Name:

INTERNATIONAL GRAMMAR SCHOOL

1999

MATHEMATICS

4 UNIT

HALF YEARLY EXAMINATION

YEAR 12

Time allowed --- 2 hours
(Plus 5 minutes reading time)

DIRECTIONS TO CANDIDATES

- Attempt ALL questions.
- Questions are NOT of equal value.
- All necessary working should be shown in every question. Marks may be deducted for careless or badly arranged work.
- Board-approved calculators may be used.
- Start each question on a *new page*. Number each question clearly.
- Label each page with your name.

Question1: **(20 marks) Start a new page.**

(a) $z_1 = 1 + 3i, \quad z_2 = 1 - i$ **[8]**

i. Find in the form a + ib, where a and b are real, the numbers z_1z_2 and $\frac{z_1}{z_2}$.

ii. On an Argand Diagram the vectors $\overrightarrow{OA}, \overrightarrow{OB}$ represent the complex numbers z_1z_2 and $\frac{z_1}{z_2}$ respectively (where z_1 and z_2 are given above). Show this on an Argand Diagram, giving the coordinates of A and B. From your diagram, deduce that $\frac{z_1}{z_2} - z_1z_2$ is real.

[6]

(b) $-3 + 4i$ has two square roots z_1 and z_2. Find z_1 and z_2 in the form a + ib and show the points representing $-3 + 4i$, z_1 and z_2 on an Argand Diagram. Show that these three points are the vertices of a right angled triangle.

[6]

(c) The complex number z is represented by the point P on an Argand Diagram. Indicate clearly on a single diagram the locus of P in each of the following cases:

i. $|z - 4| = |z + 2i|$

ii. $\arg(z + 3) = \frac{\pi}{4}$

Show that there is a point representing a complex number of the form ib, where b is real, which lies on both loci.

ion2: **(15 marks) Start a new page**

[6]

(a) i. Expand $z = (1 + ic)^6$ in powers of c.

ii Hence find the five real values of c for which z is real.

[4]

(b)

Let $w = \frac{3 + 4i}{5}$ and $z = \frac{5 + 12i}{13}$, so that $|w| = |z| = 1$.

i. Find wz and $w\bar{z}$ in the form x + iy.

ii. Hence find two distinct ways of writing 65^2 as the sum $a^2 + b^2$, where a and b are integers and $0 < a < b$

[5]

(c)

i. Show that $(1 - 2i)^2 = -3 - 4i$

ii. Hence solve the equation $z^2 - 5z + (7 + i) = 0$.

ion 3: **(15 marks)** **Start a new page**

[8]

(a)i. Sketch the graph of $f(x) = x^3 - 3x$ showing clearly the coordinates of any points of intersection with the x axis and the coordinates of any turning points.

ii. Use the graph of $y = f(x)$ in part (i) to sketch the graph of $y = |f(x)|$ showing clearly the coordinates of any critical points (where $\frac{dy}{dx}$ is not defined) and the coordinates of any turning points.

iii. Use the graph of $y = f(x)$ in part (i) to sketch the graph of $y = \frac{1}{f(x)}$ showing clearly the equations of any asymptotes and the coordinates of any turning points.

[7]

(b)i. On the same set of axes, sketch and label clearly the graphs of the functions $y = x^{\frac{1}{3}}$ and $y = e^x$.

ii. Hence, on a different set of axes, without using calculus, sketch and label clearly the graph of the function $y = x^{\frac{1}{3}}e^x$.

iii. Use your sketch to determine for which values of m the equation $x^{\frac{1}{3}}e^x = mx + 1$ has exactly one solution.

stion 4: **(15 marks)** **Start a new page**

[10]

Let $f(x) = -x^2 + 6x - 8$. On separate diagrams, and without using calculus, sketch the following graphs. Indicate clearly any asymptotes and intercepts with the axes.

i. $y = f(x)$

ii. $y = |f(x)|$

iii. $y^2 = f(x)$

iv. $y = \frac{1}{f(x)}$

v. $y = e^{f(x)}$.

[5]

i. If α is a double zero of the polynomial $P(x)$, show that α is a zero of $P'(x)$.

ii $(x - 1)^2$ is a factor of $x^5 + 2x^4 + ax^3 + bx^2$. Find the values of a and b.

stion 5: **(15 marks)** **Start a new page**

[3]

i. Express $\sqrt{3} - i$ in modulus-argument form.

ii. Hence evaluate $(\sqrt{3} - i)^8$

[5]

$\sqrt{3} + i$ is one root of $x^4 + px^2 + q = 0$, where p and q are real. Find p and q and factor $x^4 + px^2 + q$ into quadratic factors with real coefficients.

[7]

) The quadratic equation $x^2 - x + k = 0$, where k is a real number, has two distinct positive real roots α and β.

i. Show that $0 < k < \frac{1}{4}$.

ii. Show that $\alpha^2 + \beta^2 = 1 - 2k$ and deduce that $\alpha^2 + \beta^2 > \frac{1}{2}$.

iii. Show that $\frac{1}{\alpha^2} + \frac{1}{\beta^2} > 8$

THE END

Question 1.

(a)(i) $z_1 z_2 = (1+3i)(1-i)$

$= 1+3+i(-1+3)$ 1

$= 4+2i$

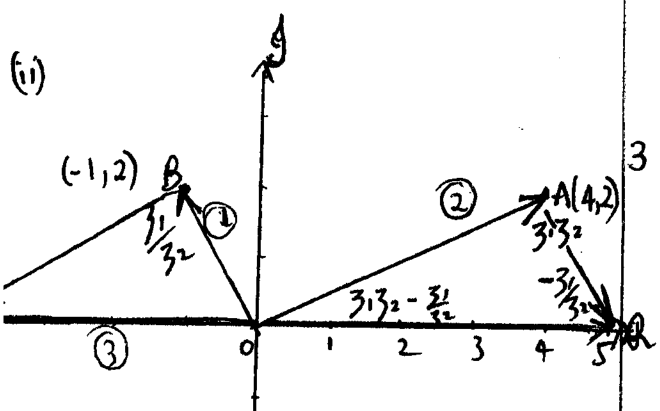
$\dfrac{z_1}{z_2} = \dfrac{(1+3i)}{1-i} \times \dfrac{1+i}{1+i}$ 1

$= \dfrac{1-3+i(1+3)}{1+1}$ 1

$= \dfrac{-2+4i}{2}$

$= -1+2i$ 1

(ii)



3 see diagram

$z_1z_2 + \dfrac{z_1}{z_2} = (4 - -1) + i(2-2)$

$= 5$

$=$ Real 1

(b) $-3+4i = (a+ib)^2$

$-3+4i = a^2 + 2aib - b^2$ 1

equating $\mathbb{R}$ and $\mathbb{I}$ parts.

$a^2 - b^2 = -3 \quad \ldots ①$

$2ab = 4 \quad \ldots ② \Rightarrow b = \dfrac{2}{a}$ 1 working

substitution ② into ①

$a^4 + 3a^2 - 4 = 0$

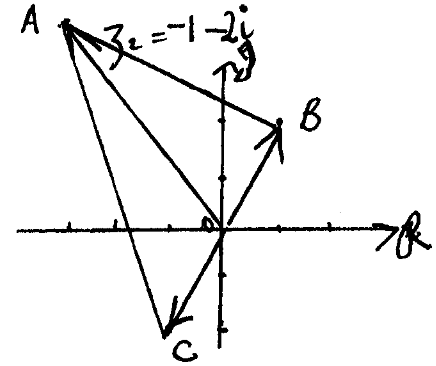
$(a^2-1)(a^2+4) = 0$

$a^2 = 1$ or ($a^2 = 4$ not possible) as a is real.

$a = \pm 1 \quad \therefore \quad b = \pm 2$

$\therefore \quad z_1 = 1+2i$ 1

$z_2 = -1-2i$ 1



let OA represent $-3+4i$

OB represent $1+2i$

OC represent $-1-2i$

$\therefore$ AB is $-3+4i-(1+2i)$

$= -4+2i$ 2 proof

and $OB \times 2i = (1+2i)\,2i$

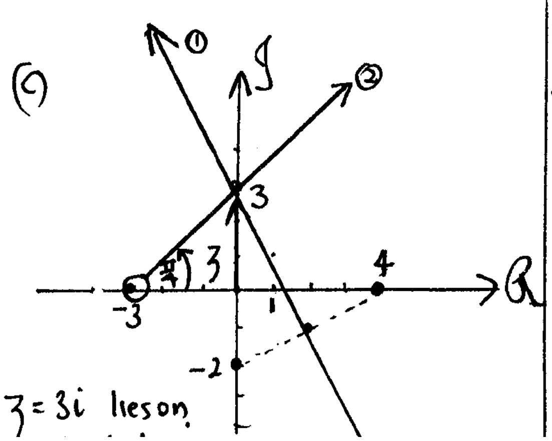
$= -4+2i$

So $AB \perp OB$ and

OB and OC are collinear

So ABC is a rt $\angle$ $\triangle$

(c)



2 2 locus

$z = 3i$ lies on 2

Question 2:

(a) $z = (1+ic)^6$

using binomial expansion

$z = 1 + \binom{6}{1} ic + \binom{6}{2}(ic)^2 + \binom{6}{3} ic^3 + \ldots \binom{6}{6}(ic)^6$

$z = 1 - 15c^2 + 15c^4 - c^6 + i(6c - 20c^3 + 6c^5)$ 2

z is real when $6c - 20c^3 + 6c^5 = 0$ 1

$2c(3 - 10c^2 + 3c^4) = 0$

$2c(3c^2 - 1)(c^2 - 3) = 0$ 1

$\therefore c = 0,\ c = \pm \frac{1}{\sqrt{3}},\ c = \pm\sqrt{3}$ 2 (1-½ each error)

b) $w \cdot z = \left(\frac{3+4i}{5}\right) \times \left(\frac{5+12i}{13}\right)$

$= \frac{1}{65}\left(15 - 48 + i[36 + 20]\right)$

$= \frac{1}{65}(-33 + 56i)$ 1

$w \cdot \bar{z} = \left(\frac{3+4i}{5}\right) \times \left(\frac{5-12i}{13}\right)$

$= \frac{1}{65}\left(15 + 48 + i[-36 + 20]\right)$ 1

$= \frac{1}{65}(63 - 16i)$

given $|w| = |z| = 1$

$\therefore |wz| = \sqrt{\left(\frac{1}{65}\right)^2[33^2 + 56^2]} = 1$

$\Rightarrow 33^2 + 56^2 = 65^2$ 1

$|w\bar{z}| = \sqrt{\left(\frac{1}{65}\right)^2[63^2 + 16^2]} = 1$

$\Rightarrow 63^2 + 16^2 = 65^2$

$16^2 + 63^2 = 65^2$ 1

(c)(i) $(1-2i)(1-2i) = 1 - 4 + i(-2-2)$

$= -3 - 4i$ qed 1

(ii) $z^2 - 5z + (7+i) = 0$

$z = \frac{5 \pm \sqrt{5^2 - 4(7+i)}}{2}$ 1

$= \frac{5 \pm \sqrt{25 - 28 - 4i}}{2}$

$= \frac{5 \pm \sqrt{-3-4i}}{2}$

$= \frac{5 + 1 - 2i}{2}$ or $\frac{5 - 1 + 2i}{2}$ 1

$z_1 = 3 - i$ or $z_2 = 2 + i$ 1, 1

15

Question 3:

(a) $f(x) = x^3 - 3x$

$f(x) = 0 \Rightarrow x^3 - 3x = 0$

$x(x^2 - 3) = 0$

$x = 0,\ x = \sqrt{3},\ x = -\sqrt{3}$ 1

$f'(x) = 3x^2 - 3$

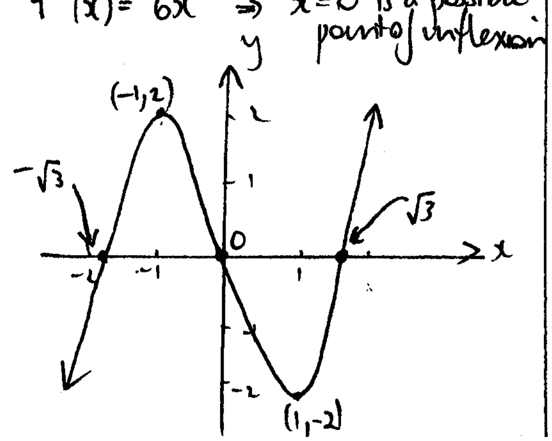
$f'(x) = 0 \Rightarrow x^2 - 1 = 0$

$\Rightarrow x = 1,\ x = -1$

$f(1) = -2,\ f(-1) = 2$ 1

$(1,-2)$ min $f''(1) > 0$ $(-1,2)$ max $f''(-1) < 0$

$f''(x) = 6x \Rightarrow x = 0$ is a possible point of inflexion.



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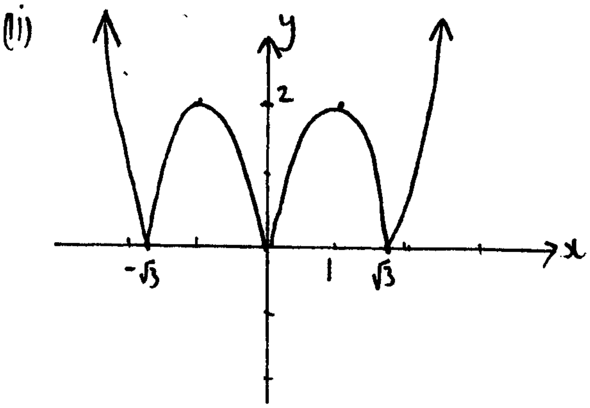
(ii)



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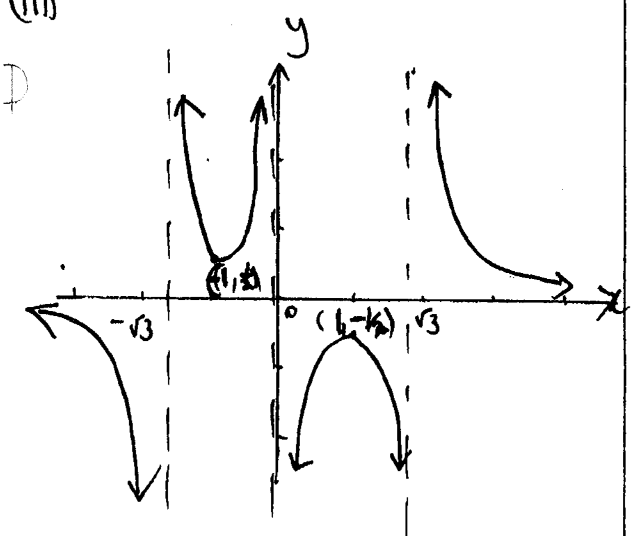
(iii)



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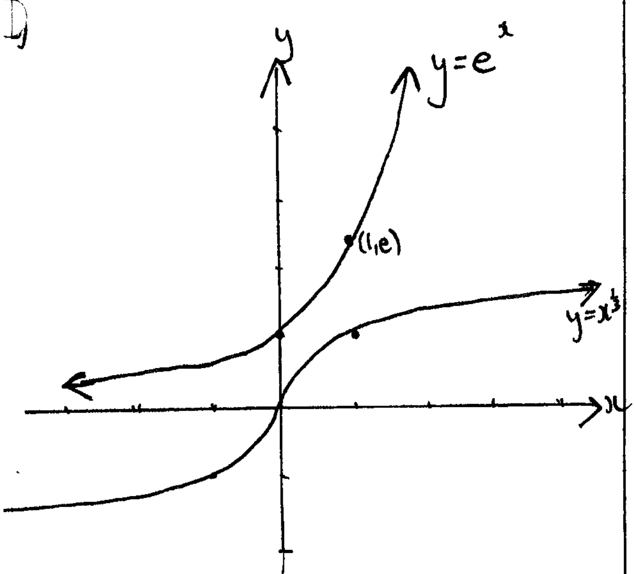

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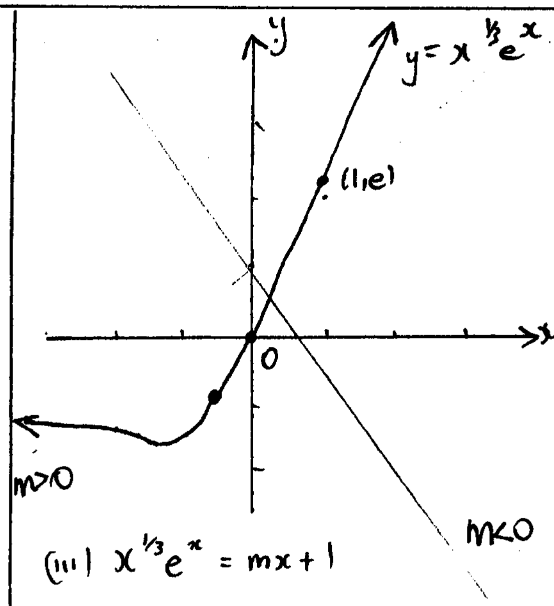

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(iii) $x^{\frac{1}{3}}e^x = mx+1$

let $y = mx+1$ (passes through (0,1))

see diagram

$\therefore$ exactly one solution for $m \le 0$

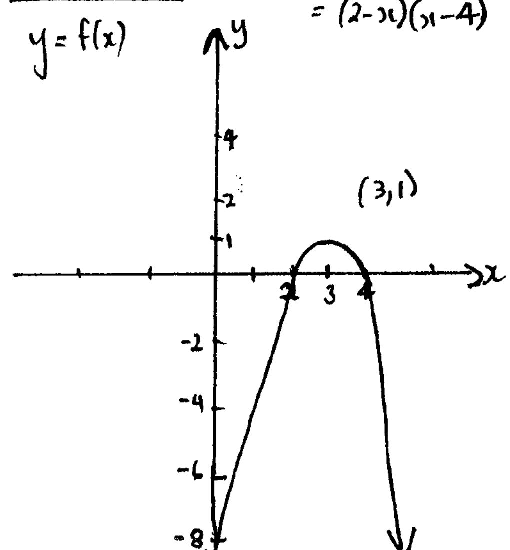
1 line

1 answ

15

Question 4: $f(x) = -x^2+6x-8$
$= (2-x)(x-4)$

$y=f(x)$



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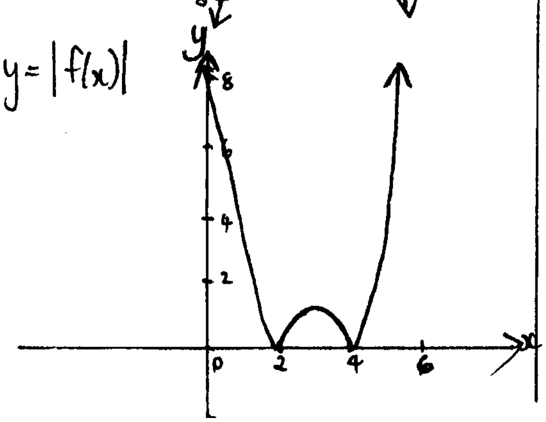
$y=|f(x)|$



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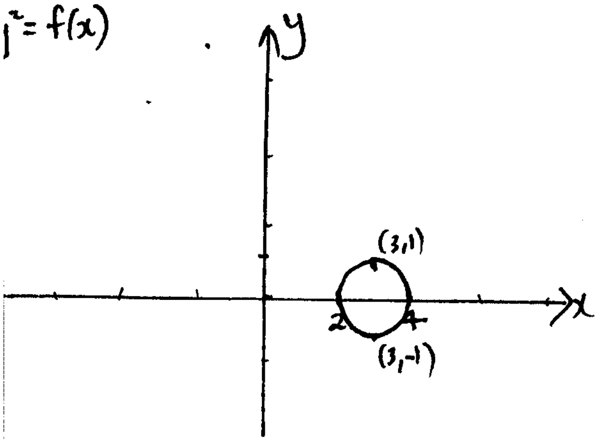
$y^2 = f(x)$



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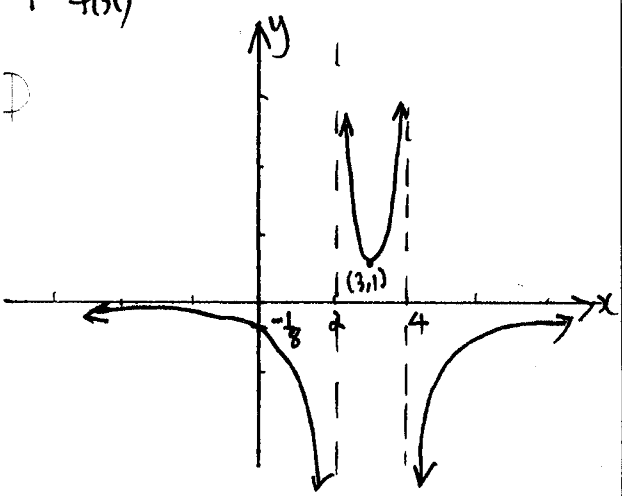
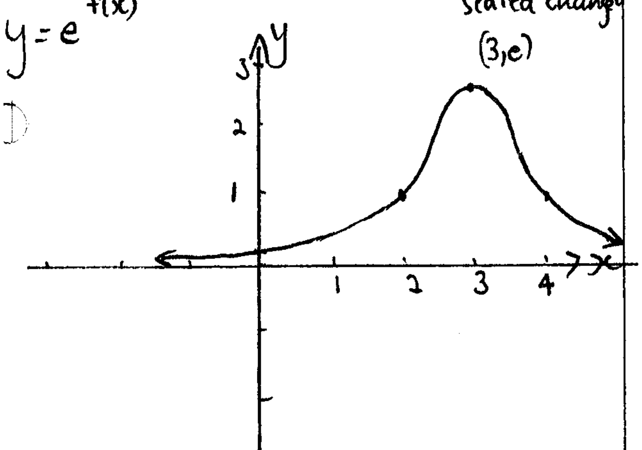
$y = \frac{1}{f(x)}$



2

$y = e^{f(x)}$

scaled changed



2

$y = e^{f(x)}$

$y' = f'(x) \cdot f(x) e^{f(x)}$

Note: $e^0 = 1$ when $x = 2, 4$

$f(x) = -x^2 + 6x - 8$

$f'(x) = -2x + 6$

$f'(x) = 0$ when $x = 3$

$f(x) = 0$ when $x = 2$ or $x = 4$

(b)(i) $P(x) = (x-\alpha)^2 Q(x)$ for some $Q(x)$ — 1

$P'(x) = 2(x-\alpha)Q(x) + (x-\alpha)^2 Q'(x)$

$= (x-\alpha)\left[2Q(x) + (x-\alpha)Q'(x)\right]$ — 1

So α is a zero of $P'(x)$

(ii) $P(x) = x^5 + 2x^4 + ax^3 + bx^2$

$P'(x) = 5x^4 + 8x^3 + 3ax^2 + 2bx$ — 1

$P(1) = 1 + 2 + a + b = 0$

$\Rightarrow a + b = -3 \quad \text{..(1)}$

$P'(1) = 5 + 8 + 3a + 2b = 0$

$\Rightarrow 3a + 2b = -13 \quad \text{..(2)}$

1 work

$2 \times (1) \Rightarrow 2a + 2b = -6$

$(2) \Rightarrow 3a + 2b = -13$

$(1) - (2) \quad -a = +7$

$\therefore a = -7$

$\therefore b = 4$ — 1

Question 5:

$\sqrt{3} - i = r \operatorname{cis}\theta$

where $r = \sqrt{\sqrt{3}^2 + 1^2} = 2$

and $\tan\theta = \frac{-1}{\sqrt{3}}$

$\theta = -\frac{\pi}{6}$

$= 2 \operatorname{cis}\left(-\frac{\pi}{6}\right)$ 1

$\therefore (\sqrt{3} - i)^8 = 2^8\left(\cos -\frac{\pi}{6} + i\sin -\frac{\pi}{6}\right)^8$ 1

$= 2^8\left(\cos \frac{8\pi}{6} - i\sin \frac{8\pi}{6}\right)$

$= 2^8\left(-\frac{1}{2} + i\frac{\sqrt{3}}{2}\right)$

$= 2^7(-1 + i\sqrt{3})$ 1

If $\sqrt{3}+i$ is a root of

$x^4 + px^2 + q = 0$ where $p, q \in \mathbb{R}$

then $\sqrt{3} - i$ is also a root 1

$\Rightarrow$ Quadratic factor $(x - (\sqrt{3}+i))(x - (\sqrt{3}-i))$

$= (x^2 - 2\sqrt{3}x + 4)$ 1

By polynomial division,

$$\begin{array}{r} x^2 + 2\sqrt{3}x + (p+8) \\ x^2 - 2\sqrt{3}x + 4 \overline{) x^4 + 0x^3 + px^2 + 0x + q} \\ \underline{x^4 - 2\sqrt{3}x^3 + 4x^2} \\ 2\sqrt{3}x^3 + (p-4)x^2 + 0x \\ \underline{2\sqrt{3}x^3 - 12x^2 + 8\sqrt{3}x} \\ (p+8)x^2 - 8\sqrt{3}x + q \\ \underline{(p+8)x^2 - 2(p+8)\sqrt{3}x + 4(p+8)} \\ 0 \quad 0 \end{array}$$

2 working

$\therefore 2(p+8)\sqrt{3} = 8\sqrt{3}$

$p + 8 = 4$

$p = -4$ $\therefore q = 4(-4+8)$ 1 answer

$q = +16$

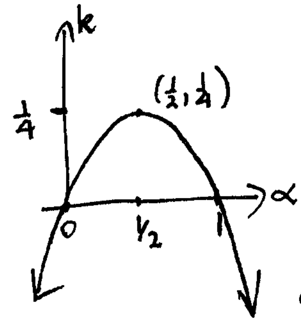
c) let $x^2 - x + k = (x-\alpha)(x-\beta)$

(i) $\therefore \alpha + \beta = 1$ .. ① 1

$\alpha\beta = k$ .. ②

sub ① into ② $\Rightarrow$

$\alpha(1-\alpha) = k$ 1



as k is a positive real 1

$\therefore 0 < k < \frac{1}{4}$

(ii) $\alpha^2 + \beta^2 = (\alpha+\beta)^2 - 2\alpha\beta$

$= 1^2 - 2k$

$= 1 - 2k$ 1

but $0 < k < \frac{1}{4}$

$\Rightarrow 0 < 2k < \frac{1}{2}$

$-\frac{1}{2} < -2k < 0$

$\frac{1}{2} < 1 - 2k < 1$

$\frac{1}{2} < \alpha^2 + \beta^2$ qed 1

(iii) $\frac{1}{\alpha^2} + \frac{1}{\beta^2} = \frac{\alpha^2 + \beta^2}{\alpha^2\beta^2} = \frac{1-2k}{k^2}$

$\frac{1-2k}{k^2} > \frac{1/2}{(1/4)^2}$ 2

$\frac{1-2k}{k^2} > \frac{16}{2}$

$\frac{1}{\alpha^2} + \frac{1}{\beta^2} > 8$ qed

15 marks