GOSFORD HIGH SCHOOL

2009
HIGHER SCHOOL CERTIFICATE.
HALF YEARLY EXAMINATION.

MATHEMATICS EXTENSION 2

General Instructions:

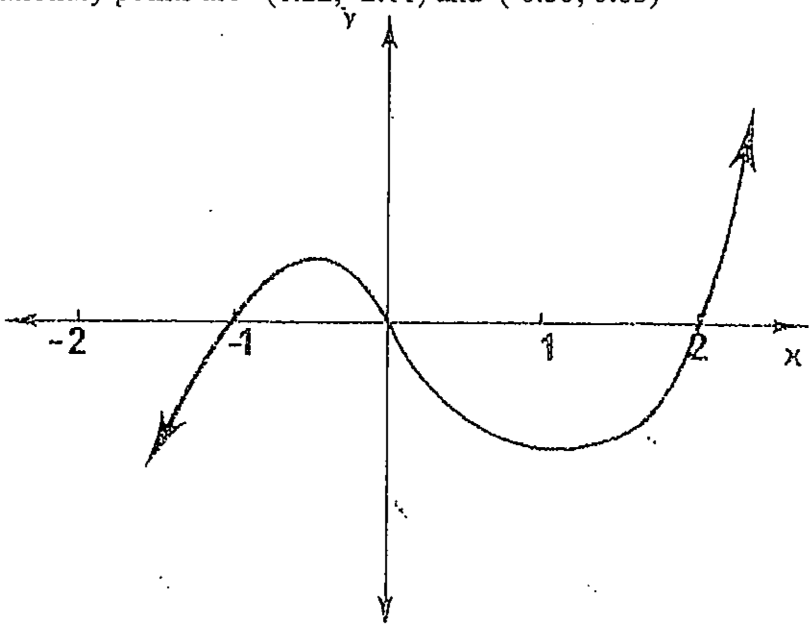
- Reading time – 5minutes.
- Working time – 2hours.
- Write using black or blue pen.
- Board-approved calculators may be used.
- Each question should be started in a new booklet.

Total marks: - 90

- Attempt Questions 1 -4
- All necessary working should be shown.

QUESTION 1: (20 marks)

a. The graph of $f(x) = x^3 - x^2 - 2x$ is given below. Note that the approximate coordinates of the stationary points are (1.22, -2.11) and (-0.55, 0.63)



Draw separate one-third page sketches of each of the following:

i. $y = -f(x)$ (1)

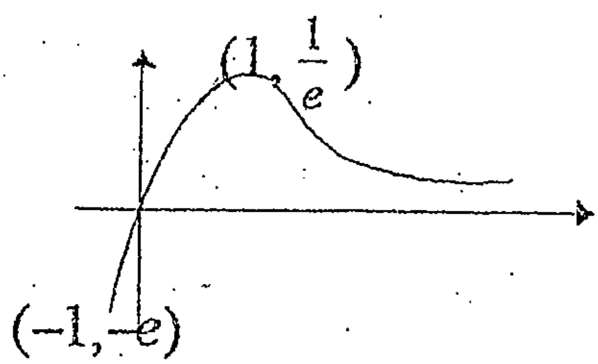
ii. $y = |f(x)|$ (1)

iii. $y = [f(x)]^2$ (2)

iv. $y = \dfrac{1}{f(x)}$ (2)

v. $y = \sqrt{f(x)}$ (2)

b. i. The graph of $g(x) = xe^{-x}$ for $x \geq -1$ is given below:



Draw separate sketches showing $g(x - 2)$ and $g(- x)$ (2)

(Question 1 continued)

ii. The function $f(x)$ is given by

$$f(x) = g(x-2),\ x \geq 1$$
$$= g(-x)\ ,\ x<1$$

Draw a neat sketch of $y = f(x)$ showing all important features. (2)

c. i. Show that if $f(x) = \dfrac{2x^2-1}{x^2-1}, x \neq \pm 1,$ then $f(x)$ is an even function. (1)

ii. Sketch the curve $y = \dfrac{2x^2-1}{x^2-1}, \quad x \neq \pm 1,$ showing the location and nature of all stationary points, the equations of all asymptotes and any intercepts with the co-ordinate axes. (7)

Question 2: (Start a new booklet) (20 marks)

a. i. For the complex number $z = 1 - \sqrt{3}i$ find:

$|z|$ and $\arg(z)$ (1)

ii. Express $\bar{z}, z^2$ and $\frac{1}{z}$ in the form $a+ib$ where a, b are real numbers (3)

iii. Plot $z, \bar{z}, z^2$ *and* $\frac{1}{z}$ on an Argand Diagram (1)

b. Solve $z^2 + 4z - 1 + 12i = 0$ (3)

c. Sketch on separate Argand Diagrams each of the following regions:

i. $2 < |z| \leq 3$ and $\frac{\pi}{4} \leq \arg(z) \leq \frac{3\pi}{4}$ (2)

ii. $|z - 2i| > |z - 2 + i|$ (2)

iii. $z\bar{z} \leq z + \bar{z}$ (2)

d. Use De Moivre's Theorem to express $\cos 5\theta$ *and* $\sin 5\theta$ in powers of $\sin\theta$ *and* $\cos\theta$ Hence express $\tan 5\theta$ as a rational function of t where $t = \tan\theta$. Deduce that:

$$\tan\frac{\pi}{5} \quad \tan\frac{2\pi}{5} \quad \tan\frac{3\pi}{5} \quad \tan\frac{4\pi}{5} \quad = \quad 5$$ (6)

Question 3: (Start a new booklet) (25 marks)

a. Without evaluating the integral explain why

$$\int_{-1}^{1} xe^{-x^2}\,dx = 0 \qquad (1)$$

b. i. Find $\displaystyle\int \frac{\cos^3 x}{\sin^2 x}\,dx$ (3)

ii. Evaluate $\displaystyle\int_{-1}^{2} x\sqrt{2-x}\,dx$ (3)

c. Use the substitution $u = sin2x$ or otherwise to evaluate

$$\int_0^{\pi/4} \frac{\sin 4x}{1+\sin^2 2x}\,dx \qquad (3)$$

d. Show that $\displaystyle\int_0^{\pi/4} x\sec^2 x\,dx = \frac{1}{2}\left(\frac{\pi}{2} - \ln 2\right)$ (4)

e. i. Use the method of partial fractions to find real numbers A, B and C such that

$$\frac{x}{(x-1)^2(x-2)} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{x-2}$$

(2)

ii. Hence show that

$$\int_0^{1/2} \left(\frac{x}{(x-1)^2(x-2)}\right) dx = 2\log_e\left(\frac{3}{2}\right) - 1 \qquad (3)$$

f. i. Given that $I_{2n+1} = \displaystyle\int_0^1 x^{2n+1} e^{x^2}\,dx$

show that $I_{2n+1} = \dfrac{1}{2}e - nI_{2n-1}$ (3)

ii. Hence or otherwise evaluate

$$\int_0^1 x^5 e^{x^2}\,dx \qquad (3)$$

Question 4: (Start a new booklet) (25 marks)

a. i. Show that $x-3$ is a factor of the polynomial

$$Q(x) = 4x^3 - 15x^2 + 8x + 3 \qquad (1)$$

ii. Given that the equation $x^4 - 5x^3 + 4x^2 + 3x + 9 = 0$ has a root of multiplicity 2, solve the equation completely. (6)

b. Show that $1+i$ is a zero of the polynomial $P(x) = x^3 - x^2 + 2$ and hence resolve $P(x)$ into irreducible factors over the field of

i. real numbers

ii. complex numbers (5)

c. When a polynomial $P(x)$ is divided by $(x-2)$ and $(x-3)$ the respective remainders are 4 and 9. Find the remainder when $P(x)$ is divided by $(x-2)(x-3)$ (4)

d. The equation $2x^3 - 3x - 1 = 0$ has three non zero roots α, β and γ. Evaluate

i. $\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma}$ (1)

ii. $\alpha^4 + \beta^4 + \gamma^4$ (4)

e. If one root of the equation $x^3 - px^2 + qx - r = 0$ is equal to the product of the other two roots prove that:

$$r(p+1)^2 = (q+r)^2 \qquad (4)$$

STANDARD INTEGRALS

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x}\,dx = \ln x, \quad x > 0$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax}, \quad a \neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax, \quad a \neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax, \quad a \neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax\,dx = \frac{1}{a}\sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a}, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \ln\left(x+\sqrt{x^2-a^2}\right), \quad x > a > 0$$

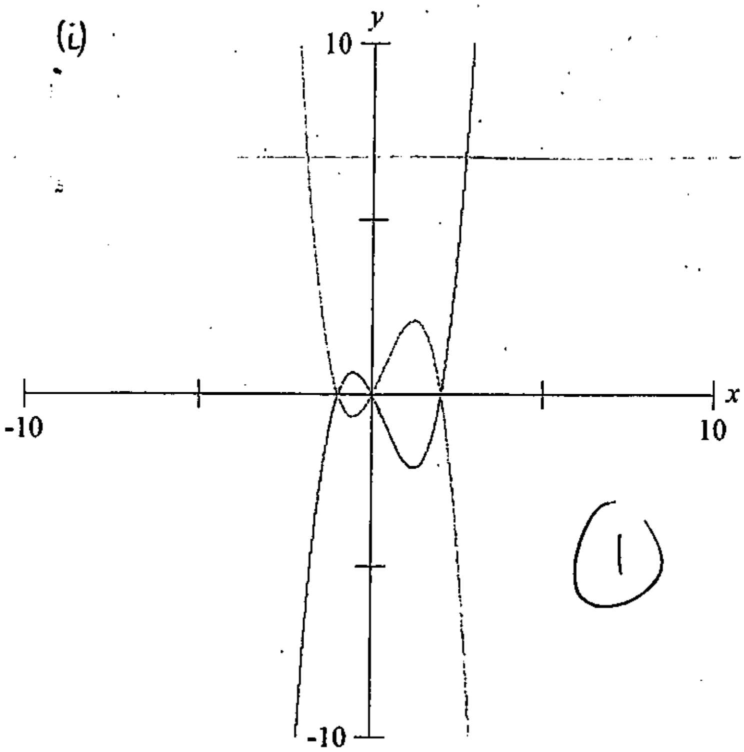
$$\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \ln\left(x+\sqrt{x^2+a^2}\right)$$

NOTE : $\ln x = \log_e x, \quad x > 0$

1 . a)

Mx2 H/Y

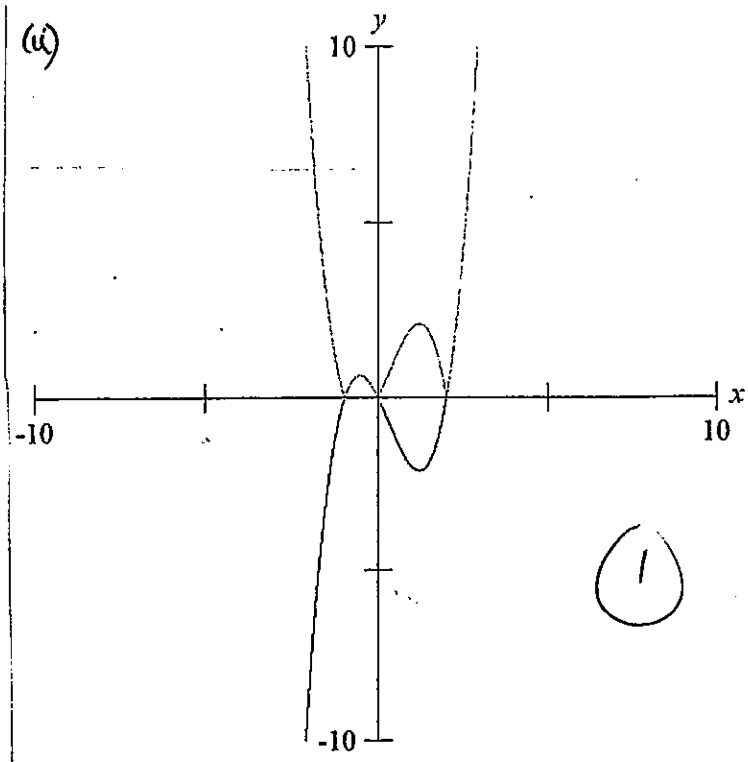
(i)



1

$y=x^3-x^2-2\cdot x$ $y=-(x^3-x^2-2\cdot x)$

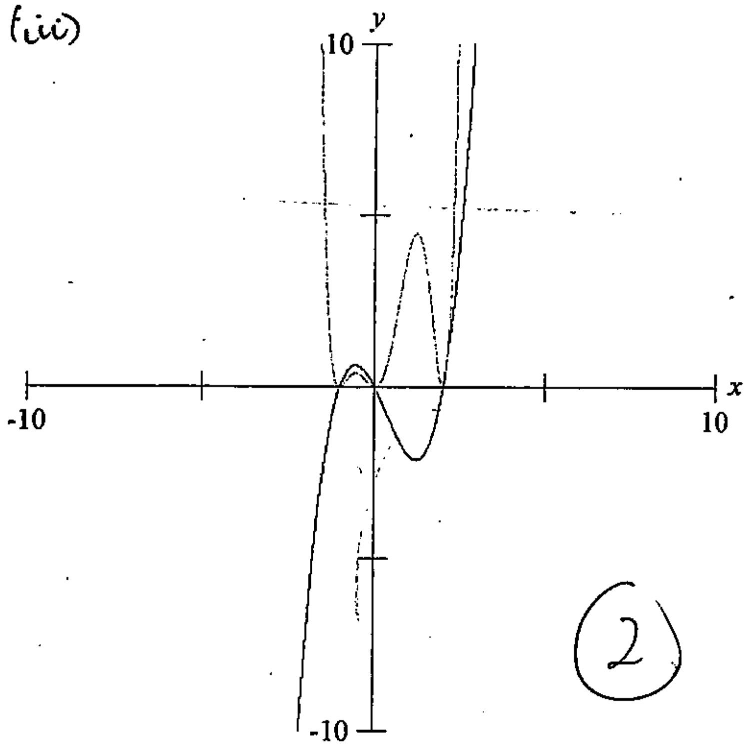
(ii)



1

$y=x^3-x^2-2\cdot x$ $y=|x^3-x^2-2\cdot x|$

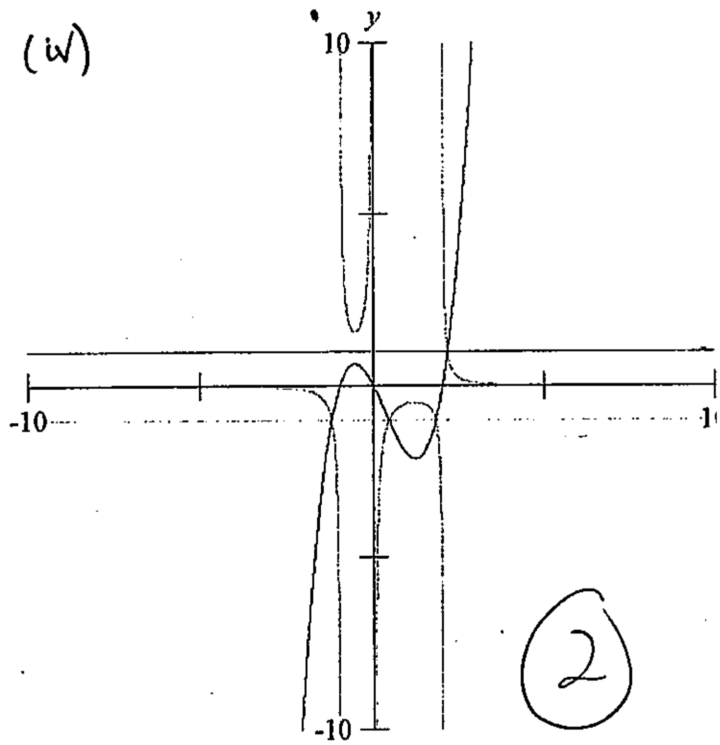
(iii)



2

$y=x^3-x^2-2\cdot x$ $y=(x^3-x^2-2\cdot x)^2$

(iv)



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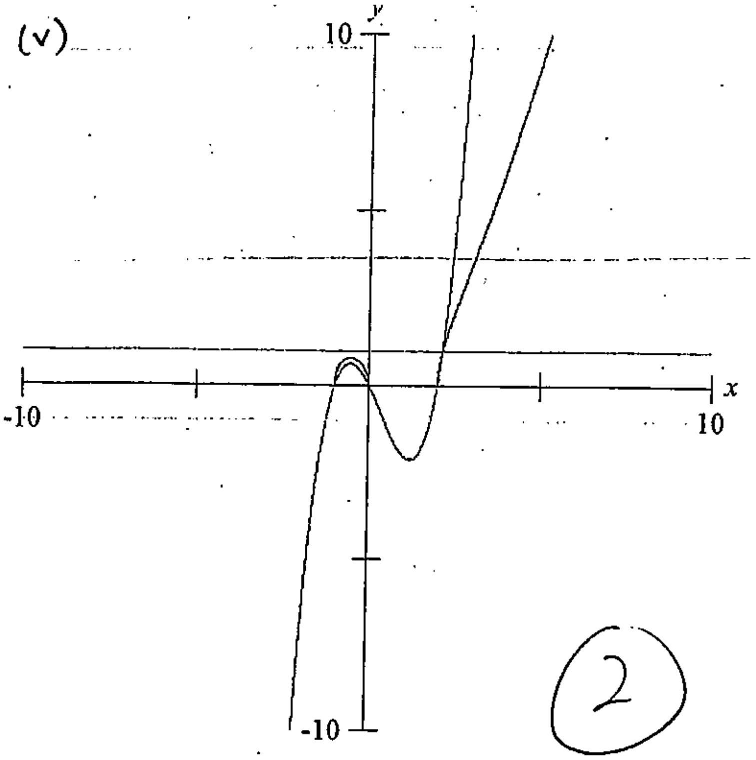
$y=x^3-x^2-2\cdot x$ $y=\frac{1}{(x^3-x^2-2\cdot x)}$ $y=1$ $y=-1$

(v)



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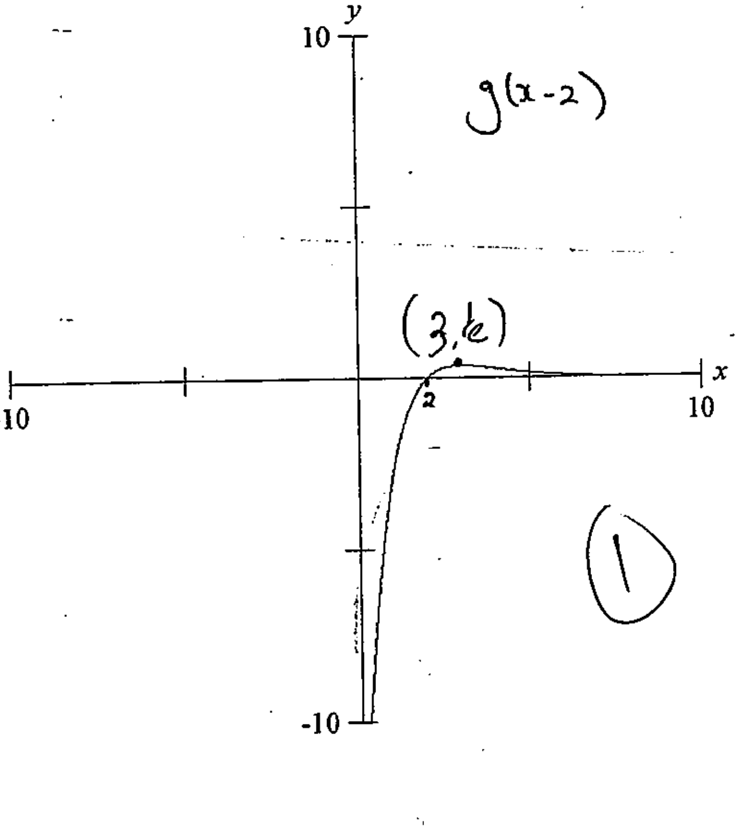
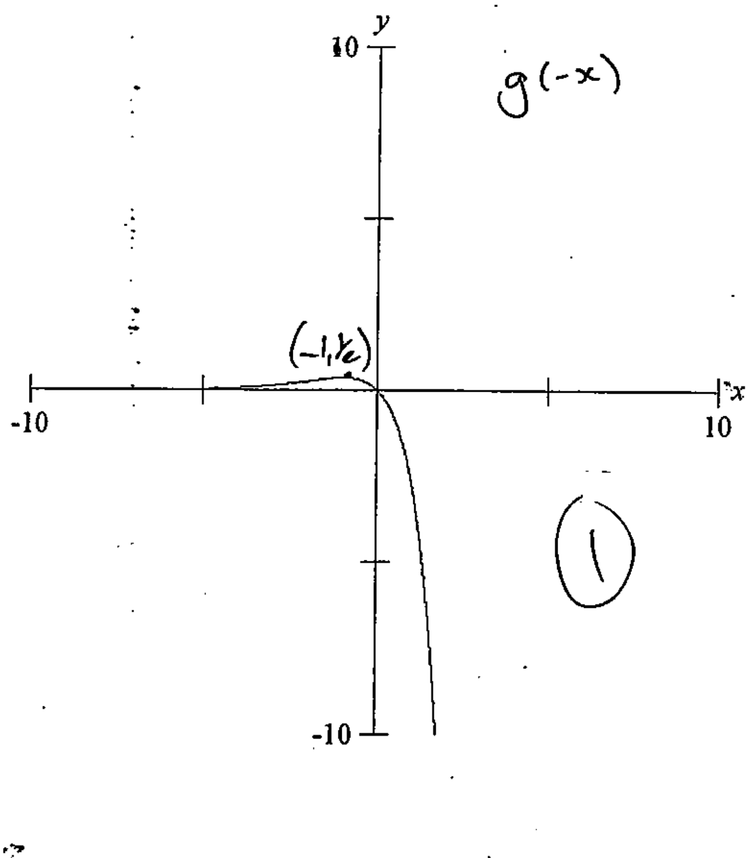
$y=x^3-x^2-2\cdot x$ $\quad y=\sqrt{(x^3-x^2-2\cdot x)}$ $\quad y=1$ $\quad y=-1$

b) i)



1

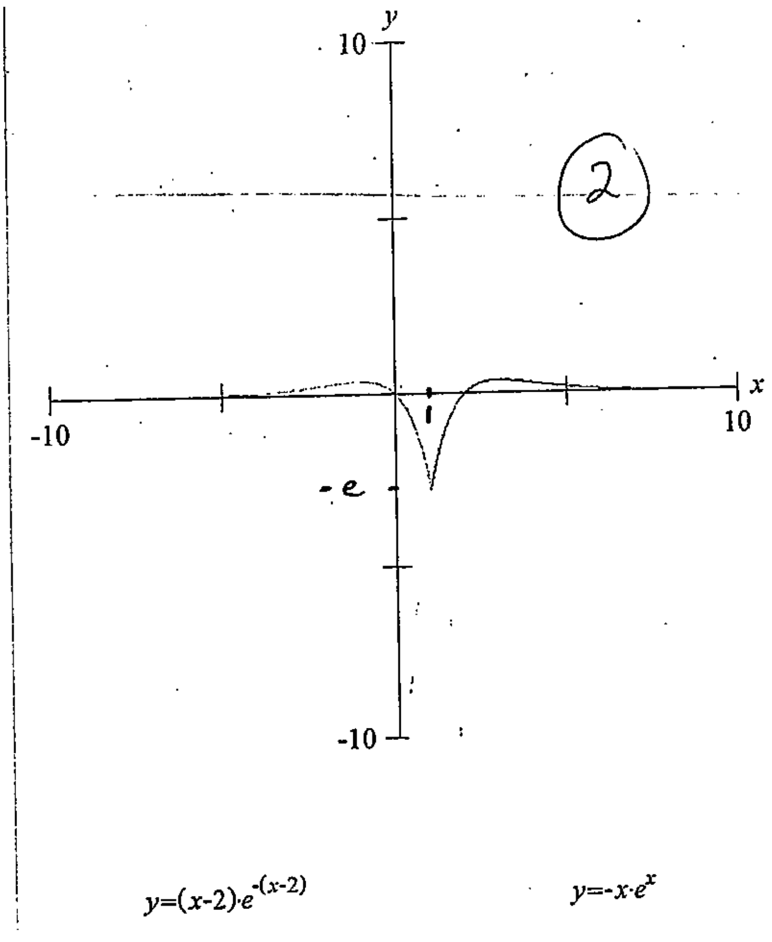
$y=(x-2)\cdot e^{-(x-2)}$



1

$y=-x\cdot e^{x}$

(ii)



2

c) (i) $f(x) = \dfrac{2x^2-1}{x^2-1}, \quad x \neq \pm 1$

$f(-x) = \dfrac{2(-x)^2-1}{(-x)^2-1}$

$= \dfrac{2x^2-1}{x^2-1}$

1

$\therefore f(x) = f(-x)$

& hence $f(x)$ is an even function.

ii) If $y = \dfrac{2x^2-1}{x^2-1}$

$= \dfrac{2x^2-2}{x^2-1} + \dfrac{1}{x^2-1}$

$= 2 + \dfrac{1}{x^2-1}$

2

$\therefore$ Horizontal asymptote at $y=2$

Vertical asymptotes at $x = \pm 1$

If $y = 2 + (x^2-1)^{-1}$

$y' = -1(x^2-1)^{-2} \cdot 2x$

$= \dfrac{-2x}{(x^2-1)^2}$

When $y' = 0$, $-2x = 0$

$x = 0$

$\therefore y = 1$.

x	$0-\varepsilon$	0	$0+\varepsilon$
y'	$+$	0	$-$

2

$(0,1)$ is a max t.p.

Also when $y=0$, $\dfrac{2x^2-1}{x^2-1} = 0$

$\therefore x^2 = \frac{1}{2}$

$x = \pm \frac{1}{\sqrt{2}}$

1

As $x \to 1^+$, $y \to \infty$

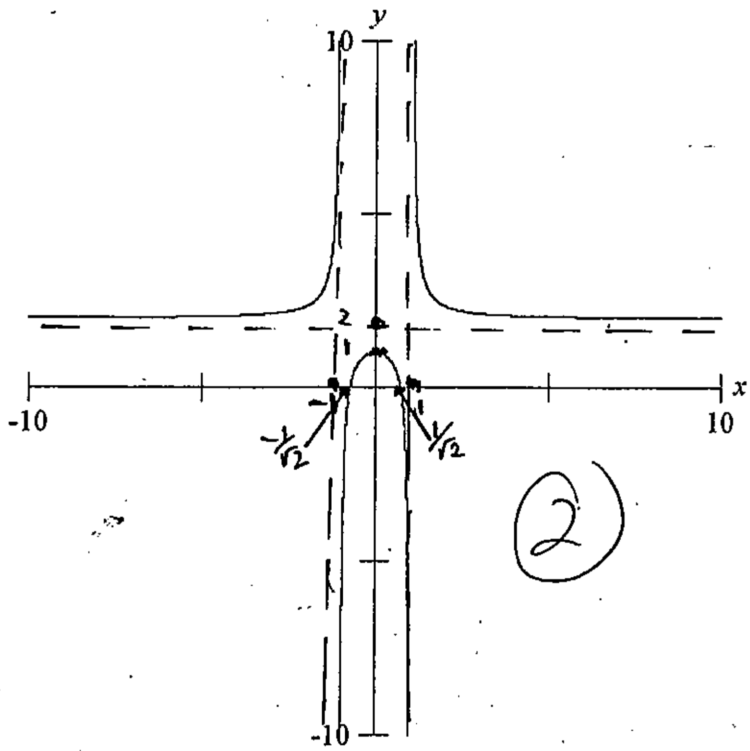
As $x \to 1^-$, $y \to -\infty$

As $x \to -1^+$, $y \to -\infty$

As $x \to -1^-$, $y \to \infty$

As $x \to \infty$, $y \to 2^+$

As $x \to -\infty$, $y \to 2^+$

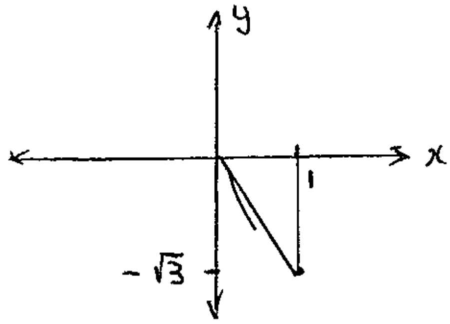


2

2/ a)

(i) $|z| = \sqrt{1^2 + (-\sqrt{3})^2}$

$= 2.$ (1)



$\text{Arg}(z) = -\frac{\pi}{3}$

(ii) $\overline{z} = 1 + \sqrt{3}i$ (1)

$z^2 = (1 - \sqrt{3}i)^2$

$= 1 - 2\sqrt{3}i + 3i^2$

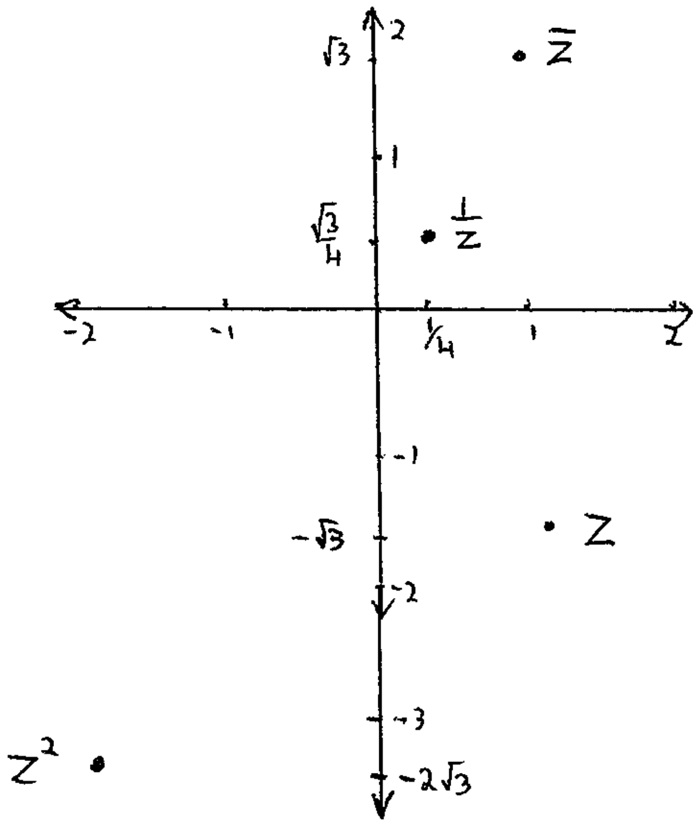
$= -2 - 2\sqrt{3}i$ (1)

$\frac{1}{z} = \frac{1}{1-\sqrt{3}i} \times \frac{1+\sqrt{3}i}{1+\sqrt{3}i}$

$= \frac{1+\sqrt{3}i}{1-(\sqrt{3}i)^2}$

$= \frac{1+\sqrt{3}i}{4}$ (1)

(iii)



(1)

b) $z^2 + 4z - 1 + 12i = 0$

$$Z = \frac{-4 \pm \sqrt{16 - 4(-1+12i)}}{2}$$

$$= \frac{-4 \pm \sqrt{20 - 48i}}{2}$$

$$= -2 \pm \sqrt{5 - 12i}$$

Let $\sqrt{5-12i} = a + ib$

$a^2 - b^2 = 5$ & $2ab = -12$

$ab = -6$ (3)

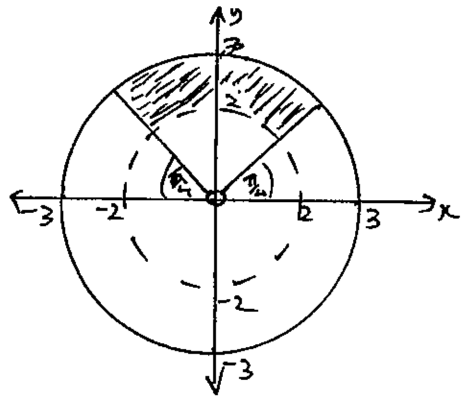
By inspection $a = \pm 3$, $b = \mp 2$.

$\therefore \sqrt{5 - 12i} = \pm(3 - 2i)$

$\therefore Z = -2 \pm (3 - 2i)$

$= 1 - 2i$ or $-5 + 2i$

c) (i)



(2)

(ii) $|z - 2i| > |z - 2 + i|$

Let $Z = x + iy$

$|x + iy - 2i| = |x + i(y-2)|$

$= \sqrt{x^2 + (y-2)^2}$

$$|x+iy-2+i| = |(x-2)+i(y+1)|$$
$$= \sqrt{(x-2)^2+(y+1)^2}$$

$$\therefore\ x^2+(y-2)^2 > (x-2)^2+(y+1)^2$$

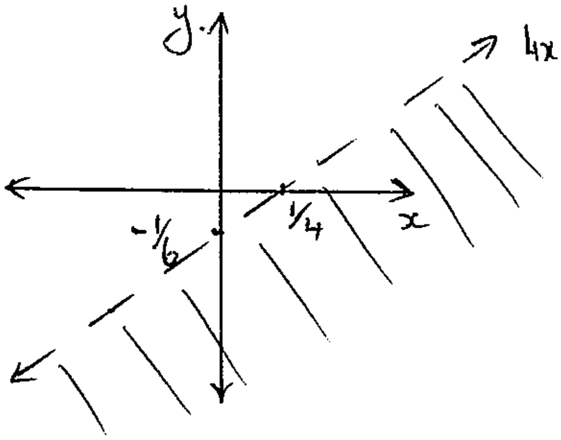
$$x^2+y^2-4y+\cancel{4} > x^2-4x+\cancel{4}+y^2+2y+1.$$

$$\therefore\ 4x-6y-1 > 0$$

If $4x-6y-1=0$

$$6y = 4x-1$$
$$y = \frac{2}{3}x-\frac{1}{6}.$$

2



Test $(0,0)$

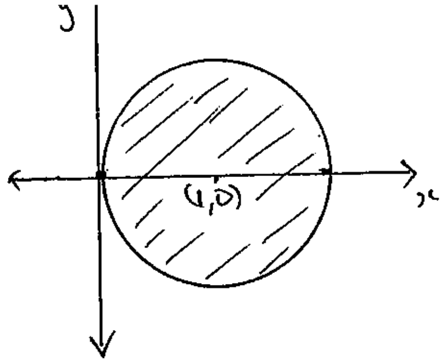
$$0-0-1 \not> 0.$$

(iii) $z\bar{z} \le z+\bar{z}$

Let $z = x+iy$

$$(x+iy)(x-iy) \le x+iy+x-iy.$$
$$x^2+y^2 \le 2x$$
$$x^2-2x+y^2 \le 0$$
$$x^2-2x+1+y^2 \le 1$$
$$(x-1)^2+y^2 \le 1$$

Circle centre $(1,0)$ & radius 1.



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d) $(\cos\theta + i\sin\theta)^5 = \cos 5\theta + i\sin 5\theta$

$$\text{LHS} = \cos^5\theta + 5\cos^4\theta\, i\sin\theta + 10\cos^3\theta\, i^2\sin^2\theta + 10\cos^2\theta\, i^3\sin^3\theta + 5\cos\theta\, i^4\sin^4\theta + i^5\sin^5\theta$$

$$= \cos^5\theta + 5\cos^4\theta\sin\theta\, i - 10\cos^3\theta\sin^2\theta - 10\cos^2\theta\sin^3\theta\, i + 5\cos\theta\sin^4\theta + \sin^5\theta\, i$$

Equating real and imaginary parts

$$\cos 5\theta = \cos^5\theta - 10\cos^3\theta\sin^2\theta + 5\cos\theta\sin^4\theta$$

& $\sin 5\theta = 5\cos^4\theta\sin\theta - 10\cos^2\theta\sin^3\theta + \sin^5\theta.$

$$\therefore \tan 5\theta = \frac{5\cos^4\theta\sin\theta - 10\cos^2\theta\sin^3\theta + \sin^5\theta}{\cos^5\theta - 10\cos^3\theta\sin^2\theta + 5\cos\theta\sin^4\theta}$$

Divide throughout by $\cos^5\theta$

$$\tan 5\theta = \frac{5\dfrac{\sin\theta}{\cos\theta} - 10\dfrac{\sin^3\theta}{\cos^3\theta} + \dfrac{\sin^5\theta}{\cos^5\theta}}{1 - 10\dfrac{\sin^2\theta}{\cos^2\theta} + 5\dfrac{\sin^4\theta}{\cos^4\theta}}$$

$$= \frac{5t - 10t^3 + t^5}{1 - 10t^2 + 5t^4} \quad \text{where } t = \tan\theta$$

If $\tan 5\theta = 0$

$$\frac{5t - 10t^3 + t^5}{1 - 10t^2 + 5t^4} = 0$$

$\therefore \quad 5t - 10t^3 + t^5 = 0$

ie $\quad t^5 - 10t^3 + 5t = 0$

Now if $\tan 5\theta = 0$, the "first" 5 sol^{ns} are

$\tan 0, \ \tan \frac{\pi}{5}, \ \tan \frac{2\pi}{5}, \ \tan \frac{3\pi}{5} \ \& \ \tan \frac{4\pi}{5}$

$\therefore \quad t = 0, \ \tan \frac{\pi}{5}, \ \tan \frac{2\pi}{5}, \ \tan \frac{3\pi}{5}, \ \tan \frac{4\pi}{5}.$

Also if $t^5 - 10t^3 + 5t = 0$

$t(t^4 - 10t^2 + 5) = 0$

$\therefore t = 0$ & the roots of $t^4 - 10t^2 + 5 = 0$

are $\tan \frac{\pi}{5}, \ \tan \frac{2\pi}{5}, \ \tan \frac{3\pi}{5}, \ \tan \frac{4\pi}{5}$

Now the product of the roots ie $\frac{e}{a}$

$\therefore \tan \frac{\pi}{5} \cdot \tan \frac{2\pi}{5} \cdot \tan \frac{3\pi}{5} \cdot \tan \frac{4\pi}{5} = \frac{5}{1}$

ie $\tan \frac{\pi}{5} \cdot \tan \frac{2\pi}{5} \cdot \tan \frac{3\pi}{5} \cdot \tan \frac{4\pi}{5} = 5.$

6

3) a) Let $f(x) = xe^{-x^2}$

$f(-x) = -xe^{-(-x)^2}$

$= -xe^{-x^2}$

$= -f(x)$

$\therefore f(x)$ is odd. 1

Hence $\int_{-1}^{1} xe^{-x^2}\,dx = 0.$

b) (i) Let $I = \int \frac{\cos^3 x}{\sin^2 x}\,dx$

Let $u = \sin x$

$du = \cos x\,dx$

$I = \int \frac{\cos^2 x}{\sin^2 x} \cdot \cos x\,dx$

$= \int \frac{1-\sin^2 x}{\sin^2 x} \cdot \cos x\,dx$

$= \int \frac{1-u^2}{u^2}\,du$

$= \int u^{-2} - 1\,du$ 3

$= \frac{u^{-1}}{-1} - u + C$

$= \frac{-1}{\sin x} - \sin x + C.$

ii) Let $I = \int_{-1}^{2} x\sqrt{2-x}\,dx$

Let $u^2 = 2 - x$

$2u\,du = -dx$

If $x = -1$, $u = \sqrt{3}$

$x = 2$, $u = 0$.

$$I = \int_{\sqrt{3}}^{0} (2-u^2) \cdot u \cdot -2u \, du$$

$$= -2 \int_{\sqrt{3}}^{0} 2u^2 - u^4 \, du$$

$$= -2 \left[\frac{2u^3}{3} - \frac{u^5}{5} \right]_{\sqrt{3}}^{0}$$

$$= -2 \left\{ 0 - \left(2 \times \frac{\cancel{3}\sqrt{3}}{\cancel{3}} - \frac{9\sqrt{3}}{5} \right) \right\}$$

3

$$= -2 \left(-2\sqrt{3} + \frac{9\sqrt{3}}{5} \right)$$

$$= -2 \left(\frac{-10\sqrt{3} + 9\sqrt{3}}{5} \right)$$

$$= \frac{2\sqrt{3}}{5}.$$

c) Let $I = \int_0^{\pi/4} \frac{\sin 4x}{1 + \sin^2 2x} \, dx$

Let $u = \sin 2x$ If $x = 0$, $u = 0$

$du = 2 \cos 2x \, dx$ $x = \frac{\pi}{4}$, $u = 1$.

$$I = \int_0^{\pi/4} \frac{2 \sin 2x \cos 2x}{1 + \sin^2 2x} \, dx$$

$$= \int_0^1 \frac{u}{1+u^2} \, du$$

3

$$= \frac{1}{2} \left[\ln (1+u^2) \right]_0^1$$

$$= \frac{1}{2} \ln 2 - \frac{1}{2} \ln 1$$

$$= \frac{1}{2} \ln 2 \quad \underline{\text{OR}} \quad \ln \sqrt{2}.$$

d) Let $I = \int_0^{\pi/4} x \sec^2 x \, dx$

$u = x, \quad v' = \sec^2 x$

$u' = 1, \quad v = \tan x$

$$I = \left[x \tan x \right]_0^{\pi/4} - \int_0^{\pi/4} 1 . \tan x \, dx$$

$$= \left(\frac{\pi}{4} . 1 - 0 \right) - \int_0^{\pi/4} \frac{\sin x}{\cos x} \, dx$$

$$= \frac{\pi}{4} + \int_0^{\pi/4} \frac{-\sin x}{\cos x} \, dx$$

$$= \frac{\pi}{4} + \left[\ln (\cos x) \right]_0^{\pi/4}$$

$$= \frac{\pi}{4} + \left(\ln \frac{1}{\sqrt{2}} - \ln 1 \right)$$

4

$$= \frac{\pi}{4} + \ln (2)^{-\frac{1}{2}} - 0 .$$

$$= \frac{\pi}{4} - \frac{1}{2} \ln 2 .$$

$$= \frac{1}{2} \left(\frac{\pi}{2} - \ln 2 \right)$$

e) (i) If $\frac{x}{(x-1)^2 (x-2)} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{x-2}$

$$x = A(x-1)(x-2) + B(x-2) + C(x-1)^2$$

Let $x = 1$, $\quad 1 = 0 - B + 0$

$\therefore \quad B = -1 .$

Let $x=2$, $\quad 2 = 0 + 0 + C$

$C = 2$

Let $x = 0$, $\quad 0 = 2A - 2B + C$

$0 = 2A + 2 + 2$

$A = -2.$

2

(ii) Let $I = \int_0^{\frac{1}{2}} \frac{x}{(x-1)^2(x-2)}\,dx$

$= \int_0^{\frac{1}{2}} \frac{-2}{x-1} - \frac{1}{(x-1)^2} + \frac{2}{x-2}\,dx$

$= \int_0^{\frac{1}{2}} \frac{-2}{x-1} - (x-1)^{-2} + \frac{2}{x-2}\,dx$

$= \left[-2\ln(x-1) - \frac{(x-1)^{-1}}{-1\times 1} + 2\ln(x-2)\right]_0^{\frac{1}{2}}$

$= \left[2\ln\left[\frac{x-2}{x-1}\right] + \frac{1}{x-1}\right]_0^{\frac{1}{2}}$

$= \left(2\ln\left(\frac{-1\frac{1}{2}}{-\frac{1}{2}}\right) + \frac{1}{-\frac{1}{2}}\right) - \left(2\ln\left(\frac{-2}{-1}\right) + \frac{1}{-1}\right)$

$= 2\ln 3 - 2 - 2\ln 2 + 1.$

$= 2\ln\left(\frac{3}{2}\right) - 1.$

3

f) $I_{2n+1} = \int_0^1 x^{2n+1} e^{x^2} dx$

$u = x^{2n}$, $v' = x e^{x^2}$

$u' = 2n x^{2n-1}$, $v = \frac{1}{2} e^{x^2}$

$I_{2n+1} = \left[\frac{1}{2} x^{2n} e^{x^2}\right]_0^1 - \int_0^1 \frac{1}{2} e^{x^2} . 2n x^{2n-1} dx$

$= \left(\frac{1}{2} . 1 . e - \frac{1}{2} . 0 . 1\right) - \int_0^1 n x^{2n-1} e^{x^2} dx$

$= \frac{e}{2} - n I_{2n-1}.$ 3

ii) N.B : If $2n+1 = 5$, $n = 2.$

$\therefore I_5 = \frac{e}{2} - 2 I_3 .$

If $2n+1 = 3$, $n = 1$

$I_5 = \frac{e}{2} - 2\left[\frac{e}{2} - 1 . I_1\right]$

If $2n+1 = 1$, $n = 0$

$\therefore I_5 = \frac{e}{2} - e + 2\int_0^1 x e^{x^2} dx$

$= -\frac{e}{2} + \cancel{2} \left[\frac{1}{\cancel{2}} e^{x^2}\right]_0^1$

$= -\frac{e}{2} + (e - 1)$

$= \frac{e}{2} - 1.$ 3

4. a)

(i) $Q(3) = 4\times(3)^3 - 15\times(3)^2 + 8\times(3) + 3$

$= 4\times 27 - 15\times 9 + 8\times 3 + 3$

$= 0.$

1

$\therefore$ $x-3$ is a factor of $Q(x)$.

(ii) Let $P(x) = x^4 - 5x^3 + 4x^2 + 3x + 9$

$P'(x) = 4x^3 - 15x^2 + 8x + 3.$

From (i) $P'(3) = 0$

Also $P(3) = (3)^4 - 5\times(3)^3 + 4\times(3)^2 + 3\times(3) + 9.$

$= 81 - 5\times 27 + 4\times 9 + 3\times 3 + 9$

$= 0.$

$\therefore$ $x-3$ is a double root of $P(x)$

$$\begin{array}{r|l} & x^2 + x + 1 \\ \hline x^2-6x+9 & x^4 - 5x^3 + 4x^2 + 3x + 9 \\ & \underline{x^4 - 6x^3 + 9x^2} \\ & \quad x^3 - 5x^2 + 3x \\ & \quad \underline{x^3 - 6x^2 + 9x} \\ & \qquad x^2 - 6x + 9 \\ & \qquad \underline{x^2 - 6x + 9} \\ & \qquad\qquad 0 \end{array}$$

6

$\therefore P(x) = (x-3)^2(x^2+x+1)$

If $(x-3)^2(x^2+x+1) = 0$

$(x-3)^2\left(x^2+x+\frac{1}{4}+\frac{3}{4}\right) = 0$

$(x-3)^2\left[\left(x+\frac{1}{2}\right)^2 - \frac{3}{4}i^2\right] = 0$

$(x-3)^2\left\{\left(x+\frac{1}{2}+\frac{\sqrt{3}}{2}i\right)\left(x+\frac{1}{2}-\frac{\sqrt{3}}{2}i\right)\right\} = 0$

$\therefore$ $x = 3,\ -\frac{1}{2}-\frac{\sqrt{3}}{2}i,\ -\frac{1}{2}+\frac{\sqrt{3}}{2}i$

b) $P(1+i) = (1+i)^3 - (1+i)^2 + 2$

$= 1 + 3i + 3i^2 + i^3 - (1 + 2i + i^2) + 2$

$= 1 + 3i - 3 - i - 1 - 2i + 1 + 2.$

$= 0$

$\therefore (1+i)$ is a zero of $P(x)$

Now if $(1+i)$ is a zero so is $(1-i)$

$[x - (1+i)][x - (1-i)] = x^2 - (1-i)x - (1+i)x + (1+i)(1-i)$

$= x^2 - x + \cancel{ix} - x - \cancel{ix} + 1 - i^2$

$= x^2 - 2x + 2$

$$\begin{array}{r|l} & x + 1. \\ \hline x^2 - 2x + 2 & x^3 - x^2 + 0x + 2 \\ & \underline{x^3 - 2x^2 + 2x} \\ & \quad x^2 - 2x + 2 \end{array}$$

5

(i) $\therefore P(x) = (x^2 - 2x + 2)(x + 1)$

(ii) $P(x) = [x - (1+i)][x - (1-i)](x+1)$

$= (x - 1 - i)(x - 1 + i)(x + 1)$

c) $(x-2)(x-3) = x^2 - 5x + 6$

$\therefore P(x) = (x^2 - 5x + 6) \cdot Q(x) + R(x)$

where the degree of $R(x)$ is less than the degree of $x^2 - 5x + 6$.

$\therefore R(x)$ is of the form $ax + b$.

Now $P(2) = 4$.

$\therefore 4 = 0 \cdot Q(x) + 2a + b$

ie $2a + b = 4$ — (1)

~~c~~ $P(3) = 9.$

$\therefore 9 = 0.\,Q(x) + 3a + b$

$3a + b = 9 \quad — ②$

4

② $-$ ①

$a = 5$

$\therefore b = -6$

Hence the remainder is $5x - 6$

d) $2x^3 - 3x - 1 = 0$

$\therefore 2x^3 + 0x^2 - 3x - 1 = 0$

(i) $\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} = \frac{\beta\gamma + \alpha\gamma + \alpha\beta}{\alpha\beta\gamma}$

$= \frac{\frac{c}{a}}{-\frac{d}{a}}$

1

$= \frac{-\frac{3}{2}}{\frac{1}{2}}$

$= -3$

(ii) If $2x^3 - 3x - 1 = 0$

$2x^3 = 3x + 1.$

If α, β, γ are zeros of $P(x)$

$2\alpha^3 = 3\alpha + 1$

$2\beta^3 = 3\beta + 1$

$2\gamma^3 = 3\gamma + 1$

Multiply by α, β & γ respectively

$$2\alpha^4 = 3\alpha^2 + \alpha \quad — (1)$$

$$2\beta^4 = 3\beta^2 + \beta \quad — (2)$$

$$2\gamma^4 = 3\gamma^2 + \gamma \quad — (3)$$

$(1) + (2) + (3)$

$$2(\alpha^4 + \beta^4 + \gamma^4) = 3(\alpha^2 + \beta^2 + \gamma^2) + \alpha + \beta + \gamma$$

$$= 3\left[(\alpha+\beta+\gamma)^2 - 2(\alpha\beta + \alpha\gamma + \beta\gamma)\right] + \alpha + \beta + \gamma$$

$$= 3\left[\left(-\frac{b}{a}\right)^2 - 2\left(\frac{c}{a}\right)\right] + -\frac{b}{a}.$$

$$= 3\left[0^2 - 2 \times -\frac{3}{2}\right] + 0$$

$$= 9.$$

4

$$\therefore \alpha^4 + \beta^4 + \gamma^4 = \frac{9}{2}.$$

e) Let the roots be α, β & $\alpha\beta$

Now $\sum \alpha = -\frac{b}{a}$

$$\therefore \alpha + \beta + \alpha\beta = p \quad — (1)$$

$$\sum \alpha\beta = \frac{c}{a}$$

$$\alpha\beta + \alpha^2\beta + \alpha\beta^2 = q \quad — (2)$$

$$\sum \alpha\beta\gamma = -\frac{d}{a}$$

$$\alpha^2\beta^2 = r \quad — (3)$$

From (1)

$$p + 1 = 1 + \alpha + \beta + \alpha\beta$$

$$\therefore r(p+1)^2 = \alpha^2\beta^2(1 + \alpha + \beta + \alpha\beta)$$

From (2) & (3)

$$(q+r)^2 = (\alpha\beta + \alpha^2\beta + \alpha\beta^2 + \alpha^2\beta^2)^2$$

4

$$= [\alpha\beta(1 + \alpha + \beta + \alpha\beta)]^2$$

$$= \alpha^2\beta^2(1 + \alpha + \beta + \alpha\beta)^2$$

$$= r(p+1)^2.$$