NAME: ______________________

TEACHER: ______________________

GOSFORD HIGH SCHOOL

2015
EXTENSION 2 MATHEMATICS
HSC ASSESSMENT TASK 2.

Time Allowed: 90 minutes (plus 5 min. reading time)

- Write using black or blue pen.
- Board-approved calculators may be used.
- Questions 7, 8, 9 and 10 should all be started on a new page.
- All necessary working should be shown in Questions 7, 8, 9 and 10.

QUESTION	QUESTION TYPE	MARKS	RESULT
1-6	MULTIPLE CHOICE	6	
7	EXTENDED RESPONSE	15	
8	EXTENDED RESPONSE	12	
9	EXTENDED RESPONSE	15	
10	EXTENDED RESPONSE	12	
	TOTAL	60	

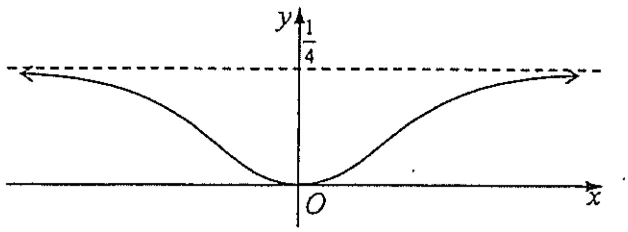
MULTIPLE CHOICE (6 marks). Answer on the multiple choice answer sheet.

1. Let $z = 5 - i$ and $w = 2 + 3i$.

What is the value of $2z + \overline{w}$?

(A) $12 + i$. (B) $12 + 2i$. (C) $12 - 4i$. (D) $12 - 5i$.

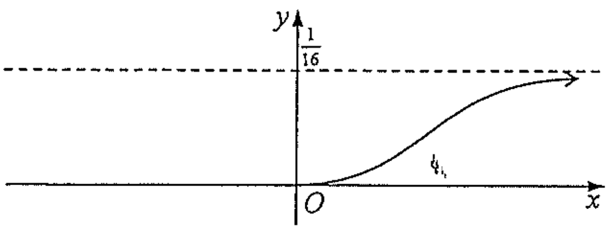
2. The diagram shows the graph of $y = f(x)$.



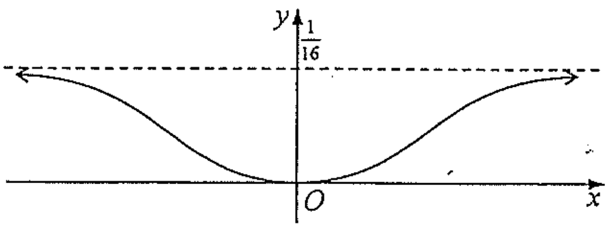
DIAGRAMS NOT TO SCALE.

Which of the following best represents the graph of $y = \sqrt{f(x)}$?

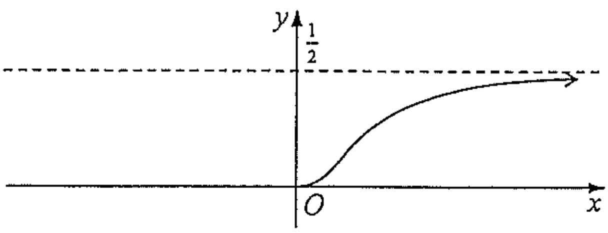
(A)



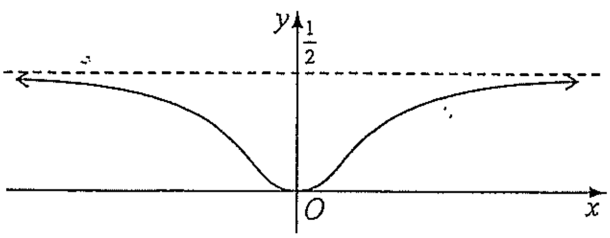
(B)



(C)



(D)



3. The equation $2x^3 - 3x^2 - 5x - 1 = 0$ has roots α, β and γ.

What is the value of $\frac{1}{\alpha^3\beta^3\gamma^3}$?

(A) $\frac{1}{8}$. (B) $\frac{-1}{8}$. (C) 8 . (D) −8 .

4. What is the eccentricity of the ellipse $4x^2 + 6y^2 = 24$?

(A) $\frac{\sqrt{10}}{2}$. (B) $\frac{\sqrt{15}}{3}$. (C) $\frac{\sqrt{3}}{3}$. (D) $\frac{\sqrt{13}}{3}$.

5. The cube roots of unity are 1, ω and ω^2. Simplify $(1-\omega+\omega^2)(1+\omega-\omega^2)$.

(A) 0. (B) 1. (C) 2. (D) 4.

6. If $\frac{4x}{x^2-x-12} \equiv \frac{a}{x-4} + \frac{b}{x+3}$, then

(A) $a = 16, b = 12$. (B) $a = 12, b = 16$. (C) $a = \frac{16}{7}, b = \frac{12}{7}$. (D) $a = \frac{12}{7}, b = \frac{16}{7}$.

Question 7. (15 marks) Start a new page.

(a) If $z = 2 + i$ and $\omega = 1 - 3i$ find in the form $x + iy$

(i) z^2. (1)

(ii) $z\bar{\omega}$. (1)

(iii) $\dfrac{z}{\omega}$. (1)

(b)

(i) Express $z = 1 + \sqrt{3}i$ in modulus-argument form. (2)

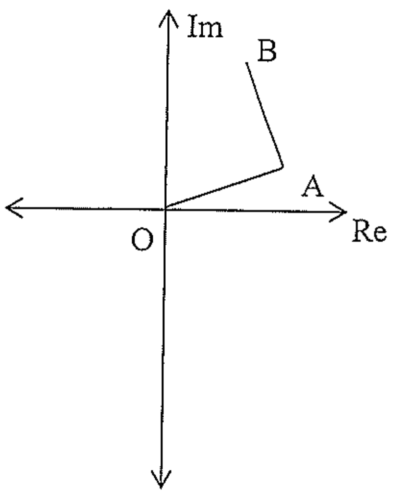
(ii) Show that $(1 + \sqrt{3}i)^6$ is a real number. (2)

(c) For the complex number $z = x + iy$, where x and y are real numbers, find and clearly sketch the curve on an Argand diagram for which

(i) $|z + \bar{z}| \leq 2.$ (2)

(ii) $|z - i| = \sqrt{2}|z + i|$ (3)

(d) The point A in the Argand diagram below represents the complex number $z = a + ib$. The point B represents the complex number $2 + 5i$.

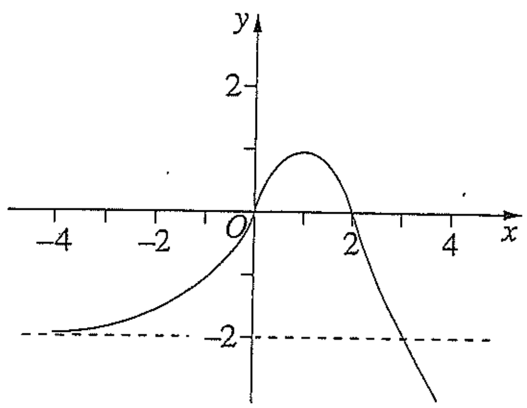


NOT TO SCALE

If the complex number represented by the point C is such that OABC is a square, find C in terms of a and b and hence evaluate a and b. (3)

Question 8 (12 marks) Start a new page.

(a) The graph of $y = f(x)$ is shown below.



Draw a neat sketch of each of the following on the template sheet provided.

(i) $y = |f(x)|$. (1)

(ii) $y = [f(x)]^2$. (2)

(iii) $y = f(|x|)$. (1)

(iv) $y^2 = f(x)$. (2)

(v) $y = \dfrac{1}{f(x)}$. (3)

(b) The equation of a curve is $4x^2 + xy + y^2 = 10$. Find the equation of the tangent to the curve at the point (1,2) on it. (3)

MAKE SURE THAT YOU ATTACH QUESTION 8 (b) TO THE BACK OF THE TEMPLATE SHEET PROVIDED FOR QUESTION 8(a).

Question 9 (15 marks) Start a new page.

(a) If α, β and γ are the roots of the equation $x^3 - 7x^2 - 7 = 0$ find the polynomial equation whose roots are α^2 , β^2 , γ^2 . (3)

(b) Express $\dfrac{2}{x^3+2x}$ in the form $\dfrac{A}{x} + \dfrac{Bx+C}{x^2+2}$. (2)

(c) Consider the equation $z^4 + pz^3 + qz + r = 0$, where p, q & r are real numbers. The sum of the roots of this equation is 6 more than the product of the roots. If $1 + i$ is a root of the equation, find p, q & r. (3)

(d) *(i)* Use DeMoivre's Theorem to express $\cos 4\theta$ and $\sin 4\theta$ in powers of $\cos\theta$ and $\sin\theta$. Hence express $\tan 4\theta$ as a rational function of t, where $t = \tan\theta$. (4)

(ii) Hence solve the equation $t^4 + 4t^3 - 6t^2 - 4t + 1 = 0$. (3)

Question 10 (12 marks) Start a new page.

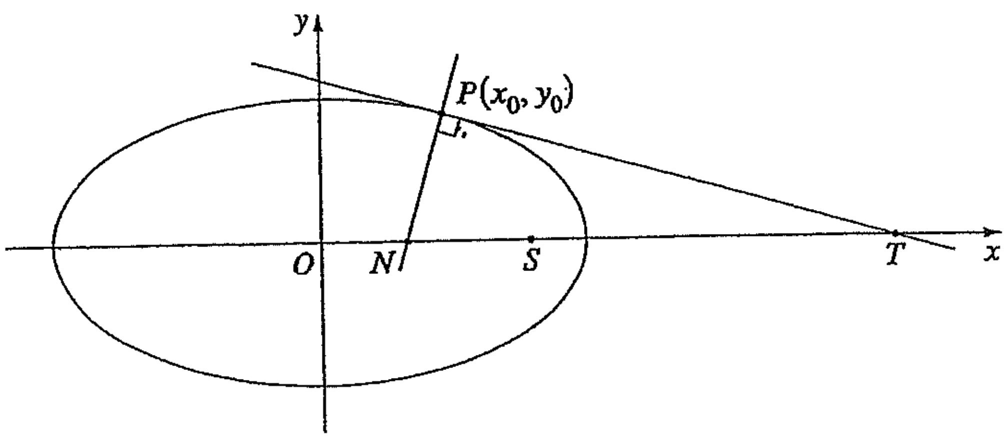
(a) Consider the ellipse $\mathcal{E}$, with equation $\frac{x^2}{100}+\frac{y^2}{64}=1$.

(i) Calculate the eccentricity of $\mathcal{E}$. (1)

(ii) Find the coordinates of the foci and the equations of the directrices of $\mathcal{E}$. (2)

(iii) Draw a neat sketch of $\mathcal{E}$ showing all important features. (2)

(b) The diagram shows the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, with $a>b$. The ellipse has focus S and eccentricity e. The tangent to the ellipse at $P(x_0, y_o)$ meets the x-axis at T. The normal at P meets the x-axis at N.



(i) Show that the tangent to the ellipse at P is given by the equation

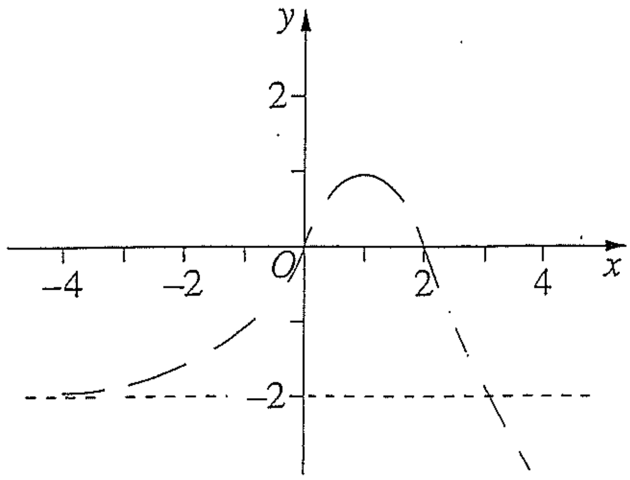
$$y-y_0=-\frac{b^2x_0}{a^2y_0}(x-x_0).$$ (2)

(ii) Show that the x-coordinate of N is x_0e^2. (2)

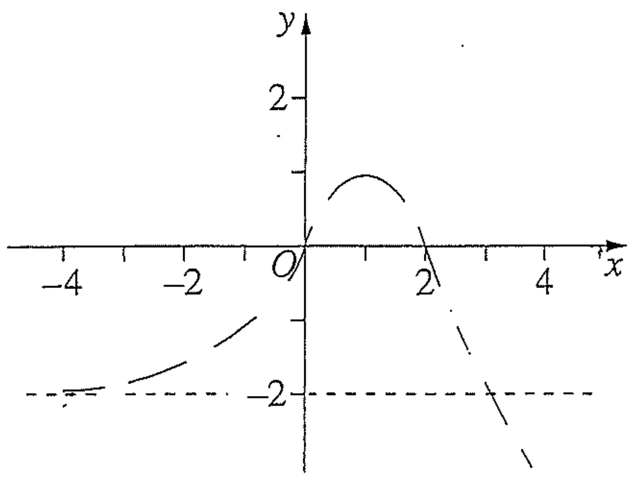
(iii) Show that $ON\times OT=OS^2$. (3)

Question 8(a) TEMPLATE SHEET

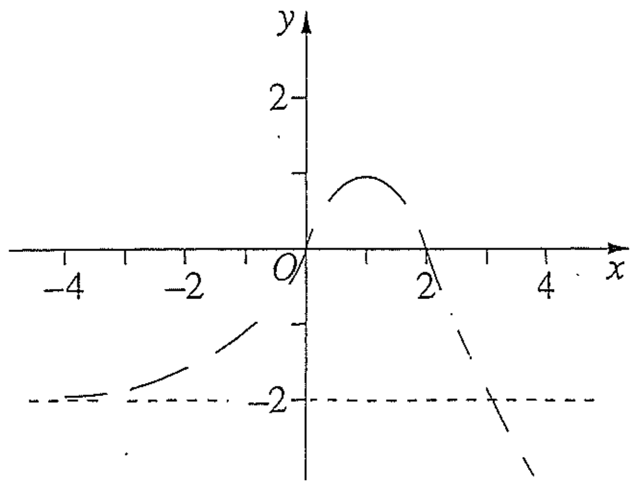
(i)



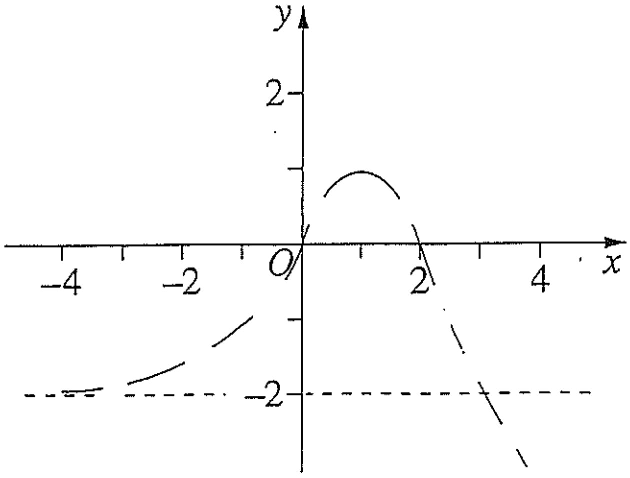
(ii)



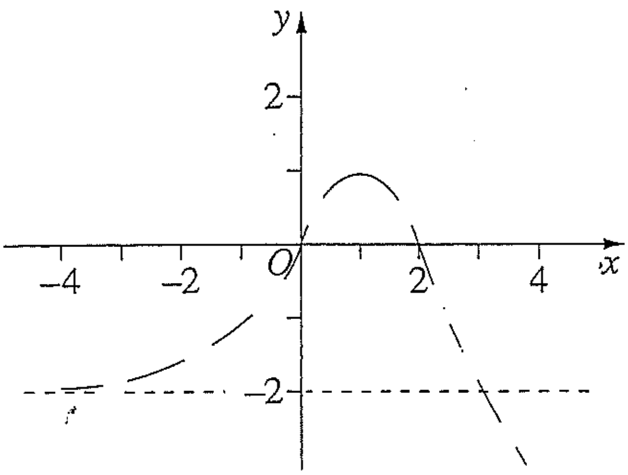
(iii)



(iv)



(v)



Multiple Choice

Q1 $2z + \bar{w} = 10 - 2i + 2 - 3i$

$= 12 - 5i$

Hence D

Q2. D N.B $\sqrt{1/4} = 1/2$

Q3 $\alpha\beta\gamma = -\frac{d}{a}$

$= - -\frac{1}{2}$

$= \frac{1}{2}$

$\alpha^3\beta^3\gamma^3 = \frac{1}{8}$

$\frac{1}{\alpha^3\beta^3\gamma^3} = 8$

Hence C

Q4. $4x^2 + 6y^2 = 24$

$\frac{x^2}{6} + \frac{y^2}{4} = 1$

$b^2 = a^2(1-e^2)$

$4 = 6(1-e^2)$

$\frac{2}{3} = 1 - e^2$

$e^2 = 1 - \frac{2}{3}$

$e^2 = \frac{1}{3}$

$e = \frac{1}{\sqrt{3}}$

$e = \frac{\sqrt{3}}{3}$

Hence C

Q5. If $z^3 - 1 = 0$

$\sum \text{roots} = -\frac{b}{a}$

$= 0$

$\therefore 1 + \omega + \omega^2 = 0$

$1 + \omega^2 = -\omega$

$\times$ $1 + \omega = -\omega^2$

$\therefore (1 - \omega + \omega^2)(1 + \omega - \omega^2)$

$= -2\omega \times -2\omega^2$

$= 4\omega^3$

$= 4$

Hence D

Q6. If $\frac{4x}{x^2 - x - 12} = \frac{a}{x-4} + \frac{b}{x+3}$

$4x = a(x+3) + b(x-4)$

If $x = -3$

$-12 = 0 - 7b$

$7b = 12$

$b = \frac{12}{7}$

If $x = 4$

$16 = 7a$

$a = \frac{16}{7}$

Hence C

Question 7

a) $z = 2 + i, \quad w = 1 - 3i$

(i) $z^2 = (2+i)^2$

$= 4 + 4i + i^2$

$= 3 + 4i$ (1)

(ii) $z\bar{w} = (2+i)(1+3i)$

$= 2 + 6i + i + 3i^2$

$= -1 + 7i$ (1)

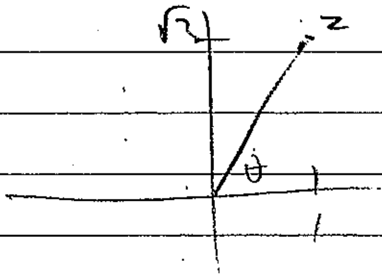
(iii) $\dfrac{z}{w} = \dfrac{(2+i)}{1-3i} \times \dfrac{(1+3i)}{1+3i}$

$= \dfrac{2 + 6i + i + 3i^2}{1 - 9i^2}$

$= \dfrac{-1+7i}{10}$ (1)

$= -\dfrac{1}{10} + \dfrac{7}{10}i$

b) (i)



$|z| = \sqrt{(\sqrt{3})^2 + 1^2}$

$= 2$

$\arg(z) = \tan^{-1}\sqrt{3}$

$= \dfrac{\pi}{3}$

$\therefore 1 + \sqrt{3}i = 2 \operatorname{cis} \dfrac{\pi}{3}$ (2)

(ii) $(1+\sqrt{3}i)^6 = \left(2 \operatorname{cis} \dfrac{\pi}{3}\right)^6$

$= 2^6 \operatorname{cis} 2\pi$

$= 2^6 \times \operatorname{cis} 0$

$= 64 + (\cos 0 + i \sin 0)$

$= 64 \times 1$

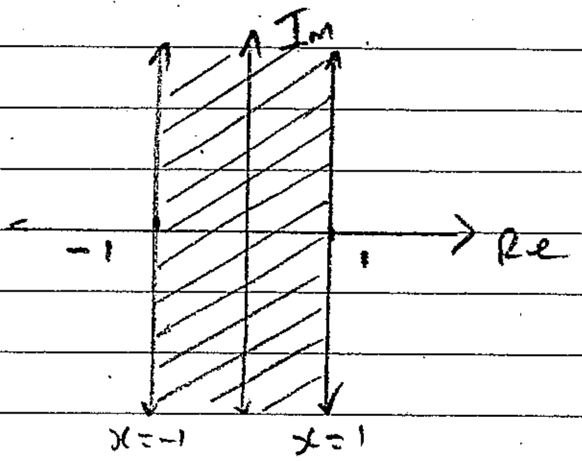
$= 64$ which is real.

c) (i) If $z = x + iy$, $\bar{z} = x - iy$

$\therefore z + \bar{z} = 2x$

So $|2x| \leq 2$

$|x| \leq 1$



(2)

(ii) Let $z = x + iy$

$|z - i| = \sqrt{2}|z + i|$

$|x + i(y-1)| = \sqrt{2}\,|x + i(y+1)|$

$\sqrt{x + (y-1)^2} = \sqrt{2}\,\sqrt{x + (y+1)^2}$

$\therefore\ x^2 + (y-1)^2 = 2(x + (y+1)^2)$

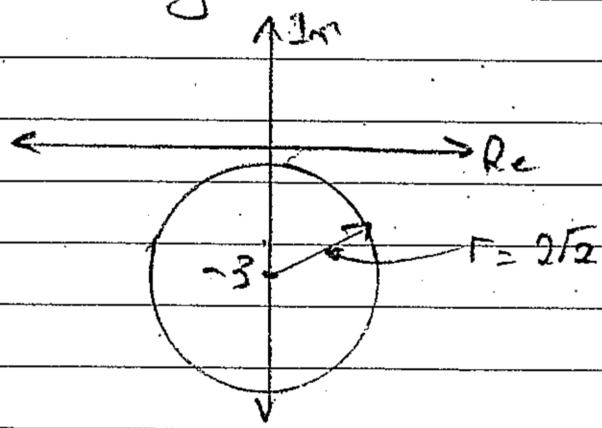
$x^2 + y^2 - 2y + 1 = 2(x^2 + y^2 + 2y + 1)$

$x^2 + y^2 - 2y + 1 = 2x^2 + 2y^2 + 4y + 2$

$0 = x^2 + y^2 + 6y + 1$

$x^2 + y^2 + 6y + 9 = 8$

$x^2 + (y+3)^2 = 8$



(3)

c) If A is $z = a+ib$
C is $iz = ai + i^2 b$
$= -b + ia$. 1

Now $2+5i = a+ib + -b+ia$
$= a-b + i(a+b)$

$a-b = 2$
$a+b = 5$
$2a = 7$
$a = \frac{7}{2}$ 2

$b = \frac{3}{2}$

Question 8a) See template attached.

1 8 b) If $4x^2 + xy + y^2 = 10$

$8x + y \times 1 + x \times \frac{dy}{dx} + 2y\frac{dy}{dx} = 0$

$8x + y + \frac{dy}{dx}(x + 2y) = 0$

$\frac{dy}{dx} = \frac{-(8x+y)}{x+2y}$

If $x = 1,\ y = 2$

$m = \frac{-(8+2)}{1+4}$

$= -2$

$\therefore$ eqn is $y-2 = -2(x-1)$
$y-2 = -2x+2$
$2x + y - 4 = 0$ 3

Question 9.

a) Let $y = x^2$ since $x = \alpha, \beta, \gamma$
$y = \alpha^2, \beta^2, \gamma^2$

$\therefore x = \sqrt{y}$

$(\sqrt{y})^3 - 7(\sqrt{y})^2 - 7 = 0$

$y^{3/2} - 7y - 7 = 0$

$y^{3/2} = 7y + 7$

$y^3 = (7y+7)^2$

$y^3 = 49y^2 + 98y + 49.$

ie $y^3 - 49y^2 - 98y - 49 = 0$ 3

$\therefore$ Required eq^n is

$x^3 - 49x^2 - 98x - 49 = 0.$

b) If $\dfrac{2}{x(x^2+2)} = \dfrac{A}{x} + \dfrac{Bx+C}{x^2+2}$

$2 = A(x^2+2) + x(Bx+C)$
$2 = Ax^2 + 2A + Bx^2 + Cx$

Let $x = 0$
$2 = 2A \implies A = 1$

Let $x = 1$
$2 = A + 2A + B + C$
$2 = 1 + 2 + B + C$
$B + C = -1$ 1

$\therefore\ -10 + q - 3p = 0$

$q - 3p = 10 \quad — ①$

$q + 2p = 0 \quad — ②$

$② - ①$

$5p = -10$

$p = -2$

$\therefore\ q = 4$

$r = -4$

Sol^n is $p = -2,\ q = 4,\ r = -4$.

d) (i) $(\cos\theta + i\sin\theta)^4 = \cos 4\theta + i\sin 4\theta$.

Also $(\cos\theta + i\sin\theta)^4 = \cos^4\theta + 4\cos^3\theta\, i\sin\theta + 6\cos^2\theta\, i^2\sin^2\theta$
$+ 4\cos\theta\, i^3\sin^3\theta + i^4\sin^4\theta$

RHS $= \cos^4\theta - 6\cos^2\theta\sin^2\theta + \sin^4\theta$ +
$i\left(4\cos^3\theta\sin\theta - 4\cos\theta\sin^3\theta\right)$

Hence $\cos 4\theta = \cos^4\theta - 6\cos^2\theta\sin^2\theta + \sin^4\theta$ ✓

$\sin 4\theta = 4\cos^3\theta\sin\theta - 4\cos\theta\sin^3\theta$ ✓

$$\tan 4\theta = \frac{4\cos^3\theta\sin\theta - 4\cos\theta\sin^3\theta}{\cos^4\theta - 6\cos^2\theta\sin^2\theta + \sin^4\theta}$$

$\div$ throughout by $\cos^4\theta$

$$\tan 4\theta = \frac{4\tan\theta - 4\tan^3\theta}{1 - 6\tan^2\theta + \tan^4\theta} \quad ✓$$

$$= \frac{4t - 4t^3}{1 - 6t^2 + t^4}.$$

4

ii) If $\tan 4\theta = 1$

$$1 = \frac{4t - 4t^3}{1 - 6t^2 + t^4}$$

$$1 - 6t^2 + t^4 = 4t - 4t^3$$

ie $t^4 + 4t^3 - 6t^2 - 4t + 1 = 0$.

3

When $\tan 4\theta = 1$

$$4\theta = \frac{\pi}{4}, \quad \pi + \frac{\pi}{4}, \quad \frac{2\pi + \pi}{4}, \quad \frac{3\pi + \pi}{4}$$

$$= \frac{\pi}{4}, \frac{5\pi}{4}, \frac{9\pi}{4}, \frac{13\pi}{4}, \ldots$$

$$\theta = \frac{\pi}{16}, \frac{5\pi}{16}, \frac{9\pi}{16}, \frac{13\pi}{16}.$$

$$\therefore t = \tan\frac{\pi}{16}, \tan\frac{5\pi}{16}, \tan\frac{9\pi}{16}, \tan\frac{13\pi}{16}$$

or $t = \tan\frac{\pi}{16}, \tan\frac{5\pi}{16}$, ~~or~~ $\tan\frac{-7\pi}{16}, \tan\frac{-3\pi}{16}$

Question 10.

a) (i) $b^2 = a^2(1-e^2)$

$64 = 100(1-e^2)$

$\frac{64}{100} = 1 - e^2$

1

$e^2 = \frac{36}{100}$

$e = \frac{3}{5}, \quad e > 0$

(ii) Foci are $(\pm ae, 0)$

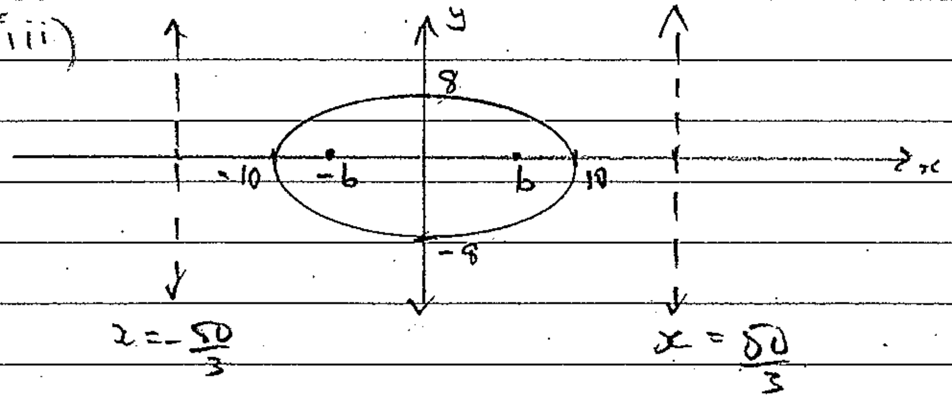
ie $(\pm 6, 0)$

Directrices are $x = \pm \frac{a}{e}$

2

ie $x = \pm \frac{50}{3}$

(iii)



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b) (i) If $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

by implicit differentiation

$\frac{2x}{a^2} + \frac{2y}{b^2} \cdot \frac{dy}{dx} = 0$

$\frac{2y}{b^2} \cdot \frac{dy}{dx} = \frac{-2x}{a^2}$

$\frac{dy}{dx} = \frac{-2x}{a^2} \times \frac{b^2}{2y}$

Let $x = -1$, $2 = A + 2A + B - C$

$2 = 1 + 2 + B - C$

$B - C = -1$ (2)

(1) + (2)

$2B = -2$

$B = -1$

$C = 0$

2

$\therefore \dfrac{2}{x^3+2x} = \dfrac{1}{x} - \dfrac{x}{x^2+2}$

c) Since $1+i$ is a root we sub $1+i$ into the eqn

N.B. $(1+i)^2 = 1+2i+i^2$

$= 2i$

$(1+i)^3 = 2i(1+i)$

$= -2+2i$

$(1+i)^4 = (2i)^2$

$= -4.$

Now $(1+i)^4 + p(1+i)^3 + q(1+i) + r = 0$

$-4 + p(-2+2i) + q(1+i) + r = 0$

Now $\sum \alpha = -p$ & $\sum \alpha\beta\gamma\delta = +r$

So $-p = 6 + r$

$r = -p - 6$

$\therefore -4 + p(-2+2i) + q(1+i) - p - 6 = 0$

$-4 - 2p + 2pi + q + qi - p - 6 = 0$

$-10 + q - 3p + i(2p+q) = 0$

At (x_0, y_0) the grad. of the tangent is

$$y' = \frac{-2x_0}{a^2} \times \frac{b^2}{2y_0}$$

$$= -\frac{b^2 x_0}{a^2 y_0}$$

2

$\therefore$ the eqn of the tanget is

$$y - y_0 = -\frac{b^2 x_0}{a^2 y_0}(x - x_0)$$

(ii) the grad of the normal is $\frac{a^2 y_0}{b^2 x_0}$

$\therefore$ the eqn of the normal is

$$y - y_0 = \frac{a^2 y_0}{b^2 x_0}(x - x_0)$$

At N, $y = 0$

$$\therefore -y_0 = \frac{a^2 y_0}{b^2 x_0}(x - x_0)$$

$$\frac{a^2 y_0}{b^2 x_0} x = \frac{a^2 y_0}{b^2} - y_0$$

$$\frac{a^2 y_0}{b^2 x_0} x = y_0\left(\frac{a^2}{b^2} - 1\right)$$

$$x = \cancel{y_0}\left(\frac{a^2}{b^2} - 1\right) \times \frac{b^2 x_0}{a^2 \cancel{y_0}}$$

$$x = x_0\left(1 - \frac{b^2}{a^2}\right)$$

Since $b^2 = a^2(1 - e^2)$

$$b^2 = a^2 - a^2 e^2$$

$$a^2 e^2 = a^2 - b^2$$

$$e^2 = 1 - \frac{b^2}{a^2}$$

Hence $x = e^2 x_0$

2

(iii) From (i), when $y = 0$

$$0 - y_0 = -\frac{b^2 x_0}{a^2 y_0}(x - x_0)$$

$$a^2 y_0^2 = b^2 x x_0 - b^2 x_0^2 \quad *$$

Since $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

$$\frac{x_0^2}{a^2} + \frac{y_0^2}{b^2} = 1$$

$$b^2 x_0^2 + a^2 y_0^2 = a^2 b^2$$

$\therefore \quad a^2 b^2 = b^2 x x_0$ from $*$

$$\frac{a^2 b^2}{b^2 x_0} = x$$

$$x = \frac{a^2}{x_0}$$

T is $\left(\frac{a^2}{x_0}, 0\right)$

Now $ON \times OT = e^2 x_0 \times \frac{a^2}{x_0}$

3

$$= a^2 e^2$$

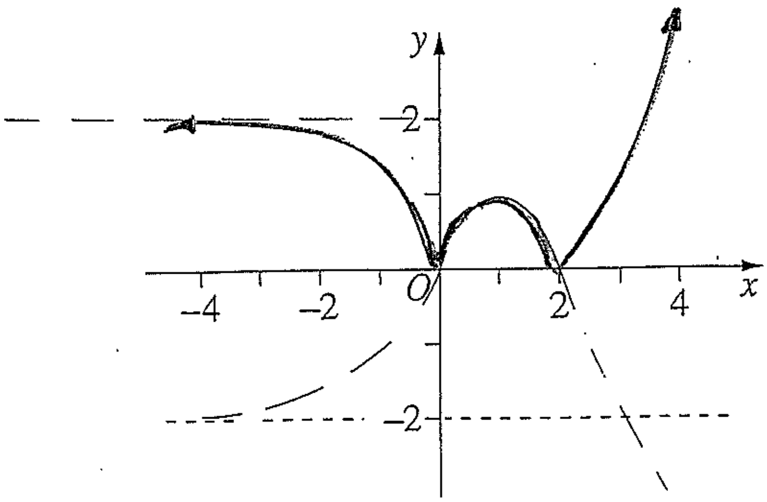
& $OS^2 = (ae)^2$

$$= a^2 e^2$$

$\therefore \quad ON \times OT = OS^2$.

Question 8(a) TEMPLATE SHEET

(i)



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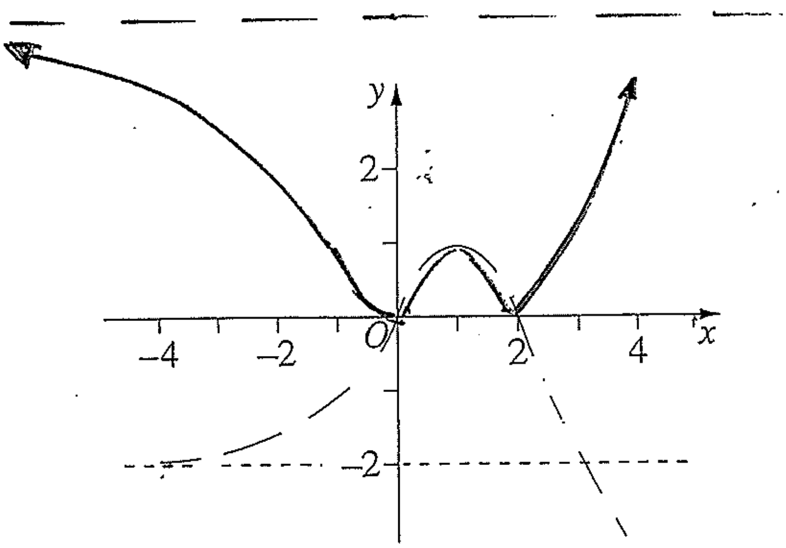
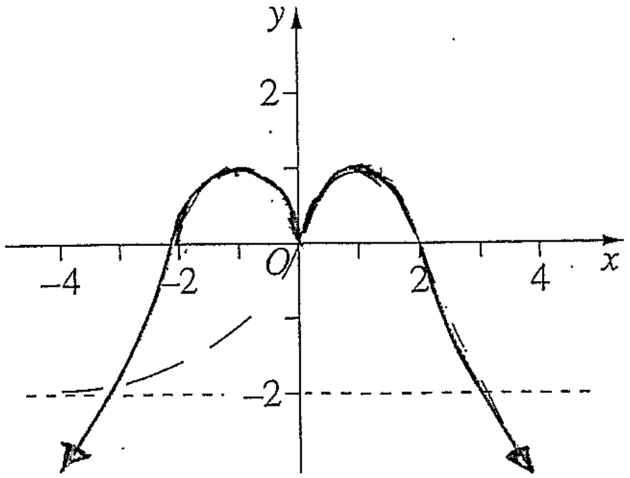
(ii)



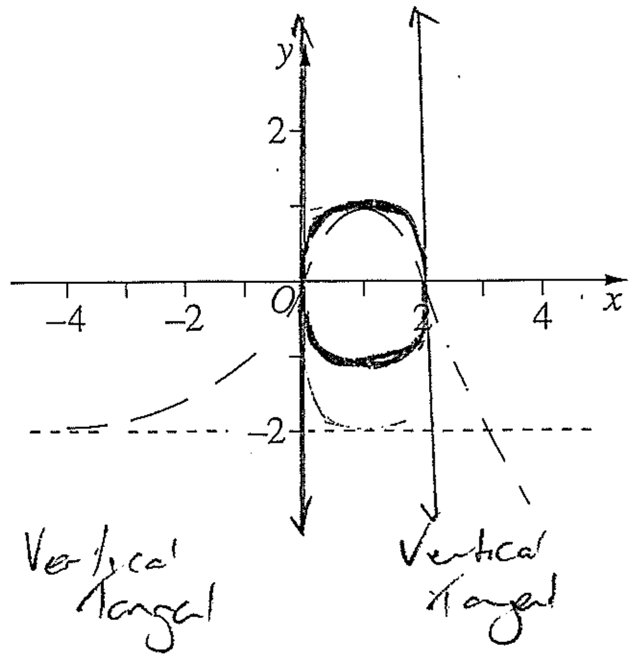
2

(iii)



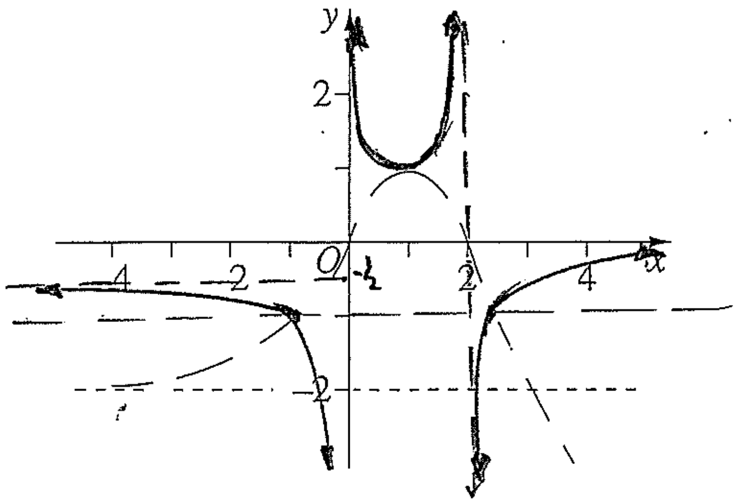
1

(iv)


y
2
−4
−2
O
2
4
x
−2
Vertical Tangent
Vertical Tangent

(2)

(v)


y
2
−4
−2
O
$-\frac{1}{2}$
2
4
x
−2

(3)