Total Marks (120)

Attempt Questions 1 – 8

All questions are of equal value

Answer each question in a SEPARATE writing booklet. Extra writing booklets are available.

Question 1 (15 marks) Use a SEPARATE writing booklet. **Marks**

(a) (i) Express $\beta = 1 - \sqrt{3}\,i$ and $z = 1 + i$ in modulus-argument form. **2**

(ii) Consider the product βz.

(α) Find $|\beta z|$. **2**

(β) Find $\arg(\beta z)$. **2**

(iii) With the aid of a diagram, give a geometric description of the transformation $z \to \beta z$. **3**

(b) Graph the locus specified by $\arg(z - 1) = \frac{\pi}{6}$ **2**

(c) By letting $z = x + iy$ describe with the use of a diagram the locus represented by $\text{Re}\left(z - \frac{1}{z}\right) = 0$. **4**

Question 2 (15 marks) Use a SEPARATE writing booklet. **Marks**

(a) (i) Determine the type of curve represented by the equation $\frac{x^2}{k} + \frac{y^2}{k-16} = 1$ in each of the following cases.

(α) $k > 16$ **1**

(β) $0 < k < 16$ **1**

(ii) Show that all curves in part (a) above have the same foci, no matter what the value of k is. Write down this value of the foci. **5**

(Hint: Use two cases—one for $k > 16$ and one for $0 < k < 16$.)

(b) (i) Sketch the hyperbola $\frac{x^2}{9} - \frac{y^2}{16} = 1$, showing foci, directrices, vertices and asymptotes. **4**

(ii) Write the parametric equation for the hyperbola $\frac{x^2}{9} - \frac{y^2}{16} = 1$. **2**

(c) If $\frac{3x-2}{(x-1)(x-2)} = \frac{A}{x-1} + \frac{B}{x-2}$ for all values of x, and if A and B are coefficients independent of x, find A and B. **2**

Question 3 (15 marks) Use a SEPARATE writing booklet. **Marks**

(a) (i) Find the square roots of $5 - 12i$. **3**

(ii) Hence, or otherwise, solve the equation $x^2 - 3x + 1 + 3i = 0$. **2**

(b) The equation $x^3 + ax^2 + bx + 15 = 0$ ($a, b \in R$) has $x = 2 + i$ as one root.

(i) Find a and b. **3**

(ii) Solve the equation. **2**

(c) Given that the equation $x^4 - 5x^3 - 9x^2 + 81x - 108 = 0$ has a triple root, find all roots of this equation. **5**

Question 4 (15 marks) Use a SEPARATE writing booklet. **Marks**

(a) If α, β and γ are the roots of $6x^3 - 2x^2 + 3x - 4 = 0$, form an equation whose roots are:

(i) $\frac{1}{\alpha}, \frac{1}{\beta}, \frac{1}{\gamma}$ **2**

(ii) $\frac{1}{\alpha^2}, \frac{1}{\beta^2}, \frac{1}{\gamma^2}$ (Hint: Use answer from part (i)). **2**

(b) (i) Use De Moivre's Theorem to express cos 5θ and sin 5θ in powers of sinθ and cosθ. **3**

(ii) Hence, express tan 5θ as a rational function of t where $t = \tan\theta$. **3**

(iii) Deduce that $\tan\frac{\pi}{5}\tan\frac{2\pi}{5}\tan\frac{3\pi}{5}\tan\frac{4\pi}{5} = 5$. **5**

(Hint: Solve tan 5θ = 0, and equate this with the rational function of t.)

Question 5 (15 marks) Use a SEPARATE writing booklet. **Marks**

(a) The complex number w, where $w \neq 1$, is a root of the equation $z^3 - 1 = 0$.

(i) Show that $1 + w + w^2 = 0$. **2**

(ii) Show that $1 + (1 + w)^3 = 0$. **3**

(b) If β is a complex number, use De Moivre's Theorem to prove that $\overline{\beta^n} = (\overline{\beta})^n$ **3**

(c) Find and plot on the Argand diagram the fifth roots of –32. **3**

(d) $P(x)$ is a polynomial of degree 4 with real coefficients. **4**

The complex number α satisfies $\text{Im}(\alpha) \neq 0$, $\text{Re}(\alpha) = a$, and $|\alpha| = r$.

Show that if α is a zero of $P(x)$, then $P(x)$ has a factor of $x^2 - 2ax + r^2$ over the field of real numbers.

Question 6 (15 marks) Use a SEPARATE writing booklet. **Marks**

(a) Given that the focus of an ellipse is S(ae, 0) and the directrix is $x = \frac{a}{e}$, show with the aid of a diagram that $b^2 = a^2(1 - e^2)$. **4**

(b) Let $z = x + iy$ be any non-zero complex number.

(i) Express $z + \frac{1}{z}$ in the form of $a + ib$ **2**

(ii) Given that $z + \frac{1}{z} = k$ where k is real, show that either $y = 0$ or $x^2 + y^2 = 1$. **3**

(iii) Using $z = x + iy$, show that, if $y = 0$, then $|k| \geq 2$ **3**

(iv) Using your result from part (ii) above, show that, if $x^2 + y^2 = 1$, then $|k| \leq 2$. **3**

Question 7 (15 marks) Use a SEPARATE writing booklet. **Marks**

(a) If α, β and γ are the roots of $2x^3 + 4x^2 - 2x - 2 = 0$, find the value of $(\alpha + \beta - 2)(\alpha + \gamma - 2)(\beta + \gamma - 2)$. **3**

(b) If one root of the equation $x^3 + ax^2 + bx + c = 0$ is the sum of the other two roots, show that $a^3 + 4ab + 8c = 0$. **5**

(c) The complex number z is a function of the real number t such that

$z = \dfrac{t-i}{t+i}$ where $\dfrac{1}{\sqrt{3}} \le t \le 1$.

(i) Show that $|z| = 1$. **1**

(ii) Find the values of z for $t = \dfrac{1}{\sqrt{3}}$ and $t = 1$. **2**

(iii) Hence, or otherwise, write down $\arg(z)$ for $t = \dfrac{1}{\sqrt{3}}$ and $t = 1$. **2**

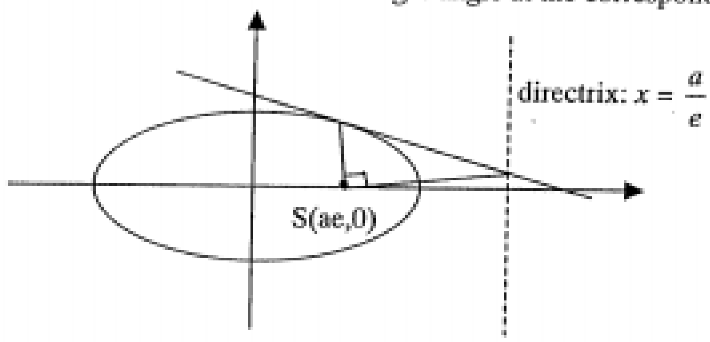
(iv) Describe the locus of z in the Argand diagram as t varies from $\dfrac{1}{\sqrt{3}}$ to 1. **2**

Show clearly where z starts and finishes.

Question 8 (15 marks) Use a SEPARATE writing booklet. **Marks**

(a) Consider the ellipse with equation $\frac{x^2}{25}+\frac{y^2}{9}=1$.

Prove that the section of the tangent between the point of contact and its point of intersection with the directrix subtends a right angle at the corresponding focus. **7**



(b) P (a cosθ, b sinθ) is a point on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$. A line L through the origin and parallel to the tangent at P meets the ellipse at R.

(i) Show that the equation of the line L is given by $\frac{x\cos\theta}{a}+\frac{y\sin\theta}{b}=0$ **2**

(ii) Show that R = ($a$sinθ, $-b$cosθ) **2**

(iii) Show that the perpendicular distance of P from line L is given by **2**

$$\frac{ab}{\sqrt{b^2\cos^2\theta+a^2\sin^2\theta}}$$

(iv) Prove that the area of ΔOPR is $\frac{1}{2}ab$. **2**

END OF EXAMINATION