)(x^3+1)$

$= (x-1)(x^2+x+1)(x+1)(x^2-x+1)$

$= (x+1)(x-1)(x^2+x+1)(x^2-x+1)$

equating this with (A)

$\Rightarrow x^4+x^2+1 = (x^2+x+1)(x^2-x+1)$

g) $2z^4+3z^3+5z^2+3z+2=0$

$= z^2\left(2z^2+3z+5+\frac{3}{z}+\frac{2}{z^2}\right)=0$

$= z^2\left(2\left(z^2+\frac{1}{z^2}\right)+3\left(z+\frac{1}{z}\right)+5\right)=0$

as $z=0$ is not a solution the solutions come from.

$2\left(z^2+\frac{1}{z^2}\right)+3\left(z+\frac{1}{z}\right)+5=0$

$2\left(z+\frac{1}{z}\right)^2-4+3\left(z+\frac{1}{z}\right)+5=0$

$2\left(z+\frac{1}{z}\right)^2+3\left(z+\frac{1}{z}\right)+1=0$

let $M = z+\frac{1}{z}$

$2M^2+3M+1=0$

$(2M+1)(M+1) = 0$

$M = -\frac{1}{2}$ or $M = -1$

$z+\frac{1}{z} = -\frac{1}{2}$, $z+\frac{1}{z} = -1$

$2z^2+2 = -z$, $z^2+1 = -z$

$2z^2+z+2=0$ $z^2+z+1=0$

$z = \frac{-1\pm\sqrt{1-16}}{4}, \frac{-1\pm\sqrt{1-4}}{2}$

$= \frac{-1\pm i\sqrt{15}}{4}, \frac{-1\pm i\sqrt{3}}{2}$

(h) $(\cos\theta + i\sin\theta)^4$

$= \cos 4\theta + i\sin 4\theta$

also $(\cos\theta + i\sin\theta)^4$

$= \cos^4\theta + 4i\cos^3\theta\sin\theta - 6\cos^2\theta\sin^2\theta$

$- 4i\cos\theta\sin^3\theta + \sin^4\theta$

$= \cos^4\theta - 6\cos^2\theta\sin^2\theta + \sin^4\theta$

$+ i\left(4\cos^3\theta\sin\theta - 4\cos\theta\sin^3\theta\right)$

equating real parts

$\cos 4\theta = \cos^4\theta - 6\cos^2\theta\sin^2\theta + \sin^4\theta$

ii) $\cos 4\theta = \cos^4\theta - 6\cos^2\theta\sin^2\theta + \sin^4\theta$

$= \cos^4\theta - 6\cos^2\theta(1-\cos^2\theta) + (1-\cos^2\theta)^2$

$= \cos^4\theta - 6\cos^2\theta + 6\cos^4\theta + 1$

$- 2\cos^2\theta + \cos^4\theta$

$= 8\cos^4\theta - 8\cos^2\theta + 1$

iii) $8\cos^4 x - 8\cos^2 x + 1 = 0$

$\Rightarrow \cos 4x = 0$

$4x = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \frac{7\pi}{2}$

$x = \frac{\pi}{8}, \frac{3\pi}{8}, \frac{5\pi}{8}, \frac{7\pi}{8}$

for $0 \le x \le \pi$