Name:____________________ Teacher:________________

HSC Mathematics

Extension 2

Assessment Task 2 - 2013

Time Allowed - **90 minutes +5 minutes reading**

Instructions: Calculators may be used in any parts of the task. For 1 Mark Questions, the correct answer is sufficient to receive full marks. For Questions worth more than 1 Mark, necessary working MUST be shown to receive full marks.

Multiple Choice	/5
Question 6	/15
Question 7	/15
Question 8	/15
Total	/50

Answer Questions 1 to 5 on the separate multiple choice answer sheet.

Question 1

Consider the following statements about a polynomial $P(x)$:

(i) If $P(x)$ is even, then $P'(x)$ is odd.

(ii) If $P'(x)$ is even, then $P(x)$ is odd.

Which statement is always true

A (i) only B (ii) only C Both (i) and (ii) D Neither (i) nor (ii)

Question 2

Let $\alpha,\ \beta,\gamma$ be the zeros of the polynomial x^3+5x-3

The value of $\alpha^3+\beta^3+\gamma^3$ is

A -125 B 0 C 9 D 34

Question 3

If $f(x)=\sqrt{1-x}$ then the common domain between $y=f(x)$ and $y=f(-x)$ is

A $-1\le x\le 1$ B $-1<x<1$ C $x\le 1$ D $x<1$

Question 4

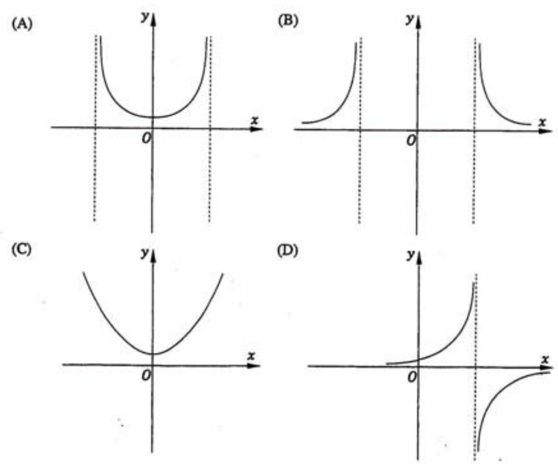
Let $z=a+ib$ where $a\neq 0$ and $b\neq 0$

Which of the following statements is false?

A $z-\bar{z}=2bi$ B $|z|^2=|z||\bar{z}|$ C $|z|+|\bar{z}|=|z+\bar{z}|$ D $\arg(z)+\arg(\bar{z})=0$

Question 5

Which of the following best represents the graph of $y=\dfrac{1}{\sqrt{4-x^2}}$?



Question 6 (15 Marks) Begin a new sheet of paper **Marks**

a) Describe, give the equation and sketch the locus defined by

(i) $\text{Arg}\left[z-(1+\sqrt{3}i)\right]=\dfrac{\pi}{3}$ **2**

(ii) $z^2-\bar{z}^2=16i$ **2**

b) Let P, Q and R represent the complex numbers w_1, w_2 and w_3 respectively. **3**

What geometric properties characterise triangle PQR if $w_2-w_1=i(w_3-w_1)$.

Give reasons for your answer.

c) For the polynomial $P(x)=x^6+x^3+1$

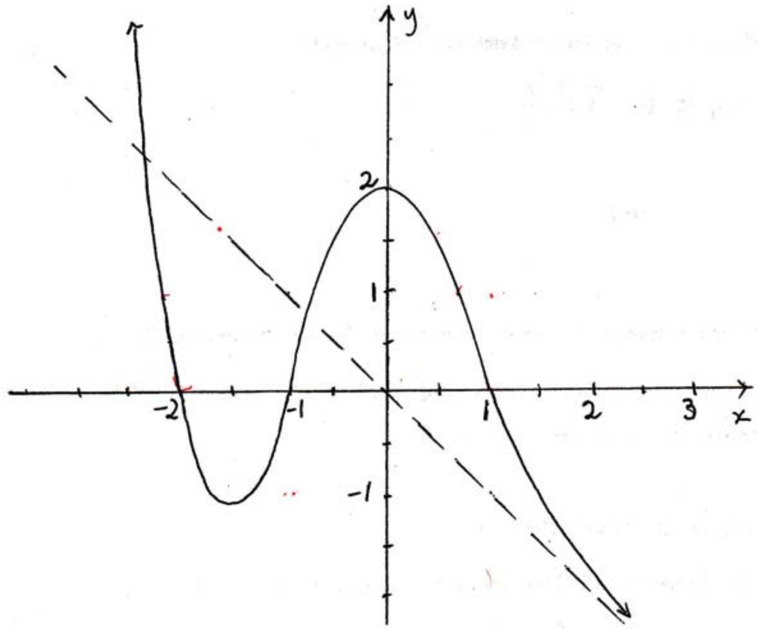
(i) Show that the roots of $P(x)=0$ are amongst the roots of $x^9-1=0$ **2**

(ii) Hence show the roots of $P(x)=0$ on the unit circle on an Argand diagram **3**

(iii) Hence show that **3**

$$P(x)=(x^2-2x\cos\frac{2\pi}{9}+1)(x^2-2x\cos\frac{4\pi}{9}+1)(x^2-2x\cos\frac{8\pi}{9}+1)$$

Question 7 (15 Marks) Begin a new sheet of paper **Marks**



a) For the curve $y=f(x)$ given above, draw individual half page sketches showing intercepts, asymptotes, turning points and other important features of

(i) $y=[f(x)]^2$ **2**

(ii) $y^2=f(x)$ **2**

(iii) $y=x\,f(x)$ **2**

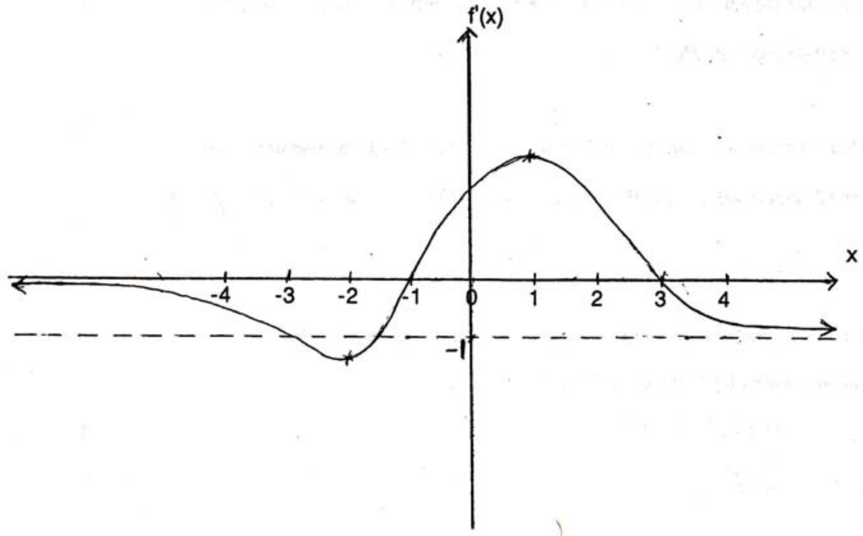
(iv) $y=f|x|$ **2**

(v) $y=\ln f(x)$ **2**

b) Part b) is on the next page.

Question 7 continued **Marks**

b) The function $y = f(x)$ has a derivative $y = f'(x)$ whose graph is given:



Sketch $y = f(x)$ given that $f(0) = 0$ and $f(-3) > 0$. **3**

Describe the behaviour of $y = f(x)$ as $x \to \infty$ and as $x \to -\infty$. **2**

Question 8 (15 Marks) Begin a new sheet of paper **Marks**

a) Given that the polynomial $P(x) = x^4 + x^3 - 3x^2 - 5x - 2$ has a 3 fold root, find all the roots of $P(x)$ **3**

b) When a certain polynomial is divided by $x+1$ and $x-3$, the remainders are 6 and 2 respectively. Find the remainder when $P(x)$ is divided by $x^2 - 2x - 3$. **3**

c) If α, β, γ are the roots of the cubic equation $2x^3 - 3x + 10 = 0$.

Find in simplest form, the equation with roots:

i) $\alpha+1, \beta+1, \gamma+1$ **2**

ii) $\alpha^3, \beta^3, \gamma^3$ **3**

d) Consider the equation $z^3 + mz^2 + nz + 6 = 0$, where m and n are real.

It is known that $1-i$ is a root of the equation.

i) Find the other two roots of the equation. **2**

ii) Find the values of m and n. **2**

END OF TEST

Name:______________________ Teacher:__________________

Multiple-choice answer sheet

Select the alternative A, B, C or D that best answers the question. Fill in the response oval completely, using a black pen.

Sample: $2 + 4 =$ (A) 2 (B) 6 (C) 8 (D) 9

A ○ B ● C ○ D ○

If you think you have made a mistake, put a cross through the incorrect answer and fill in the new answer.

A ● B ✗ C ○ D ○

If you change your mind and have crossed out what you consider to be the correct answer, then indicate the correct answer by writing the word correct and drawing an arrow as follows.

correct

A ✗ B ✗ C ○ D ○

Start here →

1. A ○ B ○ C ○ D ○
2. A ○ B ○ C ○ D ○
3. A ○ B ○ C ○ D ○
4. A ○ B ○ C ○ D ○
5. A ○ B ○ C ○ D ○

STANDARD INTEGRALS

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1},\quad n\neq -1;\ x\neq 0,\ \text{if } n<0$$

$$\int \frac{1}{x}\,dx = \ln x,\quad x>0$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax},\quad a\neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax,\quad a\neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax,\quad a\neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax,\quad a\neq 0$$

$$\int \sec ax \tan ax\,dx = \frac{1}{a}\sec ax,\quad a\neq 0$$

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a},\quad a\neq 0$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a},\quad a>0,\ -a<x<a$$

$$\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \ln\left(x+\sqrt{x^2-a^2}\right),\quad x>a>0$$

$$\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \ln\left(x+\sqrt{x^2+a^2}\right)$$

NOTE: $\ln x = \log_e x,\ x>0$

Solutions 2013 Ext 2 Task 2

Q1 i) true ii) $P(x)$ has a constant added $\therefore$ false. (A)

Q2 Consider
$$x^3+5x-3=0$$
$$x^3 = 3-5x$$
$$\alpha^3 = 3-5\alpha$$
$$\beta^3 = 3-5\beta$$
$$\gamma^3 = 3-5\gamma$$
$$\alpha^3+\beta^3+\gamma^3 = 9-5(\alpha+\beta+\gamma)$$
$$= 9 - 0$$
(C)

Q3 Domain of $f(x)$: $x \le 1$

$f(-x) = \sqrt{1+x}$ D: $x \ge -1$

common domain $-1 \le x \le 1$ (A)

Q4 $|z|+|\bar{z}| = \sqrt{a^2+b^2}+\sqrt{a^2+b^2}$; $|z+\bar{z}| = 2a$ (C)

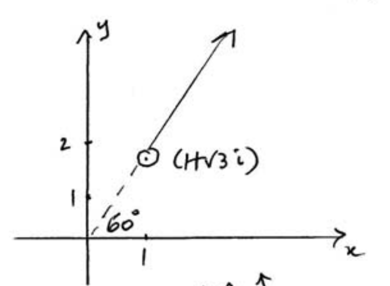
Q5 Domain $4-x^2 > 0 \Rightarrow -2<x<2$ (A)

Question 6

a) i) $\operatorname{Arg}[z-(1+\sqrt{3}i)] = \frac{\pi}{3}$

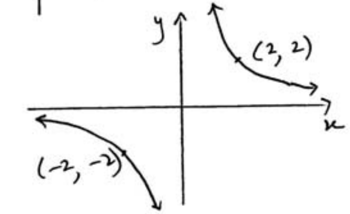
Locus is a ray beginning at $(1+\sqrt{3}i)$ + heading away from O.

Equations: $y=\sqrt{3}x$, $x>1$

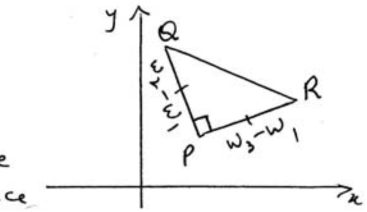


ii) $(x+iy)^2-(x-iy)^2 = 16i$
$$(x^2-y^2+2xyi)-(x^2-y^2-2xyi) = 16i$$
$$4xyi = 16i$$
$$xy = 4$$
Locus is a hyperbola in quadrants (1) + (3)

Equation $xy=4$



b) ΔPQR is right angled at $P(w_1)$ and is isosceles $PQ=PR$.
Multiplication of (w_3-w_1) by i rotates this vector $90°$ anticlockwise + does not change its length since
$$|w_2-w_1| = |i(w_3-w_1)|$$
$$= |i||w_3-w_1|$$
$$= |w_3-w_1|$$



c) i) $x^9-1 = (x^3)^3-1$ using $x^3-y^3=(x-y)(x^2+xy+y^2)$

Factorising $= (x^3-1)(x^6+x^3+1)$
$$\therefore x^9-1=0 = (x^3-1)(x^6+x^3+1)$$
$x^3-1=0$ $x^6+x^3+1=0$

Cube roots of unity remaining roots of the equation

$\therefore$ Roots of $x^6+x^3+1=0$ are all the roots of $x^9-1=0$ which are not cube roots of 1.

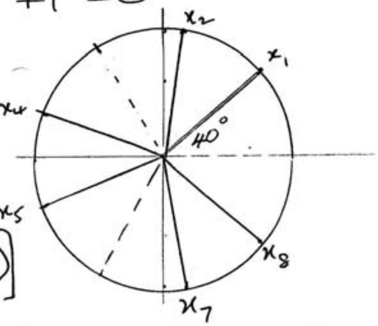
ii) Solving $x^9=1$
$$(\cos\theta+i\sin\theta)^9 = 1\operatorname{cis}0$$
$$\operatorname{cis}9\theta = 1\operatorname{cis}0$$

ii)

$$9\theta = 0 + 2n\pi$$

$n=0$	$\theta = 0$	$x_0 = \text{cis}\,0 = 1$
$n=1$	$\theta = \frac{2\pi}{9}$	$x_1 = \text{cis}\,\frac{2\pi}{9}$
$n=2$	$\theta = \frac{4\pi}{9}$	$x_2 = \text{cis}\,\frac{4\pi}{9}$
$n=3$	$\theta = \frac{6\pi}{9}$	$x_3 = \text{cis}\,\frac{6\pi}{9} = \text{cis}\,\frac{2\pi}{3}$
$n=4$	$\theta = \frac{8\pi}{9}$	$x_4 = \text{cis}\,\frac{8\pi}{9}$
$n=5$	$\theta = \frac{10\pi}{9}$	$x_5 = \text{cis}\,\frac{10\pi}{9} = \text{cis}\,\frac{-8\pi}{9}$
$n=6$	$\theta = \frac{12\pi}{9}$	$x_6 = \text{cis}\,\frac{12\pi}{9} = \text{cis}\,\frac{4\pi}{3}$
$n=7$	$\theta = \frac{14\pi}{9}$	$x_7 = \text{cis}\,\frac{14\pi}{9} = \text{cis}\,\frac{-4\pi}{9}$
$n=8$	$\theta = \frac{16\pi}{9}$	$x_8 = \text{cis}\,\frac{16\pi}{9} = \text{cis}\,\frac{-2\pi}{9}$

Now x_0, x_3, x_6 are cube roots of unity + are not roots of $x^6 + x^3 + 1 = 0$



iii) $P(x) = (x-x_1)(x-x_2)(x-x_4)(x-x_5)(x-x_7)(x-x_8)$

$= [(x-x_1)(x-x_8)][(x-x_2)(x-x_7)][(x-x_4)(x-x_5)]$

$= \left[\left(x - \text{cis}\,\frac{2\pi}{9}\right)\left(x - \text{cis}\,\frac{-2\pi}{9}\right)\right]\left[\left(x - \text{cis}\,\frac{4\pi}{9}\right)\left(x - \text{cis}\,\frac{-4\pi}{9}\right)\right]\left[\left(x - \text{cis}\,\frac{8\pi}{9}\right)\left(x - \text{cis}\,\frac{-8\pi}{9}\right)\right]$

$= \left(x^2 - 2x\cos\frac{2\pi}{9} + 1\right)\left(x^2 - 2x\cos\frac{4\pi}{9} + 1\right)\left(x^2 - 2x\cos\frac{8\pi}{9} + 1\right)$

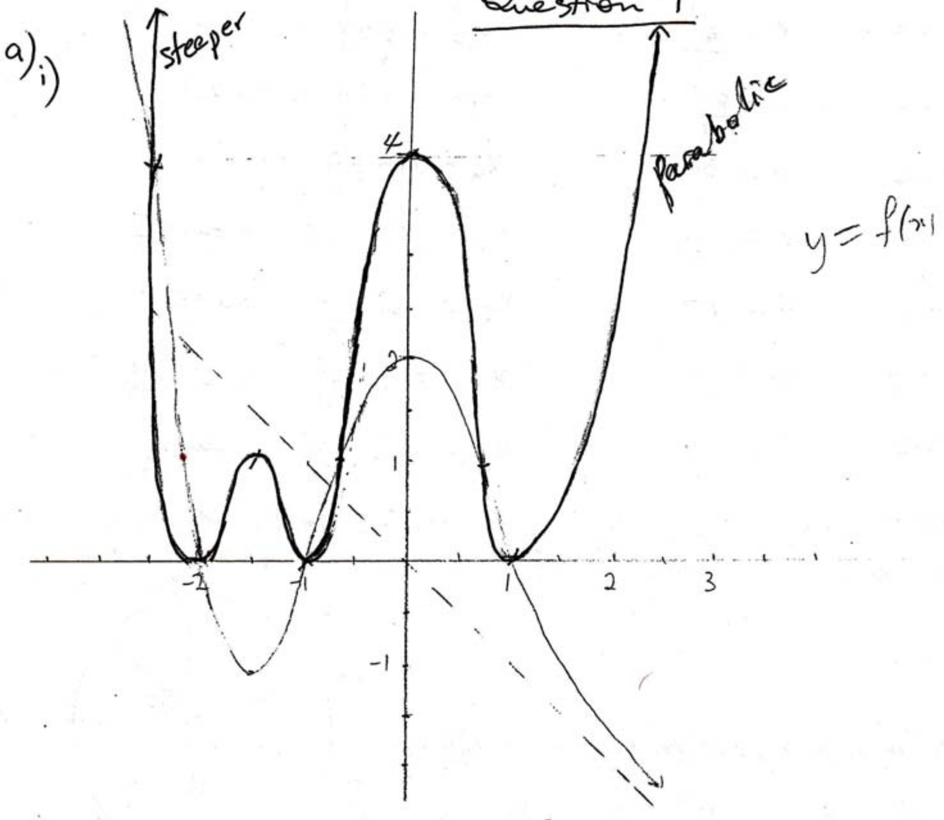
as $(x - z)(x - \bar{z}) = x^2 - (z + \bar{z})x + z\bar{z}$

$z + \bar{z} = 2a = 2\cos\frac{n\pi}{9}$

$z\bar{z} = |z|^2 = 1$

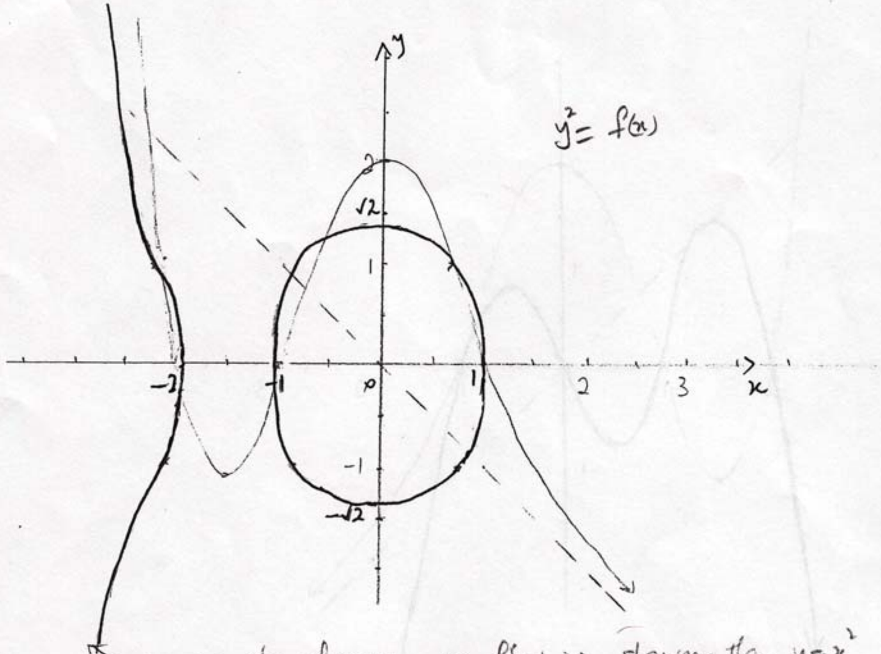
Question 7

a) i)



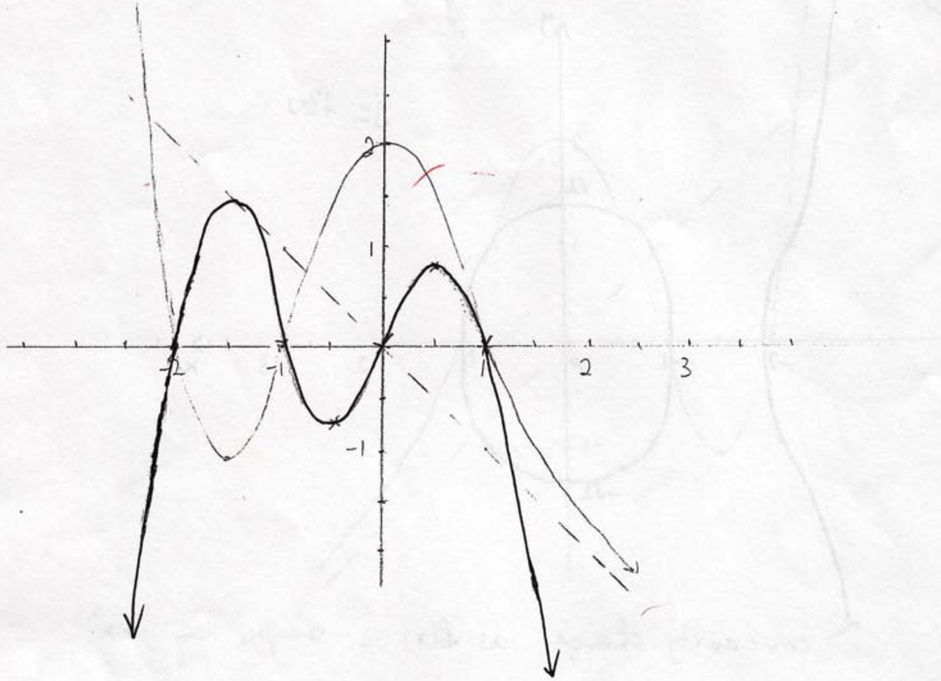
$y = \{f(x)\}^2$

ii)



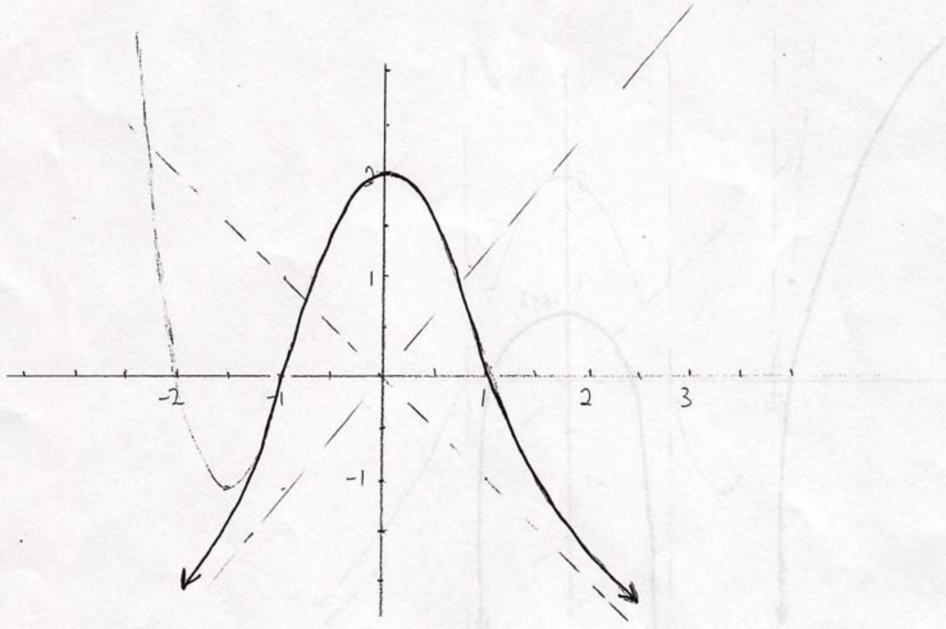
concavity change as f(x) is steeper than $y = x^2$

iii)



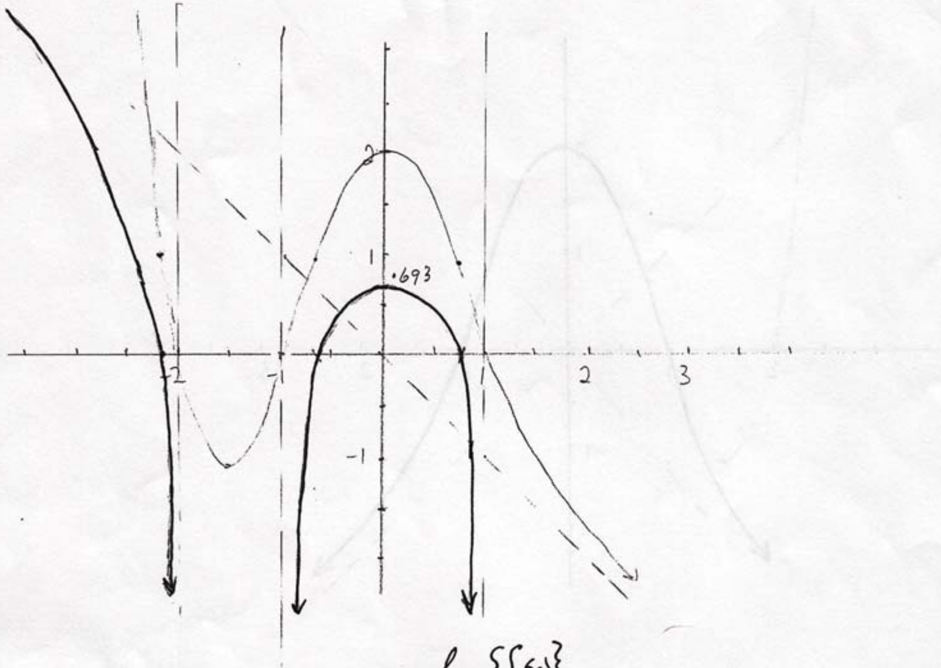
$y = x f(x)$

iv)



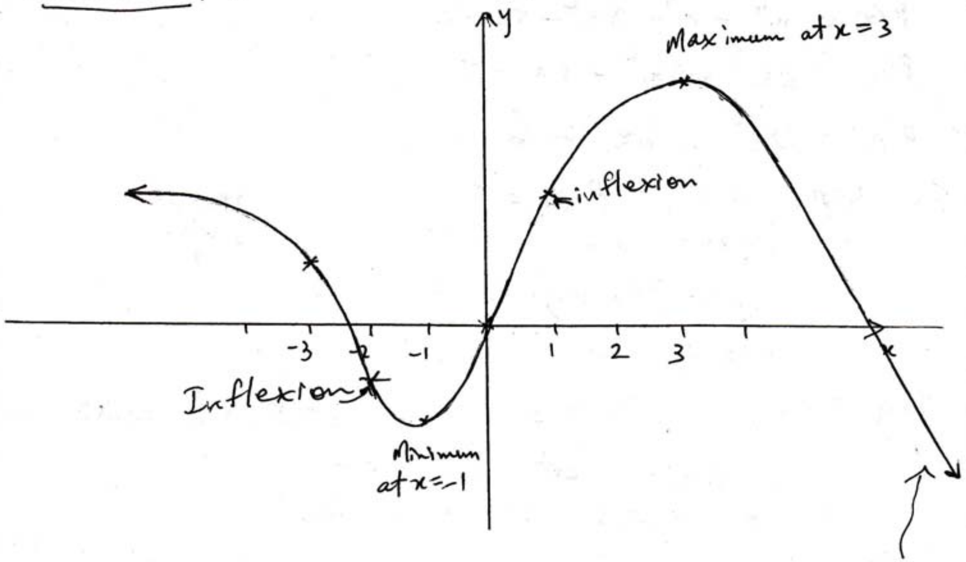
$y = f|x|$

v)



$y = \ln\{f(x)\}$

Question 7 b)



As $x \to \infty$ curve approaches line of gradient -1

As $x \to -\infty$ curve approaches $y = c$ where c is a positive constant as $f(-3) > 0$ + there are no more stationary points or inflexions.

Question 8

$P(x) = x^4 + x^3 - 3x^2 - 5x - 2$

$P'(x) = 4x^3 + 3x^2 - 6x - 5$

$P''(x) = 12x^2 + 6x - 6$

For triple root, $P''(x) = 0$

$6(2x^2 + x - 1) = 0$

$(2x-1)(x+1) = 0$

$x = \frac{1}{2}$ or -1

Try $P(-1)$ $\quad P(-1) = 1 - 1 - 3 + 5 - 2 = 0$

$\therefore$ Triple root is $x = -1$

or $(x+1)^3$ is a factor

By inspection remaining factor is $(x-2)$

$\therefore P(x) = (x+1)^3(x-2)$

roots are $x = -1$ (triple) + $x = 2$

b) $P(-1) = 6 \qquad P(3) = 2$

Let $P(x) = (x^2 - 2x - 3)Q(x) + R(x)$

$P(x) = (x+1)(x-3)Q(x) + ax + b$

$R(x) = ax + b$ as it must be of degree less than 2.

$P(-1) = 6 \qquad 6 = -a + b \quad — ①$

$P(3) = 2 \qquad 2 = 3a + b \quad — ②$

$② - ① \qquad -4 = 4a \implies a = -1$

$b = 6$

$\therefore$ Remainder is $-x + 6$.

8 c) $P(x) = 2x^3 - 3x + 10 = 0$

i) Put $y = \alpha + 1 \implies \alpha = y - 1$

Equation is $2(y-1)^3 - 3(y-1) + 10 = 0$

$2\{y^3 - 3y^2 + 3y - 1\} - 3y + 3 + 10 = 0$

$$\underline{\underline{2y^3 - 6y^2 + 3y + 11 = 0.}}$$

ii) Put $y = \alpha^3 \implies \alpha = \sqrt[3]{y}$

$2y - 3y^{1/3} + 10 = 0$

$2y + 10 = 3y^{1/3}$

$(2y+10)^3 = 27y$

$8y^3 + 3 \times 4y^2 \times 10 + 3.2y.100 + 1000 = 27y$

$8y^3 + 120y^2 + 600y + 1000 = 27y$

$$\underline{\underline{8y^3 + 120y^2 + 573y + 1000}} = 0$$

d) $z^3 + mz^2 + nz + 6 = 0$

$1-i$ is a root. Coefficients are real $\therefore 1+i$ is a root

$\therefore$ Product of roots $(1-i)(1+i).\alpha = -6$

$2\alpha = -6$

$\alpha = -3$

roots are $(1-i)$, $(1+i)$, -3.

ii) Sum of roots $1-i+1+i-3 = -m$

$-1 = -m$

$m = 1$

$\therefore z^3 + z^2 + nz + 6 = 0$

Sub $z = -3$

$-27 + 9 - 3n + 6 = 0$

$-12 = 3n$

$-4 = n$

$$\therefore \underline{\underline{m = 1 \text{ \& } n = -4}}$$