GOSFORD HIGH SCHOOL

2011
HIGHER SCHOOL CERTIFICATE

EXTENSION 2 MATHEMATICS

ASSESSMENT TASK 2

General Instructions:

- Reading time: 5minutes.
- Working time: 90 minutes
- Write using black or blue pen.
- Board-approved calculators may be used.
- Each question should be started on a new page.
- All necessary working should be shown in every question.

Total marks: - 63

Attempt all Questions 1- 4.

Question 1:

(a) If $z = 2 - 3i$ and $\omega = 1 - i$ find in the form $a + ib$

(i) $z + \omega$ (1)

(ii) $z\omega$ (1)

(iii) $\frac{z}{\omega}$ (2)

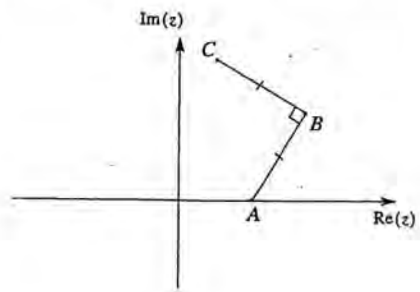
(b) (i) Express $1 - \sqrt{3}i$ in modulus – argument form. (1)

(ii) Hence evaluate $(1 - \sqrt{3}i)^4$ in the form $a + ib$ (2)

(c) Sketch the region in the complex plane where:

$|z + 2 - i| \leq 3$ and $\frac{-\pi}{3} \leq \arg z \leq \frac{\pi}{4}$ (2)

(d) The diagram below shows the fixed points A, B & C in the Argand plane, where AB = BC and $\angle ABC = \frac{\pi}{2}$. The point A represents the complex number $z_1 = 2$ and the point B represents the complex number $z_2 = 3 + \sqrt{5}i$.



(i) Find the complex number z_3 represented by the point C (2)

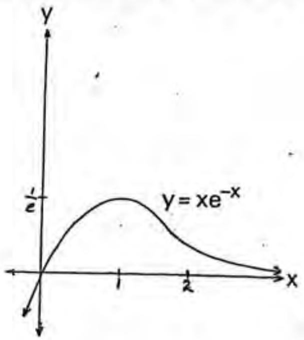
(ii) D is the point in the Argand plane such that ABCD is a square. Find the complex number z_4 represented by the point D. (1)

(e) Find, in Cartesian form, the equation of the locus of the point z if

$Re\left(\frac{z-2}{z}\right) = 0.$ (4)

Question 2:

(a) The function defined by $f(x) = xe^{-x}$ is drawn below.



Draw separate one third page sketches showing the important features of each of the following.

(i) $y = |f(x)|$ (2)

(ii) $y = f(|x|)$ (2)

(iii) $y = \frac{1}{f(x)}$ (2)

(iv) $y = \sqrt{f(x)}$ (2)

(v) $|y| = f(x)$ (2)

(b)

(i) If $y^2 = x^3(x-2)$ use implicit differentiation to show that for $y > 0,\ \frac{dy}{dx} = (2x-3)\sqrt{\frac{x}{x-2}}$. (2)

(ii) Draw a neat sketch of $y = x^3(x-2)$ clearly showing the coordinates of the stationary point between $x = 0$ & $x = 2$. (3)

(iii) On the same number plane sketch the graph of $y^2 = x^3(x-2)$ showing all important features. (2)

Question3:

(a)

(i) Prove that if a polynomial $P(x)$ has a root α of multiplicity r, then $P'(x)$ has a root α of multiplicity $r-1$. (2)

(ii) If $P(x) = x^4 - 5x^3 + 6x^2 + 4x - 8$ has a root of multiplicity 3, find all the roots. (3)

(b) It is given that $1+i$ is a root of $P(z) = 2z^3 - 3z^2 + rz + s$ where r & s are real numbers.

(i) Explain why $1-i$ is also a root of $P(z)$. (1)

(ii) Factorise $P(z)$ over the Real field $\mathbb{R}$. (3)

(c) Factorise $x^3 - 3x^2 + 4x - 2$ over the complex field $\mathbb{C}$. (3)

(d) Two of the roots of $3x^4 - 10x^3 + px^2 + qx - 12$ are reciprocals while the other two roots are opposites of each other.
Find the roots and evaluate p & q. (4)

(e) If $z = cos\theta + isin\theta$, show that $z^n + z^{-n} = 2cosn\theta$. Hence show that $32cos^5\theta = 2cos5\theta + 10cos3\theta + 20cos\theta$. (4)

Question 4:

(a) Find $\int \frac{e^x}{\sqrt{1-e^{2x}}}dx$. (2)

(b) Use the substitution $u = \sqrt{2x-1}$ to evaluate $\int_1^2 \frac{dx}{x\sqrt{2x-1}}$. (4)

(c)

(i) Find real numbers a and b such that:

$$\frac{7}{2x^2+5x-3} = \frac{a}{x+3} + \frac{b}{2x-1}$$ (3)

(ii) Hence find $\int \frac{7}{2x^2+5x-3}dx$ (1)

SOLUTIONS

QUESTION 1.

a) (i) $z + w = 2 - 3i + 1 - i$

$= 3 - 4i$ (1)

(ii) $zw = (2-3i)(1-i)$

$= 2 - 2i - 3i + 3i^2$

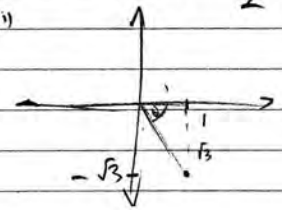
$= -1 - 5i$ (1)

(iii) $\frac{z}{w} = \frac{(2-3i)}{(1-i)} \times \frac{(1+i)}{(1+i)}$

$= \frac{2 + 2i - 3i - 3i^2}{2}$

$= \frac{5}{2} - \frac{1}{2}i$ (2).

b) (i)



Let $z = 1 - \sqrt{3}i$

$|z| = 2 \quad \arg(z) = -\frac{\pi}{3}$.

$\therefore 1 - \sqrt{3}i = 2\,\text{cis}\left(-\frac{\pi}{3}\right)$ (1)

(ii) $(1-\sqrt{3}i)^4 = \left[2\,\text{cis}\left(-\frac{\pi}{3}\right)\right]^4$

$= 16\,\text{cis}\left(-\frac{4\pi}{3}\right)$

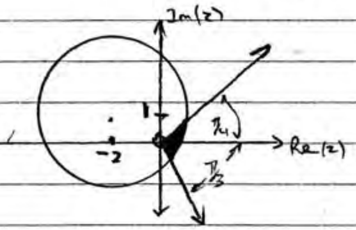
$= 16\,\text{cis}\,\frac{2\pi}{3}$

$= 16\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right)$

$= 16\left(-\frac{1}{2} + i \cdot \frac{\sqrt{3}}{2}\right)$

$= -8 + 8\sqrt{3}i.$ (2)

c)



$|z + 2 - i| \le 3$

$|z - (-2+i)| \le 3$

(2)

d) (i) $\overrightarrow{OC} = \overrightarrow{OA} + \overrightarrow{AB} + \overrightarrow{BC}$

$= 2 + 1 + \sqrt{5}i + i(1 + \sqrt{5}i)$

$= 2 + 1 + \sqrt{5}i + i - \sqrt{5}$

$= (3 - \sqrt{5}) + (1 + \sqrt{5})i$ (2)

$\therefore z_3 = (3-\sqrt{5}) + (1+\sqrt{5})i$

(ii) $\overrightarrow{OD} = \overrightarrow{OA} + \overrightarrow{BC}$

$= 2 + i - \sqrt{5}$

$= (2 - \sqrt{5}) + i$

$\therefore z_4 = (2-\sqrt{5}) + i$ (1)

e) Let $z = x + iy$.

$\frac{z-2}{z} = \frac{x+iy-2}{x+iy} \times \frac{x-iy}{x-iy}$.

$= \frac{x^2 - i\cancel{xy} + i\cancel{xy} - i^2y^2 - 2x + 2iy}{x^2+y^2}$

$= \frac{x^2 - 2x + y^2}{x^2+y^2} + \frac{i \cdot 2y}{x^2+y^2}$

If $\text{Re}\left(\frac{z-2}{z}\right) = 0$,

$\frac{x^2 - 2x + y^2}{x^2+y^2} = 0$

$x^2 - 2x + y^2 = 0, \quad x \ne 0, y \ne 0$ (4)

$x^2 - 2x + 1 + y^2 = 1$

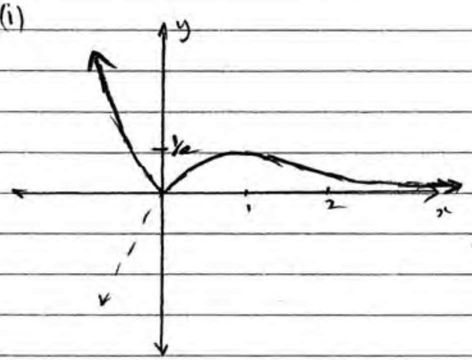
$(x-1)^2 + y^2 = 1$

This is a circle centre $(1,0)$ & radius 1 unit excluding the origin.

QUESTION 2.

a)

(i)

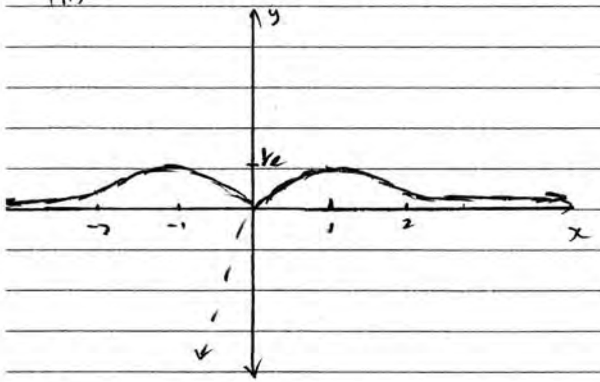


Reflect f(x) where f(x) < 0 in the x-axis

$y = |f(x)|$

2

(ii)

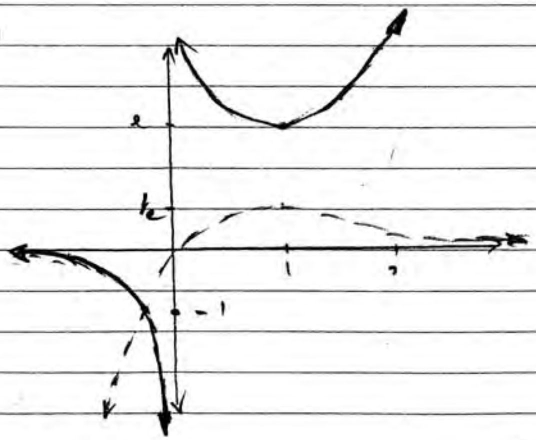


Reflect f(x) where $x > 0$ in the y-axis

$y = f(|x|)$

2

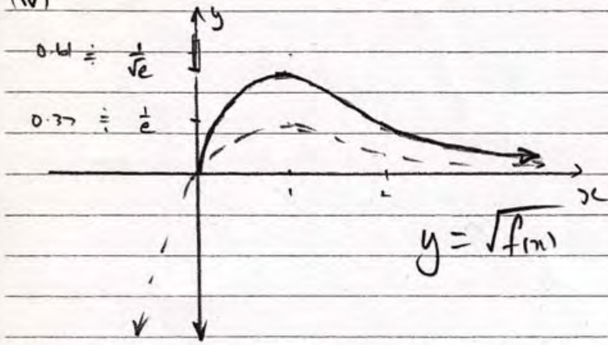
(iii)



$y = \frac{1}{f(x)}$

2

(iv)



$\sqrt{f(x)}$ is only defined where $f(x) \geq 0$

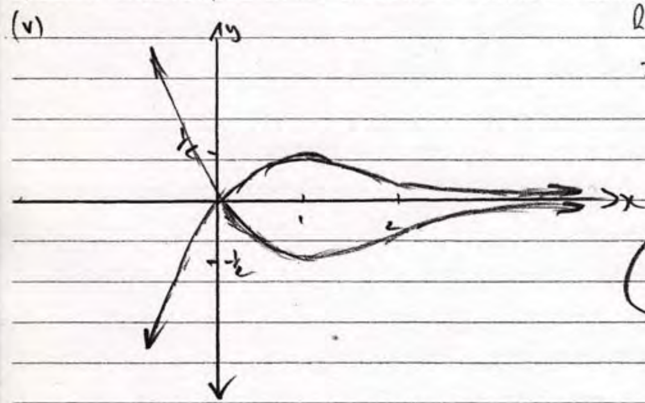
$\sqrt{f(x)} > f(x)$ if $f(x) < 1$

$y = \sqrt{f(x)}$

2

$|y| = f(x)$ is f(x) & the reflection of f(x) in the x-axis.

(v)



$|y| = f(x)$

2

b)

(i) If $y^2 = x^3(x-2)$

$y^2 = x^4 - 2x^3$

Diff w.r.t. x

$\therefore\ 2y\frac{dy}{dx} = 4x^3 - 6x^2$

$y\frac{dy}{dx} = 2x^3 - 3x^2$

$\frac{dy}{dx} = \frac{x^2(2x-3)}{y}$

$$\therefore \frac{dy}{dx} = \frac{x^2(2x-3)}{\sqrt{x^3(x-2)}}$$
$$= \frac{x^2(2x-3)}{x\sqrt{x(x-2)}}$$
$$= \frac{x(2x-3)}{\sqrt{x(x-2)}}$$
$$= \frac{\sqrt{x}(2x-3)}{\sqrt{x-2}}$$
$$= (2x-3)\sqrt{\frac{x}{x-2}}$$

2

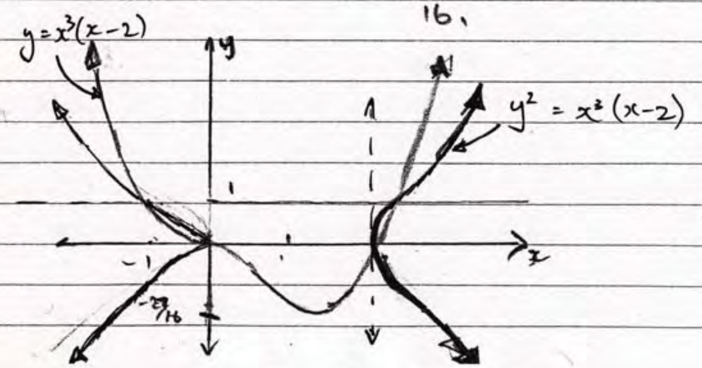
(ii) If $y = x^3(x-2)$
$= x^4 - 2x^3$
$y' = 4x^3 - 6x^2$
If $y' = 0$, $4x^3 - 6x^2 = 0$
$2x^3 - 3x^2 = 0$
$x^2(2x-3) = 0$
$x = 0$ or $\frac{3}{2}$

3

When $y = 3/2$, $x = \frac{27}{8}\left(-\frac{1}{2}\right)$
$= -\frac{27}{16}$

(iii) See graph.

2



QUESTION 3.

a) (i) Let $P(x) = (x-\alpha)^r . Q(x)$

$P'(x) = Q(x) . r(x-\alpha)^{r-1} + (x-\alpha)^r . Q'(x)$
$= (x-\alpha)^{r-1}[r . Q(x) + (x-\alpha) . Q'(x)]$
$= (x-\alpha)^{r-1} . A(x)$ for some polynomial $A(x$

$\therefore P'(x)$ has a root of multiplicity $r-1$.

2

(ii) $P(x) = x^4 - 5x^3 + 6x^2 + 4x - 8$
$P'(x) = 4x^3 - 15x^2 + 12x + 4$
$P''(x) = 12x^2 - 30x + 12$
$= 6(2x^2 - 5x + 2)$
$= 6(2x-1)(x-2)$

$P''(x)$ has roots $\frac{1}{2}$ or 2
One of these is also a root of $P'(x)$ & $P(x)$

Now $P'(2) = 4(2)^3 - 15(2)^2 + 12(2) + 4 = 0$
$P(2) = (2)^4 - 5(2)^3 + 6(2)^2 + 4(2) - 8 = 0$

$\therefore$ 2 is the root of multiplicity 3.

$\therefore P(x) = (x-2)^3(x-b)$ for some value of b
The product of the roots is $(-2)^3 \times -b = 8b$
$\therefore 8b = -8$
$b = -1$
$\therefore P(x) = (x-2)^3(x+1)$.

3

b). (i) Since the coefficients of $P(x)$ are real, if $1+i$ is a root $\overline{1+i}$ is also a root i.e $1-i$ is a root of $P(x)$

1

(ii) $P(z) = 2z^3 - 3z^2 + rz + s$

Now $[z-(1+i)][z-(1-i)]$

$\alpha+\beta = 1+i+1-i$ $\qquad$ $\alpha\beta = (1+i)(1-i)$

$= 2$ $\qquad$ $= 2$

$\therefore P(z) = (z^2-2z+2)(az+b)$ for some values a, b

$= az^3 + bz^2 - 2az^2 + 2bz + 2az + 2b)$

$= az^3 - (2a-b)z^2 + (2a-2b)z + 2b$

$\therefore$ Equating coefficients

$a = 2$ $\qquad$ & $\quad 2a - b = 3$

$4 - b = 3$

$\therefore b = 1$

$\therefore P(z) = (z^2-2z+2)(2z+1)$ $\qquad$ 3

c) Let $P(x) = x^3 - 3x^2 + 4x - 2$

$P(1) = (1)^3 - 3(1)^2 + 4(1) - 2$

$= 0$

$\therefore x-1$ is a factor

$$\begin{array}{r} x^2 - 2x + 2 \\ x-1 \;\big)\; x^3 - 3x^2 + 4x - 2 \\ x^3 - x^2 \\ \hline -2x^2 + 4x \\ -2x^2 + 2x \\ \hline 2x - 2 \\ 2x - 2 \\ \hline 0 \end{array}$$

$P(x) = (x-1)(x^2-2x+2)$

$= (x-1)[(x^2-2x+1)+1]$ $\qquad$ 4

$= (x-1)[(x-1)^2 - i^2]$

$= (x-1)(x-1+i)(x-1-i)$

d) Let the roots be $\alpha, \frac{1}{\alpha}, \beta, -\beta$.

Now $\Sigma\alpha = \alpha + \frac{1}{\alpha} + \beta - \beta$

$\therefore \alpha + \frac{1}{\alpha} = \frac{10}{3}$

$3\alpha^2 + 3 = 10\alpha$

$3\alpha^2 - 10\alpha + 3 = 0$

$(3\alpha - 1)(\alpha - 3) = 0$

$\alpha = \frac{1}{3}$ or 3

Also $\Sigma\alpha\beta\gamma\delta = \alpha \cdot \frac{1}{\alpha} \cdot \beta \cdot -\beta$

$\therefore -\beta^2 = \frac{-12}{3}$

$\beta^2 = 4$

$\beta = \pm 2$.

$\therefore$ Roots are $\frac{1}{3}, 3, 2, -2$.

Also $\Sigma\alpha\beta = \frac{1}{3}.3 + \frac{1}{3}.2 + \frac{1}{3}.-2 + 3.2 + 3.-2 + 2.-2$

$= 1 + \frac{2}{3} - \frac{2}{3} + 6 - 6 - 4$

$= -3$

$\therefore -3 = \frac{p}{3}$

$p = -9$

& $\Sigma\alpha\beta\gamma = \frac{1}{3}.3.2 + \frac{1}{3}.3.-2 + \frac{1}{3}.2.-2 + 3.2.-2$

$= 2 - 2 - \frac{4}{3} - 12$

$= \frac{-40}{3}$ $\qquad$ 4

$\therefore \frac{-40}{3} = \frac{-q}{3}$

$q = 40$.

e) If $z = \cos\theta + i\sin\theta$

$$z^n = (\cos\theta + i\sin\theta)^n$$
$$= \cos n\theta + i\sin n\theta$$

$$z^{-n} = (\cos\theta + i\sin\theta)^{-n}$$
$$= \cos(-n\theta) + i\sin(-n\theta)$$
$$= \cos n\theta - i\sin n\theta$$

$$\therefore z^n + z^{-n} = 2\cos n\theta.$$

Now $z + \frac{1}{z} = 2\cos\theta$

$$\left(z+\frac{1}{z}\right)^5 = (2\cos\theta)^5$$
$$= 2^5\cos^5\theta$$
$$= 32\cos^5\theta$$

But $\left(z+\frac{1}{z}\right)^5 = z^5 + \binom{5}{1}z^4 \cdot \frac{1}{z} + \binom{5}{2}z^3 \cdot \frac{1}{z^2} + \binom{5}{3}z^2 \cdot \frac{1}{z^3}$

$$+ \binom{5}{4} z \cdot \frac{1}{z^4} + \frac{1}{z^5}$$

$$= z^5 + 5z^3 + 10z + 10 \cdot \frac{1}{z} + 5 \cdot \frac{1}{z^3} + \frac{1}{z^5}$$

$$= \left(z^5 + \frac{1}{z^5}\right) + 5\left(z^3 + \frac{1}{z^3}\right) + 10\left(z + \frac{1}{z}\right)$$

$$= 2\cos 5\theta + 5 \cdot 2\cos 3\theta + 10 \cdot 2\cos\theta$$
$$= 2\cos 5\theta + 10\cos 3\theta + 20\cos\theta.$$

4

QUESTION 4.

a) $$\int \frac{e^x}{\sqrt{1-e^{2x}}}\,dx = \int \frac{f'(x)}{\sqrt{1-[f(x)]^2}}\,dx$$

where $f(x) = e^x$

$$= \sin^{-1} f(x) + c$$

$$= \sin^{-1} e^x + c.$$

2

b) $$\int_1^2 \frac{dx}{x\sqrt{2x-1}}$$

If $u = (2x-1)^{1/2}$

$$du = \frac{1}{2}(2x-1)^{-1/2} \cdot 2\,dx$$

$$= \frac{dx}{\sqrt{2x-1}}.$$

$$u^2 = 2x-1$$
$$u^2+1 = 2x$$
$$x = \frac{1}{2}(1+u^2).$$

4

When $x=1$, $u=1$

$x=2$, $u=\sqrt{3}$.

$$\therefore I = \int_1^{\sqrt{3}} \frac{du}{\frac{1}{2}(1+u^2)}$$

$$= 2\left[\tan^{-1} u\right]_1^{\sqrt{3}}$$

$$= 2(\tan^{-1}\sqrt{3} - \tan^{-1} 1)$$
$$= 2\left(\frac{\pi}{3} - \frac{\pi}{4}\right)$$
$$= \frac{\pi}{6}$$

c) (i) $\frac{7}{2x^2+5x-3} = \frac{7}{(2x-1)(x+3)}$

If $\frac{7}{(x+3)(2x-1)} = \frac{a}{x+3} + \frac{b}{2x-1}$

$7 = a(2x-1) + b(x+3)$

Let $x = -\frac{1}{2}$, $7 = b \cdot 3\frac{1}{2}$

$\therefore b = 2$

Let $x = -3$, $7 = a \cdot -7$

$\therefore a = -1$

3

$\therefore \frac{7}{(x+3)(2x-1)} = \frac{-1}{x+3} + \frac{2}{2x-1}$

(ii) $\int \frac{7}{2x^2+5x-3}\,dx = \int \frac{-1}{x+3} + \frac{2}{2x-1}\,dx$

$= -\ln(x+3) + \ln(2x-1) + c$

$= \ln\left|\frac{2x-1}{x+3}\right| + c$

4