Question 1. **Marks**

(a) Find (i) $\int \sin(ax + \frac{\pi}{6})\, dx.$ **1**

(ii) $\int \sin^2(ax + \frac{\pi}{6})\, dx,$ where a is a constant. **2**

(b) Find $\int \frac{dx}{\sqrt{x^2 - 6x + 13}}.$ **3**

(c) Evaluate $\int_{\frac{1}{2}}^{\frac{\sqrt{3}}{2}} \sin^{-1}(x)\, dx.$ **3**

(d) (i) Given that $\frac{2x^2 - 2x + 1}{(x+1)(2x^2+3)} = \frac{1}{x+1} + \frac{ax+b}{2x^2+3},$ **2**

find the real numbers a and b.

(ii) Hence, or otherwise evaluate $\int_0^{\frac{\sqrt{3}}{\sqrt{2}}} \frac{2x^2 - 2x + 1}{(x+1)(2x^2+3)}\, dx.$ **4**

Question 2. **[START A NEW PAGE]**

(a) Given $f(x) = x^2 - 2x$, sketch the graph of the following on separate axes

(i) $y = f(x).$ **1**

(ii) $y = f(x) + |f(x)|.$ **2**

(iii) $y^2 = f(x).$ **2**

(iv) $y = [f(x)]^{-1}.$ **2**

(v) $y = f(2^x).$ **2**

(b) (i) For the Hyperbola: H: $\frac{x^2}{9} - \frac{y^2}{16} = 1,$ find its eccentricity e. **2**

(ii) Hence, neatly sketch the graph of H, clearly showing the vertices, foci, directrices and asymptotes. **4**

Question 3. **[START A NEW PAGE]** **Marks**

(a) An ellipse E has the equation: $E:\dfrac{x^2}{100}+\dfrac{y^2}{75}=1$.

(i) Sketch this ellipse, clearly showing on your diagram the coordinates of the foci and the equation of each directrix. **4**

(ii) Show that the equation of the normal to the ellipse E is **2**

$4x-2y=5$ at point $P(5, 7\frac{1}{2})$.

(iii) A circle is tangential to the ellipse E at P and at $Q(5, -7\frac{1}{2})$. **1**

Show that the centre of the circle is $(1\frac{1}{4}, 0)$.

(iv) Hence find the equation of this circle. **2**

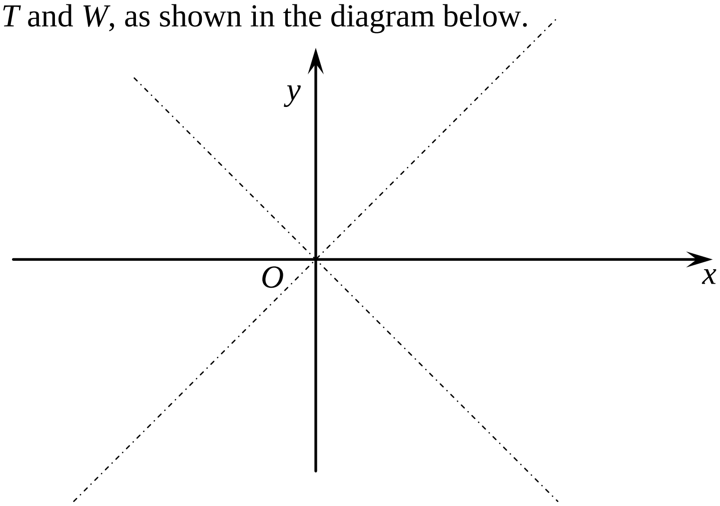
(b) Using the substitution: $t=\tan\dfrac{x}{2}$, where $0\le x<\pi$, show that: **4**

$$\int_0^{\frac{2\pi}{3}}\frac{dx}{5+3\sin x+4\cos x}=\frac{\sqrt{3}-1}{3}.$$

(c) For $x > 0$, find: $\displaystyle\int\frac{dx}{x^4+x}$. **2**

Question 4. **[START A NEW PAGE]** **Marks**

(a) The point $S(3e, 0)$ is a focus of the hyperbola H: $x^2 - y^2 = 9$.
The tangent to the hyperbola, at the point $P(p, q)$, meets the asymptotes of H in T and W, as shown in the diagram below.



Not to scale

(i) Show that the equation of the tangent TW is given by: $px - qy = 9$. **1**

(ii) Show that the gradient of the line through SW is given by **1**

$$m_{SW} = \frac{3}{e(p+q)-3}.$$

(iii) By letting $\angle WST = \theta$, find the value of $\tan\theta$. **3**

(b) Let $I_n = \int_o^{\frac{\pi}{4}} \tan^n x\, dx$, for $n = 0, 1, 2, \ldots$

(i) Show that $I_1 = \frac{1}{2}\ln 2$. **2**

(ii) Show that, $I_n + I_{n-2} = \frac{1}{n-1}$, for $n = 2, 3, 4, \ldots$ **3**

(iii) Explain why $I_n < I_{n-2}$, for $n \geq 2$, and deduce that **3**

$$\frac{1}{2n+2} < I_n < \frac{1}{2n-2}.$$

(iv) By using the recurrence relation in part (b) (ii), **2**
find I_5, and deduce that: $\frac{2}{3} < \ln 2 < \frac{3}{4}$.

THE END
☺ 😐 ☹

Trm 1 — Ext II — 2005

Question 1

a) i) $\int \sin(ax+\frac{\pi}{6})\,dx = -\frac{1}{a}\cos(ax+\frac{\pi}{6})+c$ ①

ii) $\int \sin^2(ax+\frac{\pi}{6})\,dx$

$= \frac{1}{2}\int [1-\cos 2(ax+\frac{\pi}{6})]\,dx$ ①

$= \frac{x}{2} - \frac{1}{4a}\sin 2(ax+\frac{\pi}{6}) + c$ ①

b) $\int \frac{1}{x^2-6x+13}\,dx$

$= \int \frac{1}{(x-3)^2+4}\,dx$ ①

$= \frac{1}{2}\tan^{-1}\left(\frac{x-3}{2}\right)+c$

① ①

c) $\int_{1/2}^{\sqrt{3}/2} \sin^{-1}x\,dx = \left[x\sin^{-1}x\right]_{\frac{1}{2}}^{\frac{\sqrt{3}}{2}} - \int_{1/2}^{\sqrt{3}/2}\frac{x}{\sqrt{1-x^2}}\,dx$

$= \left(\frac{\sqrt{3}}{2}\sin^{-1}\frac{\sqrt{3}}{2} - \frac{1}{2}\sin^{-1}\frac{1}{2}\right) + \int_{1/2}^{\sqrt{3}/2}\frac{d(1-x)}{\sqrt{1-x^2}}$

$= \frac{\sqrt{3}\pi}{6} - \frac{\pi}{12} + \frac{1}{2}\left[(1-x^2)^{1/2}\right]_{\frac{1}{2}}^{\sqrt{3}/2}$

$= \frac{\sqrt{3}\pi}{6} - \frac{\pi}{12} + \frac{1}{2}\left(\frac{1}{2} - \frac{\sqrt{3}}{2}\right)$

$= \frac{2\sqrt{3}\pi - \pi + 3 - 3\sqrt{3}}{12}$

d) i) $\frac{2x^2-2x+1}{(x+1)(2x^2+3)} = \frac{1}{x+1} + \frac{ax+b}{2x^2+3}$

$RHS = \frac{2x^2+3+(ax+b)(x+1)}{(x+1)(2x^2+3)}$ ①

$= \frac{(2+a)x^2 + x(a+b) + b+3}{(x+1)(2x^2+3)}$

equating coefficients:

$2 = 2+a$

$-2 = a+b$

$1 = b+3$

$\therefore a=0$ and $b=-2$ ①

ii) $\int_0^{\frac{\sqrt{3}}{\sqrt{2}}}\left[\frac{1}{x+1} + \frac{-2}{2x^2+3}\right]dx$ ①

$= \left[\ln(x+1) + \frac{-1}{\frac{\sqrt{3}}{\sqrt{2}}}\tan^{-1}\left(\frac{\sqrt{2}}{\sqrt{3}}x\right)\right]_0^{\sqrt{3}/\sqrt{2}}$

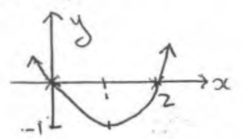
$= \left[\ln(x+1)\right]_0^{\sqrt{3}/\sqrt{2}} + -\frac{\sqrt{2}}{\sqrt{3}}\left[\tan^{-1}\frac{\sqrt{2}x}{\sqrt{3}}\right]_0^{\sqrt{3}/\sqrt{2}}$ ①

$= \ln\left(\frac{\sqrt{3}}{\sqrt{2}}+1\right) + -\frac{\sqrt{2}}{\sqrt{3}}\left(\tan^{-1}1 - \tan^{-1}0\right)$ ①

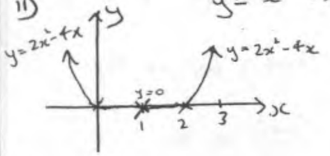
$= \ln\left(\frac{\sqrt{3}}{\sqrt{2}}+1\right) + -\frac{\sqrt{2}}{\sqrt{3}}\cdot\frac{\pi}{4}$ ①

Question 2

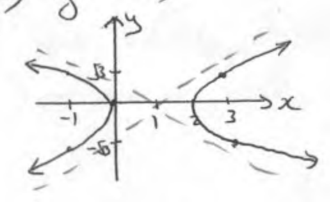
a) i) $y = x^2-2x$ ①



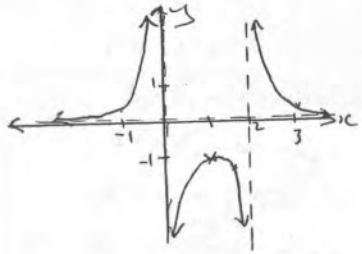
ii) $y = x^2-2x+|x^2-2x|$ ②



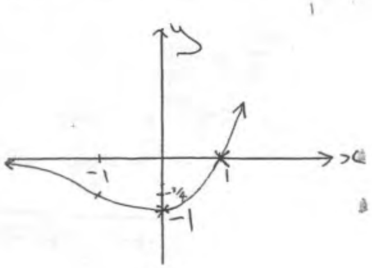
iii) $y^2 = f(x)$ ②



iv) $y = [f(x)]^{-1} = \frac{1}{x^2-2x}$ ②



v) $y = f(2^x) = 2^{2x} - 2^{x+1}$ ②



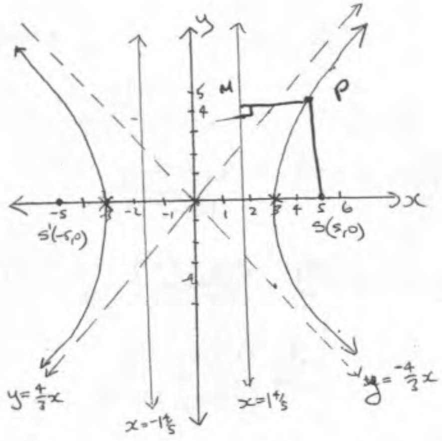
b) i) $b^2 = a^2(e^2-1)$

$16 = 9(e^2-1)$ ①

$\frac{16}{9}+1 = e^2$

$e = \frac{5}{3}$ ①

ii)



① for directrices

① for vertices

① for asymptotes

① for foci/shape

Question 3

a) i) $\frac{x^2}{100} + \frac{y^2}{75} = 1$

$a = 10 \quad b = \sqrt{75} = 5\sqrt{3}$

$b^2 = a^2(1-e^2)$

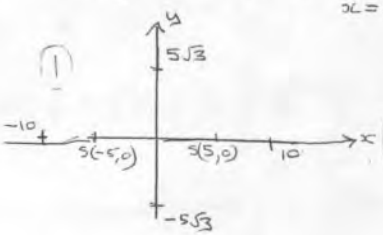
$\frac{75}{100} = 1 - e^2$

$e^2 = \frac{1}{4}$ (1)

$e = \frac{1}{2}$

* foci are $(\pm 5, 0)$
* directrices are $x = \pm 20$ (1)

(1)



(1)

ii) Differentiate with respect to x:

$\frac{2x}{100} + \frac{2y}{75} \cdot \frac{dy}{dx} = 0$

$\frac{dy}{dx} = \frac{-x}{50} \cdot \frac{75}{2y} = \frac{-3x}{4y}$

(1) At $P(5, 7\frac{1}{2})$ $\frac{dy}{dx} = \frac{-3(5)}{4(7\frac{1}{2})} = -\frac{1}{2}$

$\therefore$ gradient of normal is 2.

eqn. of normal is:

$y - 7\frac{1}{2} = 2(x-5)$

$y = 2x - 2\frac{1}{2}$

$2y = 4x - 5$

(1) ie $4x - 2y = 5$

iii) The intersection of the normals at P and Q will lie on the x-axis, the centre of the tangential circle is point of intersection of these 2 normals.

(1) sub. in $y = 0$ (lies on x-axis

iv) radius $= \sqrt{(5-1\frac{1}{4})^2 + (7\frac{1}{2}-0)^2}$

$= \sqrt{3\frac{3}{4}^2 + 7\frac{1}{2}^2}$

$= \sqrt{\frac{225}{16} + \frac{900}{16}}$

$= \frac{\sqrt{1125}}{4}$ (1)

eqn. of circle is:

$(x - 1\frac{1}{4})^2 + y^2 = \frac{1125}{16}$ (1)

$(4x-5)^2 + 16y^2 = 1125$

b) $\int_0^{\frac{2\pi}{3}} \frac{dx}{5 + 3\sin x + 4\cos x}$

$t = \tan \frac{x}{2}$ when $x = 0, t = 0$; $x = \frac{2\pi}{3}, t = \sqrt{3}$

$\frac{dt}{dx} = \frac{1}{2}\sec^2 \frac{x}{2}$

$\therefore dx = 2\cos^2 \frac{x}{2} dt$ (1)

$\therefore \int_0^{\sqrt{3}} \frac{\frac{2}{1+t^2}\,dt}{5 + \frac{6t}{1+t^2} + \frac{4-4t^2}{1+t^2}} = \int_0^{\sqrt{3}} \frac{2\,dt}{5+5t^2+6t+4-4t^2}$

$= 2\int_0^{\sqrt{3}} \frac{dt}{t^2+6t+9} = 2\int_0^{\sqrt{3}} (t+3)^{-2}\,dt$

(1) (1) $= -2\left[\frac{1}{t+3}\right]_0^{\sqrt{3}}$

$= \frac{-2}{\sqrt{3}+3} + \frac{2}{3}$

$= \frac{-2\sqrt{3}+6}{3-9} + \frac{2}{3}$

(1) $= \frac{2\sqrt{3}-2}{6}$

$= \frac{\sqrt{3}-1}{3}$ QED.

c) $\int \frac{dx}{x^4+x}$

$= \int \frac{\frac{1}{x^4}\,dx}{1 + \frac{1}{x^3}}$ (1)

$u = \frac{1}{x^3}$

$\frac{du}{dx} = \frac{-3}{x^4}$ $\therefore -\frac{1}{3}du = \frac{1}{x^4}dx$

$= -\frac{1}{3}\int \frac{du}{1+u} = -\frac{1}{3}\ln(1+u) + c$ (1)

Question 4

a) i) $x^2 - y^2 = 9$

$2x - 2y\frac{dy}{dx} = 0$

$\therefore \frac{dy}{dx} = \frac{x}{y}$

at $P(p,q)$ $\frac{dy}{dx} = \frac{p}{q}$

eqn. of tangent at P is:

$y - q = \frac{p}{q}(x-p)$

(½) $qy - q^2 = px - p^2$

$p^2 - q^2 = px - qy$

but P lies on the Hyperbola

(½) $\therefore p^2 - q^2 = 9$

So tangent is $px - qy = 9$

QED.

ii) W lies on tangent at P and the asymptote $y = -x$

$\therefore px + qx = 9$

$x = \frac{9}{p+q}$

$\therefore$ W is $\left(\frac{9}{p+q}, \frac{-9}{p+q}\right)$

Gradient of SW $= \frac{0 - \frac{-9}{p+q}}{3e - \frac{9}{p+q}}$

$= \frac{9}{3e(p+q) - 9}$

$= \frac{3}{e(p+q)-3}$

iii) T lies on $y = x$ and on the tangent at P

$\therefore px - qx = 9$

$x = \frac{9}{\ }$

$\therefore$ T is $\left(\frac{9}{p-q}, \frac{9}{p-q}\right)$

Grad. of ST $= \frac{3}{3 - e(p-q)}$

If $\angle WST = \theta$ then

$\tan\theta = \frac{m_2 - m_1}{1 + m_2 m_1}$

$\tan\theta = \frac{\frac{3}{e(p+q)-3} - \frac{3}{3-e(p-q)}}{1 + \frac{9}{[e(p+q)-3][3-e(p-q)]}}$

$\tan\theta = \frac{3[3-e(p-q)] - 3[e(p+q)-3]}{[e(p+q)-3][3-e(p-q)] + 9}$

$= \frac{9 - 3ep + 3eq - 3ep - 3eq + 9}{(ep+eq-3)(3-ep+eq) + 9}$

(1) $= \frac{18 - 6ep}{3ep - e^2p^2 + e^2pq + 3eq - e^2pq + e^2q^2 + 3ep - 3eq}$

$= \frac{6(3-ep)}{6ep - e^2(p^2-q^2)}$

(1) $= \frac{6(3-ep)}{6ep - 9e^2}$ [as $p^2 - q^2 = 9$]

$= \frac{6(3-ep)}{6ep - 18}$ [as $e^2 = 2$]

$= \frac{6(3-ep)}{6(ep-3)}$

$\therefore \tan\theta = -1$ (1)

b) i) I_1 is when $n = 1$

$I_1 = \int_0^{\pi/4} \tan x\,dx$

$= \int_0^{\pi/4} \frac{\sin x}{\cos x}\,dx$

$= \left[-\ln(\cos x)\right]_0^{\pi/4}$ (1)

$= -\ln\cos\frac{\pi}{4} + \ln\cos 0$

$= -\ln\frac{1}{\ }$

b) cont. Q4(b)

ii) show $I_n + I_{n-2} = \frac{1}{n-1}$

$$\begin{aligned}
L.H.S. &= I_n + I_{n-2} \\
&= \int_0^{\pi/4} (\tan^n x)\,dx + \int_0^{\pi/4} \tan^{n-2} x\,dx \\
&= \int_0^{\pi/4} (\tan^n x + \tan^{n-2} x)\,dx \quad 1 \\
&= \int_0^{\pi/4} \tan^{n-2} x\,(\tan^2 x + 1)\,dx \\
&= \int_0^{\pi/4} \tan^{n-2} x \,.\, \sec^2 x\,dx \quad 1 \\
&= \left[\frac{1}{n-1}\tan^{n-1} x\right]_0^{\pi/4} \\
&= \frac{1}{n-1}\left[\tan^{n-1}\pi/4 - \tan^{n-1} 0\right] \quad 1 \\
&= \frac{1}{n-1}(1-0) \\
&= \frac{1}{n-1}
\end{aligned}$$

QED.

iv) when $n=5 \Rightarrow \frac{1}{10+2} < I_5 < \frac{1}{10-2}$

$\therefore \frac{1}{12} < I_5 < \frac{1}{8}$

now $I_5 + I_3 = \frac{1}{4}$ (from ii)

$I_3 + I_1 = \frac{1}{2}$

$I_3 + \frac{1}{2}\ln 2 = \frac{1}{2}$ (as $I_1 = \frac{1}{2}$

$I_3 = \frac{1}{2} - \frac{1}{2}\ln 2$ 1

$\therefore I_5 = \frac{1}{4} - (\frac{1}{2} - \frac{1}{2}\ln 2)$

$= -\frac{1}{4} + \frac{1}{2}\ln 2$

so $\frac{1}{12} < -\frac{1}{4} + \frac{1}{2}\ln 2 < \frac{1}{8}$ 1

$\frac{1}{3} < \frac{1}{2}\ln 2 < \frac{3}{8}$

$\therefore \frac{2}{3} < \ln 2 < \frac{3}{4}$

iii) let area represent I_n $\therefore A_n = I_n$

now $0 \le x \le \pi/4$

$\therefore 0 \le \tan x \le 1$

so $0 \le \tan^n x \le \tan^2 x < \tan x \le 1$ for $n \geq 2$ 1

ie $0 \le \tan^{n+2} x < \tan^n x < \tan^{n-2} x < \ldots < \tan^2 x < \tan x \le 1$ for $n \geq 2$

now for $\int_0^{\pi/4} \ldots dx$

becomes

$0 \le I_{n+2} < I_n < I_{n-2} < \ldots < I_2 < I_1 \le \pi/4$ 1

$I_n < I_{n-2}$

now $I_{n+2} < I_n < I_{n-2}$

$I_{n+2} + I_n < 2I_n < I_n + I_{n-2}$

$\frac{1}{n+1} < 2I_n < \frac{1}{n-1}$ 1

$\frac{1}{2(n+1)} < I_n < \frac{1}{2(n-1)}$ for $n \geq 2$