QUESTION 1: START A NEW PAGE

Marks

(a) Evaluate: $\displaystyle\int_0^{\frac{\pi}{2}} \cos^7 x \sin x\, dx$ **2**

(b) Find $\displaystyle\int \frac{dx}{(x+1)(x^2+2)}$ **4**

(c) Use the substitution $t = \tan\frac{\theta}{2}$ to evaluate **4**

$$\int_0^{\frac{\pi}{3}} \frac{1}{1+\sin\theta}\, d\theta$$

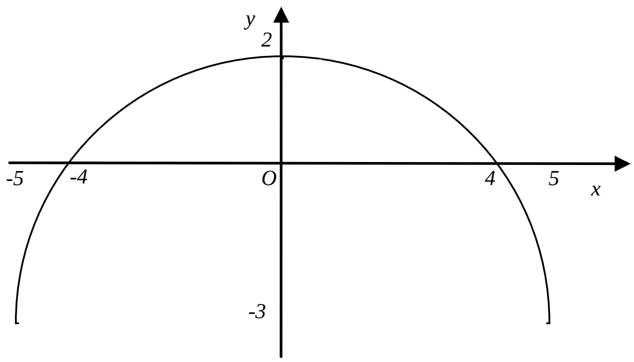
(d) (i) Find the equations to the two tangents to the hyperbola $\frac{x^2}{4} - y^2 = 1$ which are parallel to the line $2x - y = 3$. **3**

(ii) Show that the distance between these two tangents is $2\sqrt{3}$ units. **2**

QUESTION 2: START A NEW PAGE

Marks

(a) The diagram shows the graph of $y = f(x)$ for $-5 \le x \le 5$.



Draw separate sketches of the graphs of the following for $-5 \le x \le 5$.

(i) $y = \dfrac{1}{f(x)}$ **2**

(ii) $y = f(x) - |\, f(x)\,|$ **2**

(iii) $y = e^{f(x)}$ **2**

(iv) $y = \left[f(x)\right]^2$ **2**

(b) Points $P(a\cos\theta, b\sin\theta)$ and $Q(a\cos\phi, b\sin\phi)$ lie on the ellipse

$$\frac{x^2}{4} + \frac{y^2}{3} = 1, \qquad \text{where } \phi = \theta + \frac{\pi}{2}.$$

(i) Show that the gradient of the tangent at Q is $\dfrac{\sqrt{3}\sin\theta}{2\cos\theta}$. **2**

(ii) If α is the acute angle between the tangents at P and Q prove that **3**

$$\tan\alpha = \frac{4\sqrt{3}}{|\sin 2\theta|}$$

(iii) Find the value of $OP^2 + OQ^2$ where O is the centre of the ellipse. **2**

QUESTION 3: START A NEW PAGE

Marks

(a) (i) If $I_n = \int \sin^n x\,dx$ show that: **3**

$$I_n = -\frac{1}{n}\cos x \sin^{n-1} x + \left(\frac{n-1}{n}\right) I_{n-2} \text{ for } n \geq 2$$

(ii) Hence find $\int \sin^5 x\,dx$. **3**

(b) Points $P(a\cos\theta, b\sin\theta)$, $Q\big(a\cos(\theta+\phi), b\sin(\theta+\phi)\big)$ and $R\big(a\cos(\theta-\phi), b\sin(\theta-\phi)\big)$ are three points on the ellipse **4**

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

Show that *OP* bisects the chord *QR*, where *O* is the origin.

(c) (i) Use the substitution of $u = \pi - x$ to show: **2**

$$\int_{\frac{\pi}{3}}^{\frac{2\pi}{3}} \frac{x}{\sin x}dx = \int_{\frac{\pi}{3}}^{\frac{2\pi}{3}} \frac{\pi - x}{\sin x}dx$$

(ii) Evaluate $\displaystyle\int_{\frac{\pi}{3}}^{\frac{2\pi}{3}} \frac{x}{\sin x}dx$ **3**

[You may assume $\int \operatorname{cosec}\theta\,d\theta = \ln|\operatorname{cosec}\theta - \cot\theta|$]

QUESTION 4: START A NEW PAGE

Marks

(a) Evaluate: $\int_0^2 xe^{2x}\,dx$ **2**

(b) (i) Sketch the following graphs for $-2\pi \le x \le 2\pi$ on the same number plane without using calculus. **2**

(α) $y = \dfrac{x}{2}$

(β) $y = \sin x$

(ii) Sketch on a separate number plane the graph of **2**

$$y = \frac{2\sin x}{x} \text{ for } -2\pi \le x \le 2\pi .$$

(c) A hyperbola has the equation:

$$\frac{x^2}{3} - \frac{y^2}{2} = 1 .$$

(i) Find the coordinates of the foci. **2**

The equation of the tangent to the hyperbola at $P(\sqrt{3}\sec\theta, \sqrt{2}\tan\theta)$ is:

$$\sqrt{2}x\sec\theta - \sqrt{3}y\tan\theta = \sqrt{6} .$$

(Do not prove this equation)

(ii) Show that the equation of the normal at P is: **2**

$$\sqrt{3}x\tan\theta + \sqrt{2}y\sec\theta = 5\sec\theta\tan\theta.$$

The tangent and normal to the hyperbola at P cut the y axis at T and N respectively.

(iii) Show that the circle with TN as diameter passes through the foci of the hyperbola. **5**

END OF EXAMINATION

Term 1 HSC Ext II 2006

(a)

$$\int_0^{\pi/2} \cos^7 x \sin x \, dx$$

$$= -\int_0^{\pi/2} -\sin x \cos^7 x \, dx$$

This is of the form: $-\int f'(x)[f(x)]^7 dx$

$$= \left[-\frac{\cos^8 x}{8}\right]_0^{\pi/2}$$ ① for correct integration

$$= -0 - \left(-\frac{1}{8}\right)$$ ① for correct substitution and answer

$$= \frac{1}{8}$$

(b) Write as partial fractions

$$\int \frac{dx}{(x+1)(x^2+2)} = \int \left[\frac{A}{(x+1)} + \frac{Bx+C}{(x^2+2)}\right] dx$$

$$\therefore 1 \equiv A(x^2+2) + (Bx+C)(x+1)$$

$$= Ax^2 + 2A + Bx^2 + Bx + Cx + C$$

Equate coefficients

$A+B=0 \quad \therefore A=-B$ ① Writing as partial fractions and producing simultaneous equations

$B+C=0 \quad \therefore C=-B$

$2A+C=1 \quad \therefore -2B-B=1$

$\therefore B = -1/3$ ① for correct solution of equations

$A = 1/3$

$C = 1/3$

$$\frac{1}{3}\int \left(\frac{1}{x+1} + \frac{-x+1}{x^2+2}\right) dx$$ ① for integration of terms producing log functions

$$= \frac{1}{3}\int \left(\frac{1}{x+1} - \frac{\frac{1}{2}(2x)}{x^2+2} + \frac{1}{x^2+2}\right) dx$$

$$= \frac{1}{3}\left[\ln|x+1| - \tfrac{1}{2}\ln(x^2+2) + \tan^{-1}\frac{x}{\sqrt{2}}\right] + C$$ ① for integration to $\tan^{-1}\frac{x}{\sqrt{2}}$

$$= \frac{1}{3}\left[\ln\left(\frac{|x+1|}{\sqrt{x^2+2}}\right) + \tan^{-1}\frac{x}{\sqrt{2}}\right] + C$$

(c)

$$\int_0^{\pi/3} \frac{1}{1+\sin\theta} \, d\theta$$

$t = \tan\frac{1}{2}\theta$ ① correct derivative in t and change of limit values

$$\frac{dt}{d\theta} = \frac{1}{2}\sec^2\frac{1}{2}\theta = \frac{1}{2}[1+t^2]$$

$\theta = \frac{\pi}{3} \quad t = \frac{1}{\sqrt{3}}$

$\theta = 0 \quad t = 0$

$$\int_0^{1/\sqrt{3}} \frac{1}{1+\left(\frac{2t}{1+t^2}\right)} \left(\frac{2}{1+t^2}\right) dt$$

$$= \int_0^{1/\sqrt{3}} \frac{2}{1+t^2+2t} \, dt$$ ① correct change of variable in integral

$$= \int_0^{1/\sqrt{3}} \frac{2}{(1+t)^2} \, dt$$

$$= \left[\frac{-2}{(1+t)}\right]_0^{1/\sqrt{3}}$$ ① correct integration

$$= -2\left[\left(\frac{1}{1+\frac{1}{\sqrt{3}}}\right) - \frac{1}{1}\right]$$

$$= -2\left[\frac{\sqrt{3}}{\sqrt{3}+1} - \frac{(\sqrt{3}+1)}{(\sqrt{3}+1)}\right]$$

$$= \frac{-2}{(\sqrt{3}+1)}(-1)$$

$$= \frac{2}{\sqrt{3}+1}$$ ① correct substitution and simplified answer

$$= \frac{2(\sqrt{3}-1)}{2}$$

$$= \sqrt{3}-1$$

(rationalised denominator not necessary for full marks)

1 (d) (i) $\frac{x^2}{4} - y^2 = 1$

tangent parallel to $2x - y = 3$ must be in the form

$y = 2x + c$

$\therefore \frac{x^2}{4} - (2x+c)^2 = 1$

$\frac{x^2}{4} - (4x^2 + 4xc + c^2) = 1$

$\therefore x^2 - 16x^2 - 16xc - 4c^2 = 4$

$\therefore 15x^2 + 16xc + 4(c^2+1) = 0$ — ① correct substitution of $y = 2x + c$

as line is a tangent $\therefore \Delta = 0$

$\therefore \Delta = (16c)^2 - 4 \times 15 \times 4(c^2+1)$ — ① correct Δ

$0 = 16^2c^2 - 16 \times 15(c^2+1)$

$0 = 16c^2 - 15c^2 - 15$ — ① correct tangent equations.

$c^2 = 15$

$c = \pm\sqrt{15}$

$\therefore$ tangents are: $y = 2x \pm \sqrt{15}$

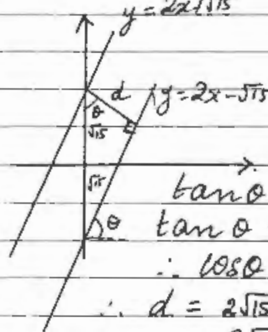
Alternative solutions using

$\frac{xx_1}{4} - yy_1 = 0$

are also possible.

① Derivative/gradient
① Equation of tangent to hyperbola.
① Equation of tangents

(ii)



$\tan\theta$ = gradient

$\tan\theta = 2$

$\therefore \cos\theta = \frac{1}{\sqrt{5}}$ — ① calculation of $\cos\theta$

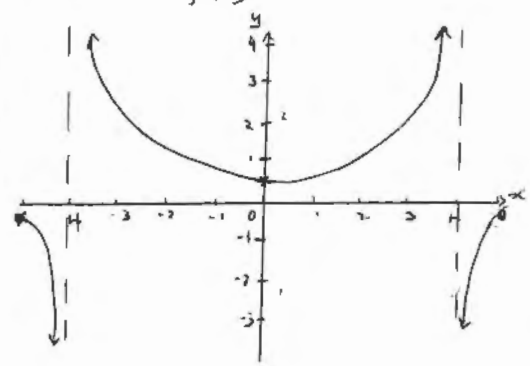
$\therefore d = 2\sqrt{15}\cos\theta$ — ① correct distance.

$= 2\sqrt{15} \times \frac{1}{\sqrt{5}} = 2\sqrt{3}$ units

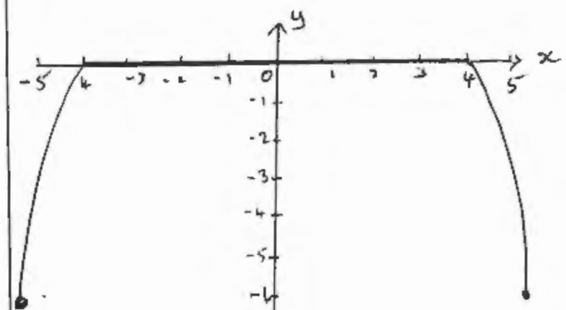
Alternative solutions using perp distance formula

① point on line
① correct distance.

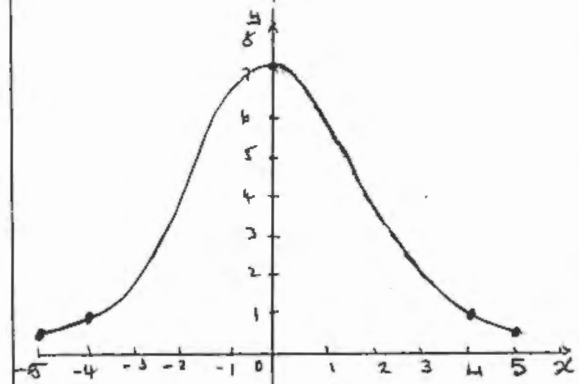
2 (a) (i) $y = \frac{1}{f(x)}$



① asymptotes + intercepts
① shape.

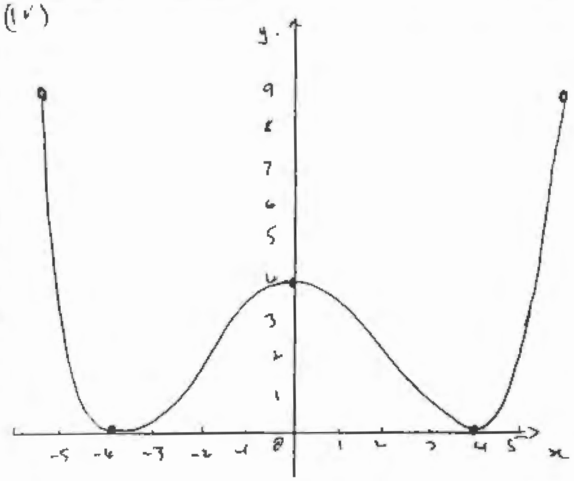


① $y = 0 \quad -4 \le x \le 4$
① graph for $-5 \le x \le 4$ and $+4 \le x \le 5$.



① values for $x = 0$ and $x = \pm 4$
① shape.

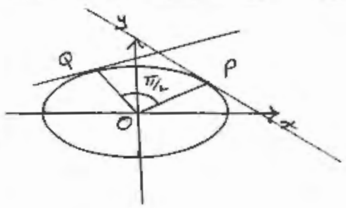
2(a) (iv)



① Endpoints and intercepts
① shape.

$y = [f(x)]^2$

2(b)



(i) $\frac{x^2}{4} + \frac{y^2}{3} = 1 \qquad a = 2,\ b = \sqrt{3}$

$\frac{2x}{4} + \frac{2y}{3}\frac{dy}{dx} = 0$

$\therefore \frac{dy}{dx} = \frac{-2x}{4} \times \frac{3}{2y}$

$= -\frac{3}{4}\frac{x}{y}.$ ① derivative

grad of tangent at Q

$m = -\frac{3}{4}\frac{a\cos\phi}{b\sin\phi}$

$= -\frac{3}{4}\frac{2\cos(\theta+\pi/2)}{\sqrt{3}\sin(\theta+\pi/2)}$

$= -\frac{\sqrt{3}}{2} \times \frac{-\sin\theta}{\cos\theta}$

$= \frac{\sqrt{3}}{2}\frac{\sin\theta}{\cos\theta}$ ① correct substitution and proof.

(ii) gradient of tangent at P is $-\frac{\sqrt{3}}{2}\frac{\cos\theta}{\sin\theta}$ ① correct gradient of tangent at P

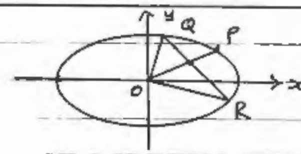
$\tan\alpha = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right|$

$= \left|\frac{\frac{\sqrt{3}}{2}\frac{\sin\theta}{\cos\theta} - \frac{-\sqrt{3}\cos\theta}{\sin\theta}}{1 + \frac{\sqrt{3}}{2}\frac{\sin\theta}{\cos\theta} \times \frac{-\sqrt{3}}{2}\frac{\cos\theta}{\sin\theta}}\right|$ ① correct substitution into formula.

$= \left|\frac{\frac{\sqrt{3}}{2}[\sin^2\theta + \cos^2\theta]}{\cos\theta\sin\theta\,(1 - \frac{3}{4})}\right|$

$= \frac{\frac{\sqrt{3}}{2}(1)}{\frac{1}{2}\sin 2\theta[\frac{1}{4}]} = \frac{4\sqrt{3}}{|\sin 2\theta|}$ ① correct simplification

	$OP^2 = a^2\cos^2\theta + b^2\sin^2\theta$	
	$OQ^2 = a\cos^2\phi + b^2\sin^2\phi$	① OP^2 and OQ^2 equations
	$= a^2\cos^2(\theta+\pi/2) + b^2\sin^2(\theta+\pi/2)$	
	$= a^2\sin^2\theta + b^2\cos^2\theta$	
	$\therefore OP^2+OQ^2 = a^2(\cos^2\theta+\sin^2\theta) + b^2(\cos^2\theta+\sin^2\theta)$	① calculations
	$= a^2+b^2$	
	$= 4+3$	
	$= 7$	
4(a)	(i) $I_n = \int \sin^n x\,dx$	
	$= -\cos x\sin^{n-1}x + \int \cos x(n-1)\sin^{n-2}x\cos x\,dx$	① integration by parts
	$= -\cos x\sin^{n-1}x + (n-1)\int \sin^{n-2}x\cos^2 x\,dx$	
	$= -\cos x\sin^{n-1}x + (n-1)\int \sin^{n-2}x(1-\sin^2 x)\,dx$	
	$= -\cos x\sin^{n-1}x + (n-1)\int \sin^{n-2}x\,dx - (n-1)\int \sin^n x\,dx$	
	$= -\cos x\sin^{n-1}x + (n-1)I_{n-2} - (n-1)I_n$	$\boxed{\text{① conversion to } I_n \text{ and } I_{n-2}}$
	$I_n(1+(n-1)) = -\cos x\sin^{n-1}x + (n-1)I_{n-2}$	
	$nI_n = -\cos x\sin^{n-1}x + (n-1)I_{n-2}$	
	$I_n = -\frac{1}{n}\cos x\sin^{n-1}x + \left(\frac{n-1}{n}\right)I_{n-2}$	① correct proof of answer.
	(ii) $\int \sin^5 x\,dx = I_5$	
	$I_5 = -\frac{1}{5}\cos x\sin^4 x + \frac{4}{5}I_3$	① correct sub into I_n for $n=5,3,1$
	$I_3 = -\frac{1}{3}\cos x\sin^2 x + \frac{2}{3}I_1$	
	$I_1 = \int \sin x\,dx$	
	$= -\cos x + C$	
	$I_5 = -\frac{1}{5}\cos x\sin^4 x + \frac{4}{5}\left[-\frac{1}{3}\cos x\sin^2 x\right] + \frac{4}{5}\times\frac{2}{3}\left[-\cos x\right] + C.$	① correct addition of terms.
	$= -\frac{1}{5}\cos x\sin^4 x - \frac{4}{15}\cos x\sin^2 x - \frac{8}{15}\cos x + C$	① Answer

3(b)		
	$P(a\cos\theta, b\sin\theta)$	
	$Q(a\cos(\theta+\phi), b\sin(\theta+\phi))$	
	$R(a\cos(\theta-\phi), b\sin(\theta-\phi))$	
	Midpoint $QR = M(x_1,y_1)$	
	$x_1 = \frac{a}{2}(\cos(\theta+\phi)+\cos(\theta-\phi))$	
	$= \frac{a}{2}[\cos\theta\cos\phi - \sin\theta\sin\phi + \cos\theta\cos\phi + \sin\theta\sin\phi]$	
	$= a/2[2\cos\theta\cos\phi]$	
	$= a\cos\theta\cos\phi$	① x co-ord midpoint QR
	$y_1 = \frac{b}{2}[\sin(\theta+\phi)+\sin(\theta-\phi)]$	
	$= b/2[\sin\theta\cos\phi + \cos\theta\sin\phi + \sin\theta\cos\phi - \cos\theta\sin\phi]$	
	$= b/2[2\sin\theta\cos\phi]$	
	$= b\sin\theta\cos\phi$	① y co-ord midpoint QR.
	gradient $OP = \frac{b\sin\theta}{a\cos\theta}$	
	equation of OP $y = \frac{b\sin\theta}{a\cos\theta}x$.	① equation of OP
	sub $M(x_1,y_1)$ $x_1 = a\cos\theta\cos\phi$	
	$RHS = \frac{b}{a}\frac{\sin\theta}{\cos\theta}a\cos\theta\cos\phi$	① correct test showing OP passes through midpoint QR.
	$= b\sin\theta\cos\phi$	
	$= y_1$	
	$\therefore OP$ passes through midpoint QR	
	$\therefore OP$ bisects QR.	

3(c)

$$\int_{\pi/3}^{2\pi/3} \frac{x}{\sin x}\,dx$$

$u = \pi - x$

$\frac{du}{dx} = -1$

$x = \frac{2\pi}{3} \quad u = \frac{\pi}{3}$

$x = \frac{\pi}{3} \quad u = \frac{2\pi}{3}$.

① derivative and change of variable limit values

$$\therefore \int_{\pi/3}^{2\pi/3} \frac{x}{\sin x}\,dx = \int_{2\pi/3}^{\pi/3} \frac{(\pi-u)(-1)\,du}{\sin(\pi-u)}$$

① correct substitution and change of variable.

$$= \int_{\pi/3}^{2\pi/3} \frac{\pi-u}{\sin u}\,du$$

(change of variable $u = x$)

$$= \int_{\pi/3}^{2\pi/3} \frac{\pi-x}{\sin x}\,dx$$

$$\therefore \int_{\pi/3}^{2\pi/3} \frac{x\,dx}{\sin x} = \int_{\pi/3}^{2\pi/3} \frac{\pi-x}{\sin x}\,dx$$

$$\int_{\pi/3}^{2\pi/3} \frac{x}{\sin x}\,dx = \int_{\pi/3}^{2\pi/3} \frac{\pi}{\sin x}\,dx - \int_{\pi/3}^{2\pi/3} \frac{x}{\sin x}\,dx$$

$$\therefore 2\int_{\pi/3}^{2\pi/3} \frac{x}{\sin x}\,dx = \pi \int_{\pi/3}^{2\pi/3} \frac{1}{\sin x}\,dx$$

$$= \pi \int_{\pi/3}^{2\pi/3} \operatorname{cosec} x\,dx$$

① splitting + combining integrals

$$\int_{\pi/3}^{2\pi/3} \frac{x}{\sin x}\,dx = \frac{\pi}{2}\Big[\ln\left|\operatorname{cosec} x - \cot x\right|\Big]_{\pi/3}^{2\pi/3}$$

① integration and substitution of values

$$= \frac{\pi}{2}\left[\ln\left|\frac{\operatorname{cosec}\frac{2\pi}{3} - \cot\frac{2\pi}{3}}{\operatorname{cosec}\frac{\pi}{3} - \cot\frac{\pi}{3}}\right|\right]$$

$$= \frac{\pi}{2}\ln\left[\frac{\frac{2}{\sqrt{3}} + \frac{1}{\sqrt{3}}}{\frac{2}{\sqrt{3}} - \frac{1}{\sqrt{3}}}\right]$$

$$= \frac{\pi}{2}\ln 3$$

① simplified answer.

4(a)

$$\int_0^2 x e^{2x}\,dx.$$

$$= \left[\tfrac{1}{2}e^{2x}\cdot x\right]_0^2 - \int_0^2 \tfrac{1}{2}e^{2x}(1)\,dx.$$

① Integration by parts

$$= \left[\tfrac{1}{2}e^4\cdot 2 - \tfrac{1}{2}e^0\cdot 0\right] - \tfrac{1}{4}\left[e^{2x}\right]_0^2$$

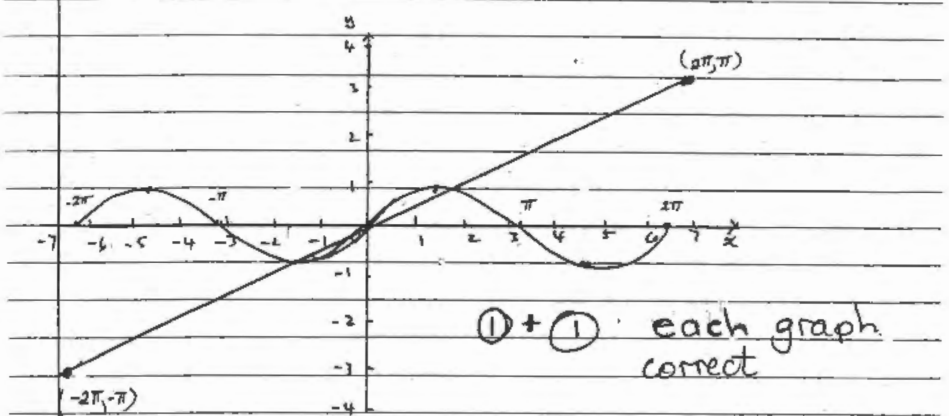
① complete integration + substitute correctly

$$= e^4 - \tfrac{1}{4}(e^4 - e^0)$$

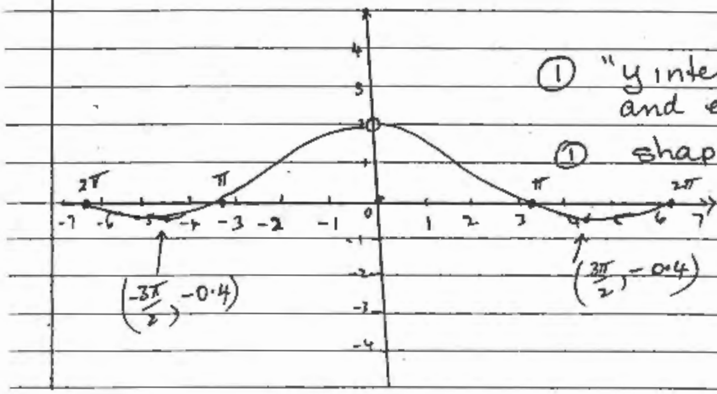
$$= e^4 - \frac{e^4}{4} + \frac{1}{4}$$

$$= \frac{3e^4+1}{4}$$

4(b)



①+① each graph correct



① "y intercept" value and exclusion of point

① shape and x intercepts

4(c) (i) $\frac{x^2}{3} - \frac{y^2}{2} = 1$	
$a^2 = 3$	
$b^2 = 2$.	
$b^2 = a^2(e^2-1)$	
$2 = 3(e^2-1)$	
$\frac{2}{3} = e^2 - 1$	
$e^2 = \frac{5}{3}$	
$e = \sqrt{5/3}$ $\quad e>0$	① for 'e' value
$\therefore$ Foci $y=0$ $\quad x = \pm ae$	
$= \pm \sqrt{3}\frac{\sqrt{5}}{\sqrt{3}}$	
$= \pm\sqrt{5}$	
foci $= (\pm\sqrt{5}, 0)$	① for correct (foci values)
(ii) Equation to tangent at P is:	
$\sqrt{2}x\sec\theta - \sqrt{3}y\tan\theta = \sqrt{6}$	
Equation to perpendicular lines:	
$\sqrt{3}x\tan\theta + \sqrt{2}y\tan\theta = c$	① perp equation or gradient
Normal passes through P	
$\therefore \sqrt{3}(\sqrt{3}\sec\theta)\tan\theta + \sqrt{2}(\sqrt{2}\tan\theta) = c$	
$\therefore 3\sec\theta\tan\theta + 2\tan\theta\sec\theta = c$	
$\therefore$ Equation to normal is:	
$\sqrt{3}x\tan\theta + \sqrt{2}y\sec\theta = 5\sec\theta\tan\theta$	① correct "c" value and equation of normal
(iii) Equation of tangent	
$\sqrt{2}x\sec\theta - \sqrt{3}y\tan\theta = \sqrt{6}$	
y intercept $x=0$	
$\therefore y = \frac{-\sqrt{6}}{\sqrt{3}\tan\theta}$ $\quad (\tan\theta \neq 0)$	① y coord of T
$\therefore T = \left(0, \frac{-\sqrt{2}}{\tan\theta}\right)$	
Equation of normal	
$\sqrt{3}x\tan\theta + \sqrt{2}y\sec\theta = 5\sec\theta\tan\theta$	
$x=0$ $\quad y = \frac{5\tan\theta}{\sqrt{2}}$ $\quad (\sec\theta \neq 0)$	① y coord of N
$\therefore N = \left(0, \frac{5\tan\theta}{\sqrt{2}}\right)$	

gradient of SN	
$m_{SN} = \dfrac{\frac{5\tan\theta}{\sqrt{2}} - 0}{0 - \sqrt{5}}$	
$= \dfrac{-\sqrt{5}\tan\theta}{\sqrt{2}}$	
gradient ST	
$m_{ST} = \dfrac{\frac{-\sqrt{2}}{\tan\theta} - 0}{0 - \sqrt{5}}$	
$= \dfrac{\sqrt{2}}{\sqrt{5}\tan\theta}$	① gradients ST and SN
$\therefore m_{SN} \times m_{ST}$	
$= \dfrac{-\sqrt{5}\tan\theta}{\sqrt{2}} \times \dfrac{\sqrt{2}}{\sqrt{5}\tan\theta}$	
$= -1$	① showing ST perp to SN, and geometric reason why S is on on the circle.
$\therefore$ SN is perpendicular to ST	
$\therefore$ Focus $S(\sqrt{5}, 0)$ is on circle with diameter TN (angle in a semicircle is 90°)	
Similarly for $S'(-\sqrt{5}, 0)$ by symmetry. (or further proof)	① Reason for S' on circle (no marks for further algebra.)
Alternative Solutions:	
(a) Find equation to circle through T and N and test S and S'	① Equation of circle ① Test for S on circle ① Test/reason S' on circle
(b) Find midpoint of TN and then distances to T, S and S'	① midpoint of TN and distance to T ① distance to S ① Reason for S' on circle