NAME.................................

INABURRA SCHOOL

HALF - YEARLY EXAMINATION, 1999

H.S.C. COURSE

4 UNIT MATHEMATICS

This paper does not necessarily reflect
the content or format of the final
HSC paper for this subject.

Time Allowed: 2 Hours (*plus 5 minutes reading time*)

INSTRUCTIONS

- Show **all necessary working** otherwise full marks marks may not be awarded.
- Answer all questions in the answer booklets provided.
- Answer each question in a **separate** booklet.
- Questions are of equal value.
- Approved calculators may be used.

Question 1

Marks

(a) (i) Given $\dfrac{1+4x}{(4-x)\,(x^2+1)} = \dfrac{A}{4-x} + \dfrac{Bx+C}{x^2+1}$ 5

Show that $A=1$, $B=1$ and $C=0$.

(ii) Hence evaluate $\displaystyle\int_0^2 \frac{(1+4x)}{(4-x)\,(x^2+1)}\,dx$

(b) Resolve $\dfrac{5}{(x^2+4)(x^2+9)}$ into Partial Fractions over the Complex Numbers. 3

(c) Evaluate $\displaystyle\int_0^{\frac{\pi}{3}} \frac{dx}{1-\sin x}$ by using the substitution $t = \tan\dfrac{x}{2}$ 5

(d) Use the table of Standard Integrals on page 8 to evaluate $\displaystyle\int_3^4 \frac{dx}{\sqrt{x^2-4}}$ 2

Question 2 Use a *separate* Writing Booklet. **Marks**

(a) If $A = 3 + 4i$ and $B = 5 - 13i$ write the following in the form $a + ib$ 5

(i) $A + B$

(ii) AB

(iii) $\frac{A}{B}$

(iv) $\sqrt{A}$

(b) If $z = 1 + \sqrt{3}\,i$ 5

(i) find the modulus of z

(ii) find the argument of z

(iii) write z in mod/arg form

(iv) write $z^{\frac{1}{2}}$ in mod/arg form

(c) If arg $\left(\frac{z-1}{z+1}\right) = \frac{\pi}{3}$ 5

(i) Find the Cartesian equation of the locus of z

(ii) Describe the locus of z

Question 3 Use a *separate* Writing Booklet.

Marks

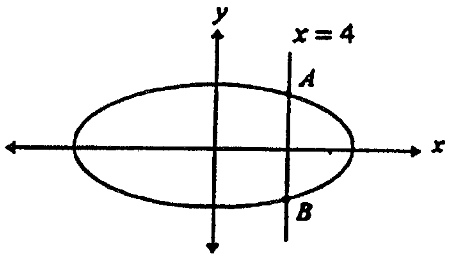
(a) The vertices of a hyperbola are located at the points (–3,0) and (3,0) . 2

The equations of the hyperbola's asymptotes are $y = \frac{5x}{3}$ and $y = \frac{-5x}{3}$

(i) Sketch a graph of the hyperbola.

(ii) Find the equation of the hyperbola.

(b) 9



The diagram above shows an ellipse with equation $\frac{x^2}{25} + \frac{y^2}{9} = 1$

The line $x = 4$ cuts the ellipse at the points A and B as shown.

(i) Find the co-ordinates of the points A and B.

(ii) Find the equation of the tangent through A.

(iii) Find the equation of the normal through A.

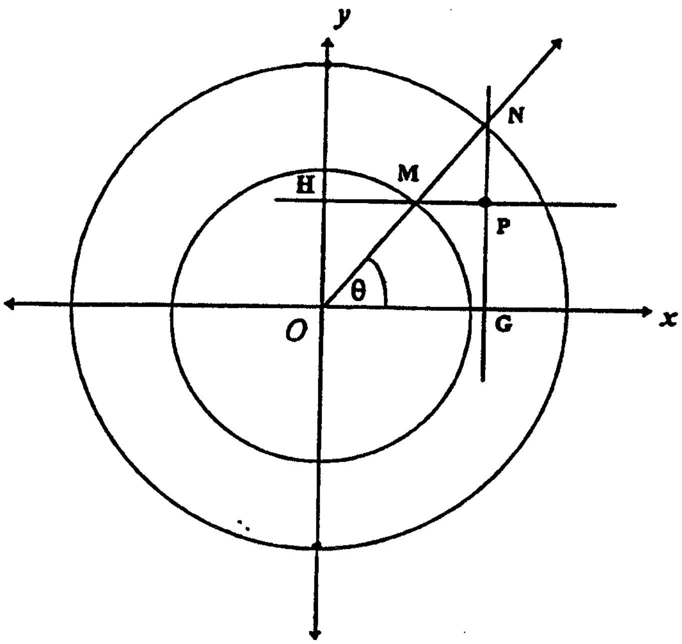
(iv) Find the eccentricity of the ellipse.

(v) Find the equation of the directrices of this ellipse.

Question 3 continues on page 4

Marks

(c) 4



The circles above have centres at O and radii of 5 units and 3 units respectively. A ray from O making an angle θ with the positive x axis, cuts the circles at the points M and N as shown.

NG is drawn parallel to the Y axis and MH parallel to the X axis.

NG and MH intersect at P.

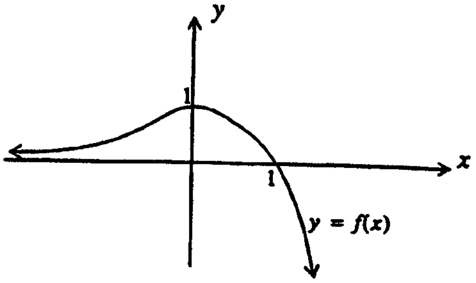
(i) Show that the parametric equations for the locus of P in terms of θ are given by $x = 5\cos\theta$ and $y = 3\sin\theta$.

(ii) By eliminating θ, find the Cartesian equation of this locus.

QUESTION 4. Use a *separate* Writing Booklet.

Marks

(a) The graph of $y = f(x)$ is sketched below. There is a stationary point at $(0, 1)$. **7**



Use this graph to sketch the following without using calculus, showing essential features.

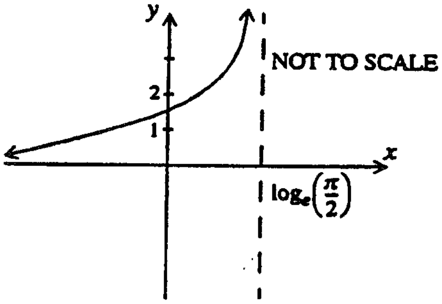
(i) $y = f\left(\frac{x}{2}\right)$.

(ii) $y = x + f(x)$.

(iii) $y = \dfrac{1}{f(x)}$.

(iv) $y = f\left(\frac{1}{x}\right)$.

(b) The diagram shows part of the curve $y = \tan(e^x)$, where $x < \log_e\left(\frac{\pi}{2}\right)$. The part to the right of $\log_e\left(\frac{\pi}{2}\right)$ has not yet been drawn. **8**



(i) By considering values of x greater than $\log_e\left(\frac{\pi}{2}\right)$, find the smallest positive solution to the equation $\tan(e^x) = 0$.

(ii) Copy the diagram and hence sketch the curve $y = \tan(e^x)$ for $x < \log_e\left(\frac{3\pi}{2}\right)$.

(iii) How many solutions are there to the equation $\tan(e^x) = 0$ in the domain $1 < x < 3$?

(iv) Find the equation of the inverse function of $y = \tan(e^x)$ for the case when

(α) $\log_e\left(\frac{\pi}{2}\right)$.

(β) $\log_e\left(\frac{\pi}{2}\right) < x < \log_e\left(\frac{3\pi}{2}\right)$.

QUESTION 5 Use a *separate* Writing Booklet. **Marks**

(a) Suppose the equation $x^3 + px^2 + qx + r = 0$, where p, q and r are real, has three distinct positive real roots α, β and γ. **6**

(i) Explain why r must be negative.

(i) Show that $\frac{\alpha+\beta}{\gamma} + \frac{\beta+\gamma}{\alpha} + \frac{\gamma+\alpha}{\beta} = \frac{pq}{r} - 3$.

(iii) Hence show that $(p+q+r)^2 > 2r(p+q+3)$.

(b) The hyperbola $\mathcal{H}$ has equation $9x^2 - 4y^2 = 36$. **9**

(i) Find the co-ordinates of the foci, S and S'.

(ii) Find the equations of the directrices.

(iii) Find the equations of the asymptotes.

(iv) Sketch the curve $\mathcal{H}$ indicating the information obtained in (i)–(iii).

(v) The point $P(x_0, y_0)$ lies on $\mathcal{H}$. Prove that the equation of the tangent at P is $9x_0x - 4y_0y = 36$.

(vi) The tangent at P meets the asymptotes at A and B. Prove that P is the midpoint of AB.

QUESTION 6. Use a *separate* Writing Booklet. **Marks**

(a) Sketch the region where the inequalities $-\frac{\pi}{2} \leq \arg(z-1-2i) \leq \frac{\pi}{4}$ and $|z| \leq \sqrt{5}$ both hold. **3**

(b) Find $\int \frac{dx}{\sqrt{3+2x-x^2}}$. **2**

END OF PAPER

STANDARD INTEGRALS

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x}\,dx = \ln x, \quad x > 0$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax}, \quad a \neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax, \quad a \neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax, \quad a \neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax\,dx = \frac{1}{a}\sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a}, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \ln\left(x+\sqrt{x^2-a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \ln\left(x+\sqrt{x^2+a^2}\right)$$

NOTE: $\ln x = \log_e x, \quad x > 0$

4 UNIT HLY SOLUTIONS '99 .

Solutions	Marks	Comments
Question 1.		
(a) (i) $\frac{1+4x}{(4-x)(x^2+1)} = \frac{A}{4-x} + \frac{Bx+c}{x^2+1}$		
∴ $1+4x = A(x^2+1)+(4-x)(Bx+c)$	1	
If $x=0$ $A+4C=1$ — ①		
$x=4$ $17A=17$ — ②		
$x=1$ $2A+3B+C=5$ — ③	1	
from ② $A=1$		
Sub in ① $C=0$	1	any correct method.
Sub in ③ $B=1$		
(ii) $\int_0^2 \frac{1+4x}{(4-x)(x^2+1)}dx = \int_0^2 \left[\frac{1}{4-x}+\frac{x}{x^2+1}\right]dx$		
$= \left[-\ln(4-x)+\frac{1}{2}\ln(x^2+1)\right]_0^2$	1	Correct integration
$= \left[\ln\frac{\sqrt{x^2+1}}{4-x}\right]_0^2$		
$= \ln\frac{\sqrt{5}}{2} - \ln\frac{1}{4}$		
$= \ln(2\sqrt{5})$	1	Evaluation
	⑤	

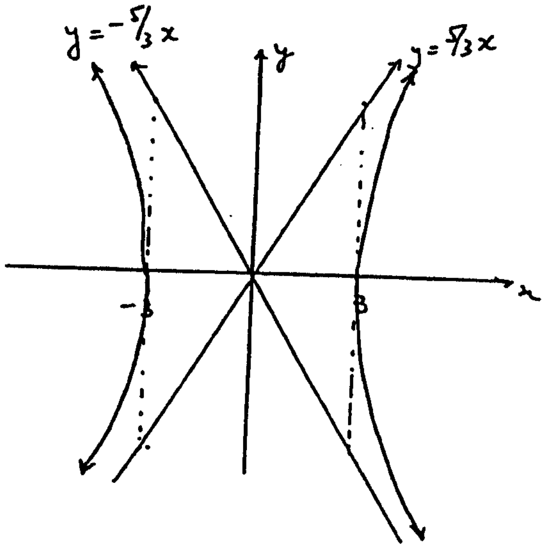
Solutions	Marks	Comments
Question 1. (contd) (b) Over the complex field C $\frac{5}{(x^2+4)(x^2+9)} = \frac{a}{x+2i} + \frac{b}{x-2i} + \frac{c}{x+3i} + \frac{d}{x-3i}$, where a, b, c, d $\in$ C	1	
Thus $5 = a(x-2i)(x+3i)(x-3i)+b(x+2i)(x+3i)(x-3i)$ $+c(x+2i)(x-2i)(x-3i)+d(x+2i)(x-2i)(x+3i)$ Letting $x = -2i,\ 2i,\ -3i,\ 3i$ in order, we obtain $a(-4i)(i)(-5i)=5$ i.e. $a=\frac{i}{4}$..(1), $b(4i)(5i)(-i)=5$, i.e. $b=\frac{-i}{4}$..(2) $c(-i)(-5i)(-6i)=5$ i.e. $c=\frac{-i}{6}$..(3), $d(5i)(i)(6i)=5$ i.e. $d=\frac{i}{6}$	1	
Thus $\frac{5}{(x^2+4)(x^2+9)} = \frac{i/4}{x+2i} - \frac{i/4}{x-2i} - \frac{i/6}{x+3i} + \frac{i/6}{x-3i}$	1	
	③	
(c) $\int_0^{\frac{\pi}{3}} \frac{dx}{1-\sin x}$ If $t = \tan\frac{x}{2}$ $\begin{cases} x=\frac{\pi}{3} & t=\frac{1}{\sqrt3} \\ x=0 & t=0 \end{cases}$	1	correct limits
$dt = \frac{1}{2}\sec^2\frac{x}{2}\,dx$ $= \frac{1}{2}\left[1+\tan^2\frac{x}{2}\right]dx$ $= \frac{1}{2}(1+t^2)\,dx$ $\therefore \frac{2dt}{1+t^2} = dx$ & $\sin x = \frac{2t}{1+t^2}$	1	correct sub for dx.
$\therefore \int_0^{\frac{1}{\sqrt3}} \frac{1}{1-\frac{2t}{1+t^2}} \cdot \frac{2dt}{1+t^2} = \int_0^{\frac{1}{\sqrt3}} \frac{2dt}{1+t^2-2t}$	1	correct sub. & simplifying
$= \int_0^{\frac{1}{\sqrt3}} \frac{2dt}{(t-1)^2}$	1	integration
$= \left[\frac{-2}{t-1}\right]_0^{\frac{1}{\sqrt3}}$ $= \frac{-2}{\frac{1}{\sqrt3}-1} - \frac{-2}{-1}$ $= \sqrt3 + 1$	1	evaluation
	⑤	

Solutions	Marks	Comments
Question 1 (contd)		
(d)		
$\int_3^4 \frac{dx}{\sqrt{x^2-4}} = \left[\ln\left\{x+\sqrt{x^2-4}\right\}\right]_3^4$	1	from table given.
$= \ln\left\{4+\sqrt{12}\right\} - \ln\left\{3+\sqrt{5}\right\}$		
$= \ln\left\{\frac{4+\sqrt{12}}{3+\sqrt{5}}\right\}$	1	evaluation
	②	

Solutions	Marks	Comments
Question 2		
(a)(i) $A+B=(3+4i)+(5-13i)$		
$=8-9i$	1	
(ii) $AB=(3+4i)(5-13i)$		
$=15-39i+20i-52i^2$	1	
$=67-19i$		
(iii) $\frac{A}{B}=\frac{3+4i}{5-13i}\times\frac{5+13i}{5+13i}$		
$=\frac{15+39i+20i+52i^2}{25+169}$	1	
$=\frac{-37+59i}{194}$		
(IV) $\sqrt{A}=\sqrt{3+4i}$		
Let $a+ib=\sqrt{3+4i}$ (a&b real)		
$\therefore a^2-b^2+2abi=3+4i$		any correct method.
Equating part		
$a^2-b^2=3$ — ①		
$2ab=4\Rightarrow(ab=2)$ — ②		
①^2+②^2 $a^4+b^4+2a^2b^2=25$		
$(a^2+b^2)^2=25$		
$a^2+b^2=5$ — ③		
①+③ $2a^2=8$		
$a=\pm2$		
$\therefore b=\pm1$	2	
$\therefore\sqrt{3+4i}=\pm(2+i)$		
	⑤	

Solutions	Marks	Comments
Question 2 (continued)		
(b) $Z = 1+\sqrt{3}i$		
(i) Mod $Z = \sqrt{1+3} = 2$.	1	
(ii) $Z = 2(\frac{1}{2}+\frac{\sqrt{3}}{2}i)$ If $\theta = \arg Z$ $\cos\theta = \frac{1}{2}$ $\sin\theta = \frac{\sqrt{3}}{2}$ $\therefore \theta = \frac{\pi}{3}$	1	
(iii) $Z = 2(\cos\frac{\pi}{3}+i\sin\frac{\pi}{3})$	1	
(iv) $\sqrt{Z} = Z^{\frac{1}{2}} = \sqrt{2}(\cos\frac{\pi}{3}+i\sin\frac{\pi}{3})^{\frac{1}{2}}$ By DeMoivre's Theorem $\sqrt{Z} = \sqrt{2}(\cos\frac{\pi}{6}+i\sin\frac{\pi}{6})$	2 ⑤	
(c) (i) Let $Z = x+iy$ $\therefore Z-1 = x-1+iy$ let $\arg(Z-1) = \theta$ $\therefore Z+1 = x+1+iy$ $\therefore \arg Z+1 = \alpha$ $\theta-\alpha = \frac{\pi}{3}$ $\therefore \tan(\theta-\alpha) = \sqrt{3}$ $\frac{\tan\theta-\tan\alpha}{1+\tan\theta\tan\alpha} = \sqrt{3}$ $\tan\theta = \frac{y}{x-1}$ $\tan\alpha = \frac{y}{x+1}$ $\therefore \frac{\frac{y}{x-1}-\frac{y}{x+1}}{1+\frac{y}{x-1}\cdot\frac{y}{x+1}} = \sqrt{3}$ $\frac{y\cancel{x}+y-y\cancel{x}+y}{x^2+y^2-1} = \sqrt{3}$ $2y = \sqrt{3}x^2+\sqrt{3}y^2-\sqrt{3}$ $\therefore$ Cartesian equation $x^2+y^2-\frac{2}{\sqrt{3}}y-1 = 0$	3	Full marks if done geometrically using angles in same segment.

Solutions	Marks	Comments
Question 2 (continued) $x^2+y^2-\frac{2}{\sqrt{3}}y = 1$ $x^2+y^2-\frac{2}{\sqrt{3}}y+\frac{1}{3} = 1+\frac{1}{3}$ $(x-0)^2+\left(y-\frac{1}{\sqrt{3}}\right)^2 = \frac{4}{3}$ ∴ The locus is a circle centre $\left(0, \frac{1}{\sqrt{3}}\right)$ radius $\frac{2}{\sqrt{3}}$ units	2 ⑤	

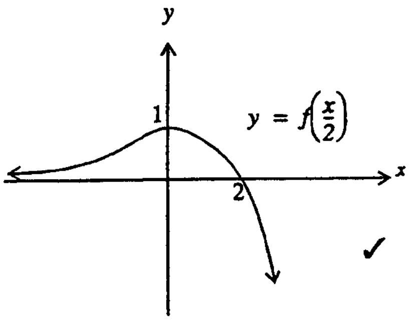
Solutions	Marks	Comments
3(a) (i) 	1	
(ii) The equation of the hyperbola is $\frac{x^2}{9}-\frac{y^2}{25}=1$	1	
	②	
b)(i) If $\frac{x^2}{25}+\frac{y^2}{9}=1$ and $x=4$ then $\frac{y^2}{9}=1-\frac{16}{25}=\frac{9}{25}$	1	
So $y^2=\frac{81}{25}$ $y=\pm\frac{9}{5}$	1	
Thus A is $(4, \frac{9}{5})$ and B is $(4, -\frac{9}{5})$.		
(ii) From $\frac{y^2}{9}=1-\frac{x^2}{25}$ Differentiating w.r.t x $\frac{2y}{9}\frac{dy}{dx}=\frac{-2x}{25}$	1	
$\frac{dy}{dx}=\frac{-2x}{25}\times\frac{9}{2y}=\frac{-9x}{25y}$	1	
At A, $\frac{dy}{dx}=\frac{-36}{45}=-4/5$	1	

Solutions	Marks	Comments
Thus the equation of the tangent through A is		
$y-\frac{9}{5} = -\frac{4}{5}(x-4)$	1	
$5y-9 = -4x+16$		
i.e. $4x+5y-25=0$.		
(iii) The equation of the normal through A is	1	
$y-\frac{9}{5} = \frac{5}{4}(x-4)$		
$4y-\frac{36}{5} = 5x-20$		
i.e. $4y-5x+12\frac{4}{5}=0$		
(iv) The eccentricity e is given by		
$e^2 = 1-\left(\frac{b}{a}\right)^2$		
$= 1-\left(\frac{3}{5}\right)^2$		
$= \frac{16}{25}$		
$\therefore e = 4/5$.	1	
(v) The directrices are given by		
$x = \pm\frac{a}{e}$		
$= \pm\frac{5}{4/5}$		
$= \pm 6\frac{1}{4}$.	1	
	⑨	

Solutions	Marks	Comments
Question 3 (contd)		
(c)(i) $x = OG$ $\quad y = OH$		
$x = 5\cos\theta$ $\quad y = 3\sin\theta$	2	1 each
(ii) $x = 5\cos\theta$ $\quad y = 3\sin\theta$		
$\frac{x}{5} = \cos\theta$ $\quad \frac{y}{3} = \sin\theta$		
$\frac{x^2}{25} = \cos^2\theta$ $\quad \frac{y^2}{9} = \sin^2\theta$		
$\therefore \frac{x^2}{25} + \frac{y^2}{9} = \cos^2\theta + \tan^2\theta$	1	
$\frac{x^2}{25} + \frac{y^2}{9} = 1$ is the	1	
cartesian equation		
	④	

QUESTION 4.

(a) (i)



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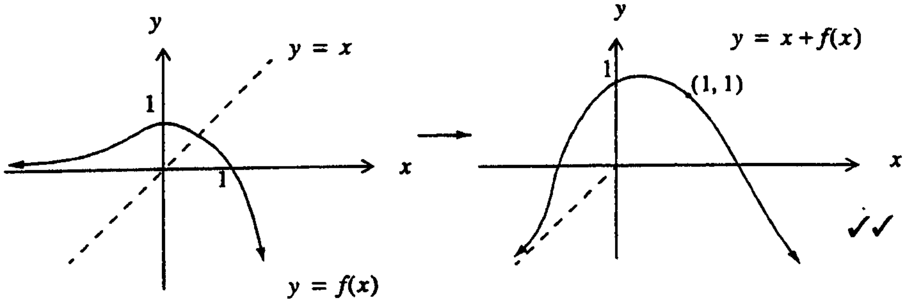
(ii)



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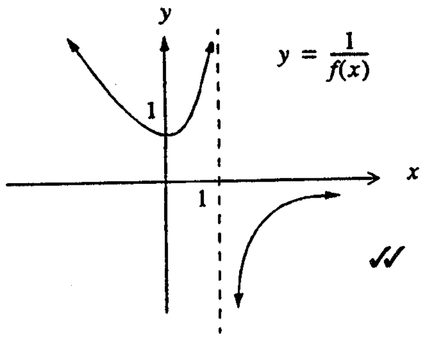
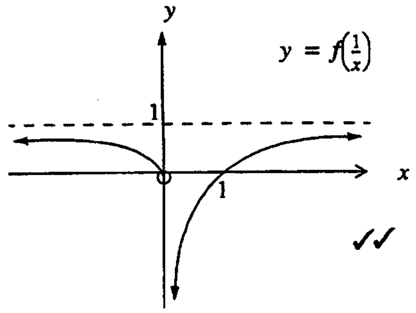
(iii)



✓✓

(iv)



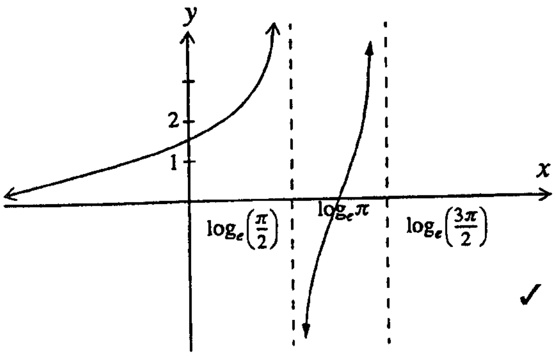
✓✓

(b) (i) $\tan e^x = 0$

$\therefore e^x = \pi$ (smallest positive solution) ✓

$\therefore x = \log_e \pi$

(ii)



✓

(iii) $$e^x = \pi, 2\pi, \ldots$$
$$x = \log_e\pi, \log_e 2\pi, \ldots$$
Now $\log_e 6\pi = 2.93$, and $\log_e 7\pi > 3$

$\therefore$ there are 6 solutions in the domain $1 < x < 3$. ✓✓

(iv) (α) $$y = \tan(e^x)$$
Inverse is $x = \tan(e^y)$ ✓ $(y < \log_e\frac{\pi}{2})$
$$e^y = \tan^{-1}x$$
$$\therefore y = \log_e(\tan^{-1}x)$$ ✓

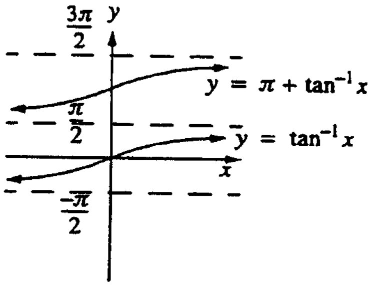
(β) When $\log_e\frac{\pi}{2} < x < \log_e\frac{3\pi}{2}$

then the equation of the original function is

$y = \tan(e^x - \pi)$,

and so the equation of the inverse function is

$y = \log_e(\pi + \tan^{-1}x)$ ✓✓



5 (b) (i) $9x^2 - 4y^2 = 36$

$$\text{i.e. } \frac{x^2}{4} - \frac{y^2}{9} = 1\text{, so that}$$

$$a = 2 \text{ and}$$

$$b = 3$$

$$b^2 = a^2(e^2 - 1)$$

$$\therefore 9 = 4(e^2 - 1)$$

$$\therefore e^2 = \frac{13}{4}$$

$$\therefore e = \frac{\sqrt{13}}{2} \; (e > 0) \quad ✓$$

$$\therefore ae = \sqrt{13}$$

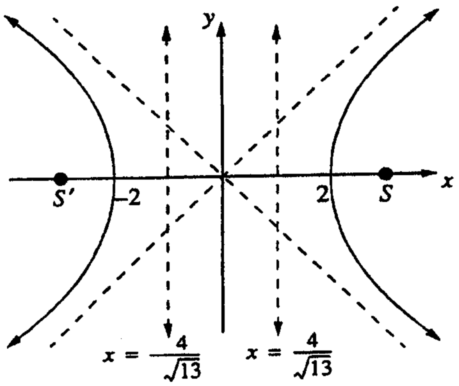
$\therefore$ the foci have coordinates $S(\sqrt{13},0)$ and $S'(-\sqrt{13},0)$ ✓

(ii) Directrices have equations $x = \frac{a}{e}$ or $-\frac{a}{e}$

i.e. $x = \frac{4}{\sqrt{13}}$ or $-\frac{4}{\sqrt{13}}$ ✓

(iii) Asymptotes have equations $y = \pm\frac{bx}{a}$, i.e. $y = \pm\frac{3x}{2}$ ✓

(iv)



✓

(v) Differentiating both sides of the equation $9x^2 - 4y^2 = 36$ with respect to x, we have:

$$18x - 8y\frac{dy}{dx} = 0$$

$$\therefore \frac{dy}{dx} = \frac{18x}{8y}$$

$$\therefore \text{ At } P(x_0, y_0), m = \frac{9x_0}{4y_0} \quad ✓$$

$$\therefore \text{Equation of tangent is } y - y_0 = \frac{9x_0}{4y_0}(x - x_0)$$

$$\text{i.e. } 4y_0y - 4y_0^2 = 9x_0x - 9x_0^2$$

$$\text{or } 9x_0x - 4y_0y = 9x_0^2 - 4y_0^2$$

$$\therefore 9x_0x - 4y_0y = 36, \quad ✓$$

since $P(x_0, y_0)$ satisfies $9x^2 - 4y^2 = 36$

(vi) Solving $9x_0x - 4y_0y = 36$ simultaneously with $y = \pm\frac{3x}{2}$, we have:

$$9x_0x - 4y_0\left(\frac{3x}{2}\right) = 36 \qquad \text{and} \quad 9x_0x - 4y_0\left(\frac{-3x}{2}\right) = 36$$

$$\text{i.e. } 9x_0x - 6y_0x = 36 \qquad \text{and} \qquad 9x_0x + 6y_0x = 36$$

$$\therefore x = \frac{36}{9x_0 - 6y_0} \quad \text{and} \qquad x = \frac{36}{9x_0 + 6y_0}$$

$$= \frac{12}{3x_0 - 2y_0} \qquad\qquad x = \frac{12}{3x_0 + 2y_0}$$

$\therefore$ the points of intersection of the tangent with the asymptotes are:

$$A\left(\frac{12}{3x_0 - 2y_0}, \frac{18}{3x_0 - 2y_0}\right) \text{ and } B\left(\frac{12}{3x_0 + 2y_0}, \frac{-18}{3x_0 + 2y_0}\right) \checkmark$$

$$\therefore \text{Midpoint } AB = \left(\frac{1}{2}\left(\frac{12}{3x_0 - 2y_0} + \frac{12}{3x_0 + 2y_0}, \frac{18}{3x_0 - 2y_0} + \frac{-18}{3x_0 + 2y_0}\right)\right)$$

$$= \left(\frac{36x_0}{9x_0^2 - 4y_0^2}, \frac{36y_0}{9x_0^2 - 4y_0^2}\right),$$

but $9x_0^2 - 4y_0^2 = 36$

$$\therefore \text{Midpoint } AB = \left(\frac{36x_0}{36}, \frac{36y_0}{36}\right)$$

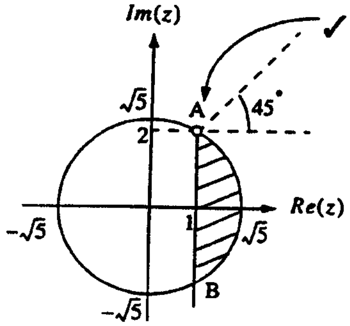
$$= (x_0, y_0)$$

$$= P \quad \checkmark$$

$\therefore$ P is the midpoint of AB.

QUESTION 6.

(a)



✓ (for showing that point A is <u>not</u> included, but B is included)

✓✓

(b) $$\int \frac{dx}{\sqrt{3+2x-x^2}} = \int \frac{dx}{\sqrt{4-(x-1)^2}}$$ ✓

$$= \sin^{-1}\frac{(x-1)}{2} + c$$ ✓

15