Question 1 (15 Marks) **Marks**

(a) Sketch the curve $(x + 1)(y - 2) = 1$ **2**

(b) Write the function in part (a) in the form $y = f(x)$. Hence, or otherwise, sketch the curve

(i) $y = f(x - 2)$ **1**

(ii) $y = \sqrt{f(x)}$ **2**

(c) Evaluate

(i) $\displaystyle\int_2^3 \frac{4}{(x-1)(4-x)}\,dx$ **3**

(ii) $\displaystyle\int_0^1 \frac{dx}{9-4x^2}$ **2**

(d) (i) The point P moves so that the sum of its distances from the points $(-2, 0)$ and $(2, 0)$ is 6 units. Prove that the equation of the locus of P is $\dfrac{x^2}{9}+\dfrac{y^2}{5}=1$. **4**

(ii) Find the eccentricity of this locus. **1**

Question 2 (15 Marks) **<u>START A NEW PAGE</u>** **Marks**

(a) The hyperbola H is given by the equation $\dfrac{x^2}{16}-\dfrac{y^2}{25}=1$.

(i) Write down the equations of the asymptotes. **2**

(ii) P is an arbitrary point with coordinates $(4\sec\theta, 5\tan\theta)$. Show that P lies on H. **2**

(iii) Prove that the tangent to H at P has equation $\dfrac{x\sec\theta}{4}-\dfrac{y\tan\theta}{5}=1$. **3**

(iv) This tangent cuts the asymptotes in L and M. Prove that $LP = PM$. **3**

Question 2 Continued **Marks**

(b) Let n be a positive integer and $I_n = \int_1^2 (\ln x)^n \, dx$.

(i) Prove that $I_n = 2(\ln 2)^n - nI_{n-1}$. **2**

(ii) Hence, evaluate $\int_1^2 (\ln x)^3 \, dx$. **3**

Question 3 (15 Marks) **<u>START A NEW PAGE</u>** **Marks**

(a) Find $\int \frac{x+1}{x^2+1} \, dx$. **2**

(b) On the attached sheet, you are given the curve of $y = f(x)$.
Sketch neatly on separate diagrams

(i) $y = |f(x)|$ **1**

(ii) $y = \frac{1}{f(x)}$ **2**

(c) (i) Evaluate $\int_{-\pi}^{\pi} x \cos x \, dx$. **2**

(ii) Evaluate $\int_0^{\frac{\pi}{2}} \cos^5 \theta \, d\theta$. **3**

(d) F is the point (4, 0) and d is th line $x = 1$. M is the foot of the perpendicular from a variable point P to d, and P moves so that $FP = 2PM$.

(i) Derive the equation of the locus of P. **3**

(ii) Find the acute angle between the asymptotes to the nearest degree. **2**

Question 4 (15 Marks) **<u>START A NEW PAGE</u>** **Marks**

(a) Use the substitution $t = \tan\frac{x}{2}$ to prove that $\int_0^{\frac{\pi}{2}} \frac{dx}{2+\sin x} = \frac{\pi}{3\sqrt{3}}$. **4**

(b) (i) Show that $\int_0^{2a} f(x)dx = \int_0^{a} \{f(x) + f(2a-x)\}dx$. **3**

(ii) Hence, or otherwise evaluate $\int_0^{\pi} \frac{x\,dx}{2+\sin x}$. **3**

(c) (i) Evaluate $\int_0^{\frac{\pi}{4}} \sec\theta\, d\theta$. **2**

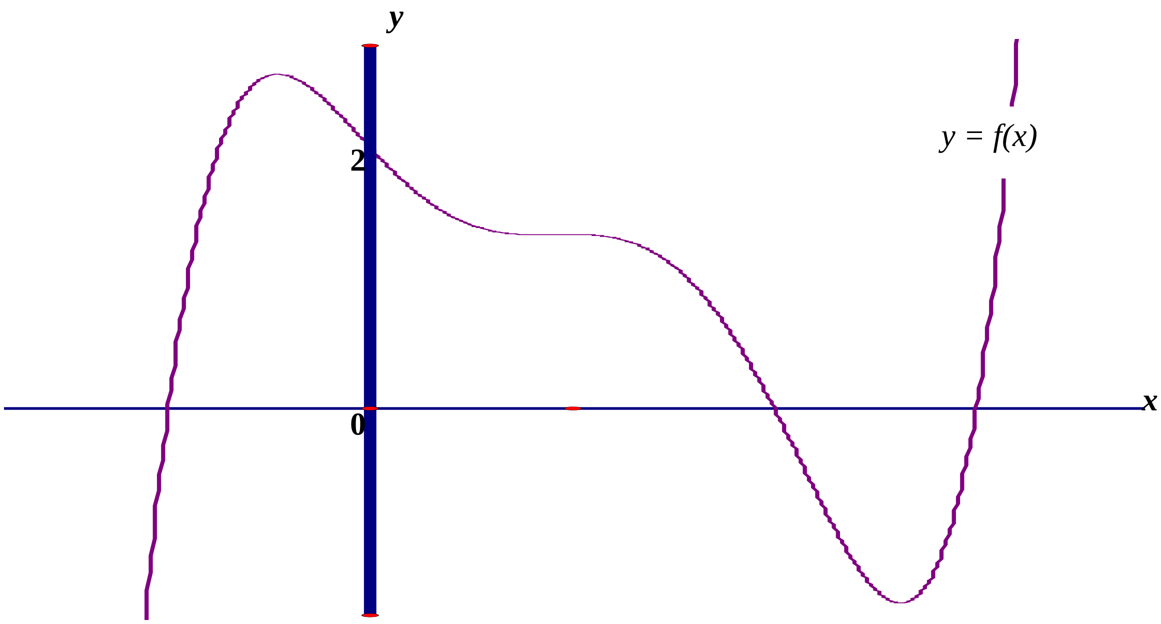
(ii) Hence, show that $\int_0^{\frac{\pi}{4}} \sec^3\theta\, d\theta = \frac{1}{2}\left[\sqrt{2} + \ln\left(\sqrt{2}+1\right)\right]$ **3**

~ END OF ASSESSMENT~

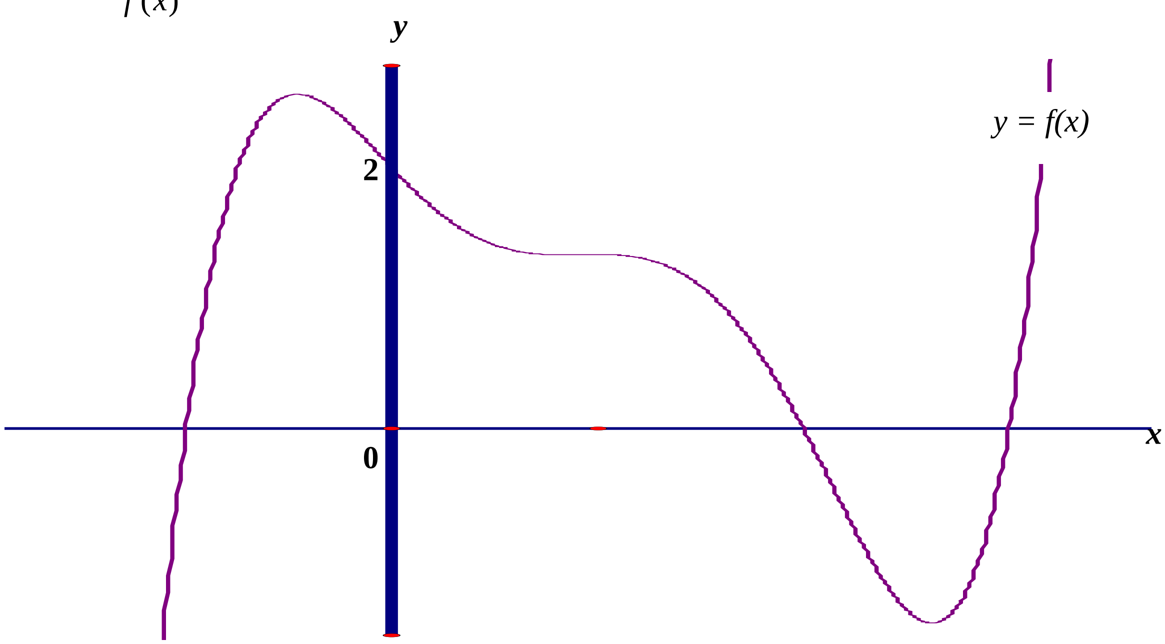
Question 3 Part (b)
HAND IN WITH YOUR ANSWERS

Sketch neatly on separate diagrams

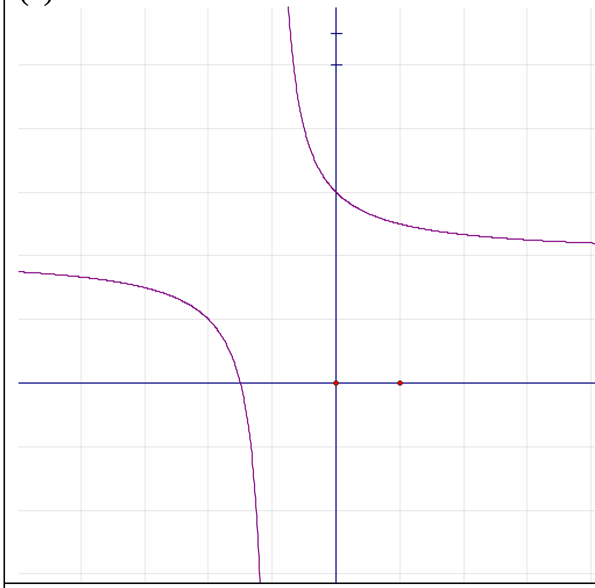
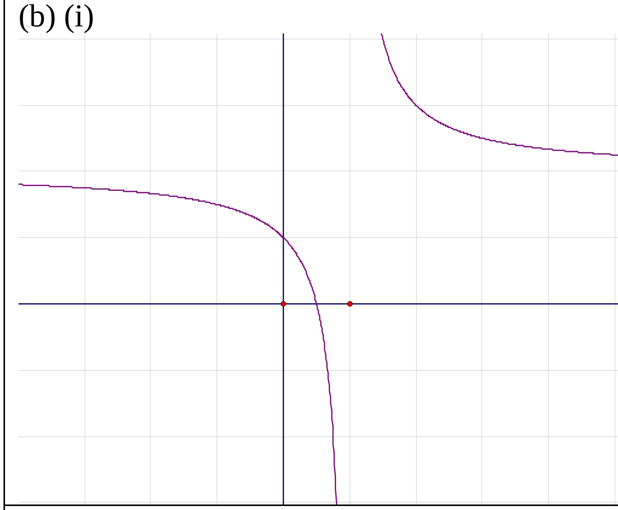
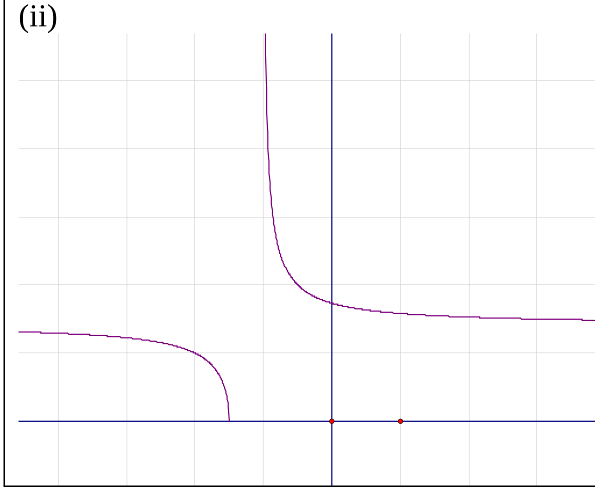
(i) $y = |f(x)|$ **1**



(ii) $y = \dfrac{1}{f(x)}$ **2**



~SOLUTIONS Year 12 Extension 2 Term 1 Assessment 2007~

<table>
<tr><td>Question 1
(a)
</td><td>Asymptotes $x = -1$ & $y = 2$ ➔ 1 mark
Shape ➔ 1 Mark</td></tr>
<tr><td>(b) (i)
</td><td>$y = f(x-2)$ is a horizontal shift of $y = f(x)$ 2 units to the right. ➔ 1 Mark
(Asymptotes at $y = 2$ & $x = 1$)</td></tr>
<tr><td>(ii)
</td><td>$y = \sqrt{f(x)}$
Asymptote at $x = -1$ & $y = \sqrt{2}$, ➔ ½ Mark
y – intercept = $\sqrt{3}$ &
x – intercept = -1.5 ➔ 1 Mark

No graph below x – axis ➔½ Mark

Shape is relatively similar to the original function ∴ no marks awarded</td></tr>
</table>

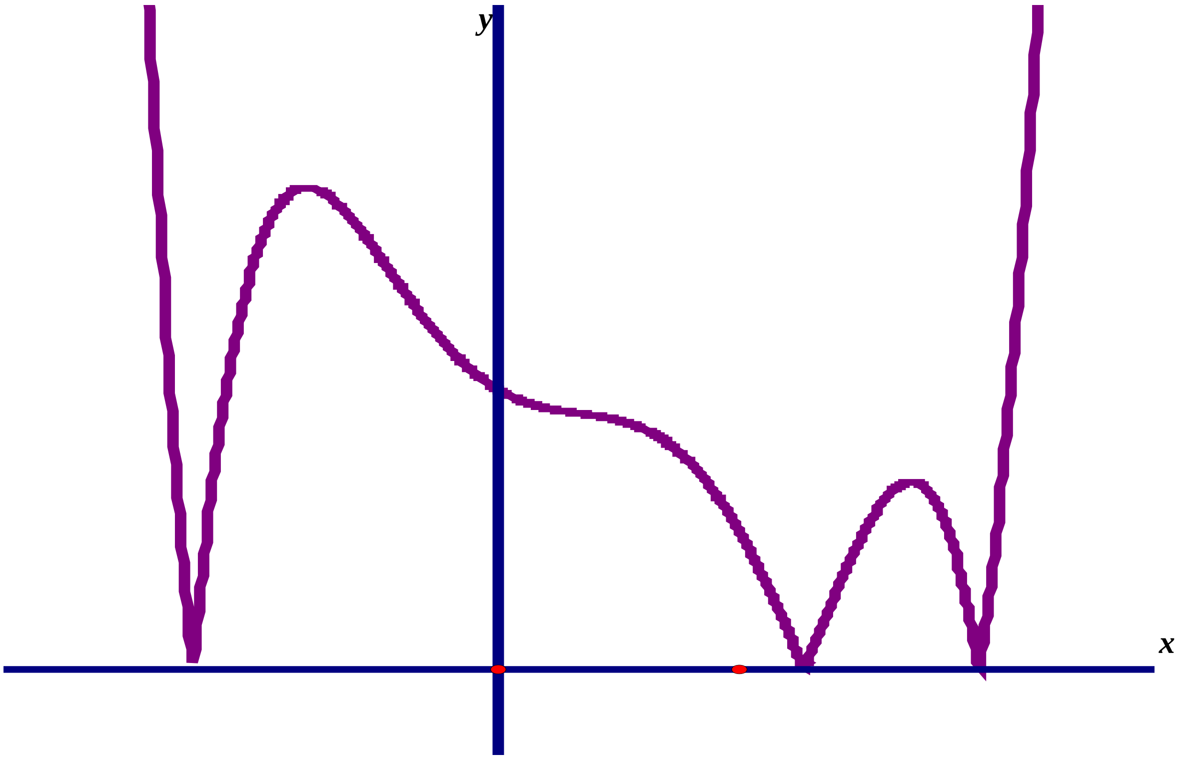
Q1 (c)	
(i) $\displaystyle\int_2^3 \frac{4}{(x-1)(4-x)}\,dx$	Correct values of A and B ➔ 1 Mark
$= \displaystyle\frac{4}{3}\int_2^3 \left(\frac{1}{x-1}+\frac{1}{4-x}\right)dx$ By Partial Fractions	
$= \displaystyle\frac{4}{3}\left[\ln(x-1)-\ln(4-x)\right]_2^3$	Correct integration ➔ 1 Mark
$= \displaystyle\frac{4}{3}\left[(\ln 2-\ln 1)-(\ln 1-\ln 2)\right]$	
$= \displaystyle\frac{4}{3}\left[2\ln 2\right] = \frac{8\ln 2}{3}$	Correct answer ➔ 1 Mark
(ii) $\displaystyle\int_0^1 \frac{dx}{9-4x^2}$	**This is not a inverse Trigonometry Question!.** No marks awarded if used incorrect method.
$= \displaystyle\frac{1}{6}\int_0^1 \left(\frac{1}{3+2x}+\frac{1}{3-2x}\right)dx$	
$= \displaystyle\frac{1}{6}\left[\frac{\ln(3+2x)}{2}-\frac{\ln(3-2x)}{2}\right]_0^1$	Correct use of partial fractions & integration➔ 1 Mark
$= \displaystyle\frac{1}{12}\left[(\ln 5-\ln 3)-(\ln 1-\ln 3)\right]$	
$= \displaystyle\frac{\ln 5}{12}$	Correct evaluation ➔1 Mark
(d) (i) P is $(x\ ,\ y)$ then by data	
$\sqrt{(x+2)^2+y^2}+\sqrt{(x-2)^2+y^2}=6$	➔1 Mark
i.e. $12\sqrt{x^2+4x+4+y^2}=36-8x$	➔1 Mark
$\therefore\ 9(x^2-4x+4+y^2)=81-36x+4x^2$	➔1 Mark
$\therefore\ 5x^2+9y^2=45$ ➔ $\displaystyle\frac{x^2}{9}+\frac{y^2}{5}=1$	➔1 Mark
(ii) Using $b^2 = a^2(1-e^2)$ since locus is an ellipse where $a = 3$ and $b = \sqrt{5}$, we get $e = \displaystyle\frac{2}{3}$.	➔1 Mark

Question 2	
(a) (i) Eqn. of asymptotes $\frac{x^2}{16} - \frac{y^2}{25} = 0$	➔1 Mark
$\therefore y = \pm\frac{5x}{4}$	➔1 Mark
(ii) If P is $(4\sec\theta, 5\tan\theta)$ then LHS $= \frac{x^2}{16} - \frac{y^2}{25}$ $= \frac{16\sec^2\theta}{16} - \frac{25\tan^2\theta}{25}$	➔1 Mark
$= \tan^2\theta + 1 - \tan^2\theta$ $= 1 = $ RHS $\therefore$ P lies on hyperbola H.	Showing LHS = RHS ➔1 Mark
(iii) $\frac{dy}{dx} = \frac{5\sec\theta}{4\tan\theta}$ at P.	Correct differential➔1 Mark
$\therefore$ eqn. of tangent at P is $y - 5\tan\theta = \frac{5\sec\theta}{4\tan\theta}(x - 4\sec\theta)$	Correct eqn. of tangent➔1 Mark
$4\tan\theta y - 5x\sec\theta = 20(\tan^2\theta - \sec^2\theta)$ $\therefore \frac{x\sec\theta}{4} - \frac{y\tan\theta}{5} = 1$	Correct rearrangement to get required answer ➔1 Mark
(iv)tangent cuts $y = \frac{5x}{4}$ when $\frac{x\sec\theta}{4} - \frac{5x}{4}\cdot\frac{\tan\theta}{5} = 1$ $\therefore x = \frac{4}{\sec\theta - \tan\theta}$ and $y = \frac{5}{\sec\theta - \tan\theta}$. $\therefore L\left[\frac{4}{\sec\theta - \tan\theta}, \frac{5}{\sec\theta - \tan\theta}\right]$	Correct coordinates of L ➔ ½ Mark
Tangent cuts $y = -\frac{5x}{4}$ at $M\left[\frac{4}{\sec\theta + \tan\theta}, \frac{-5}{\sec\theta + \tan\theta}\right]$	Coordinates of M ➔ ½Mark
If P is the midpoint of LM then $LP=PM$.	

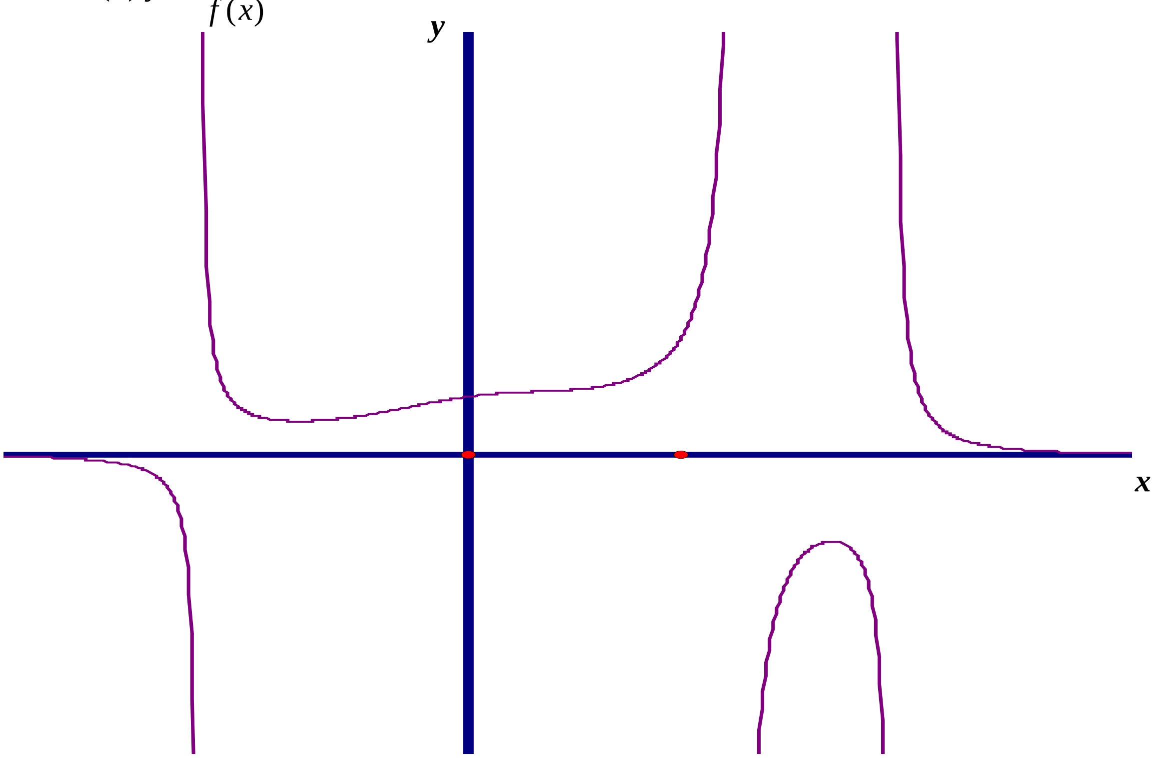
$\therefore$ $x-$ coordinate of midpoint of LM $= \frac{1}{2}\left[\frac{4}{\sec\theta - \tan\theta} + \frac{4}{\sec\theta + \tan\theta}\right]$ $= \frac{1}{2}\left[\frac{8\sec\theta}{\sec^2\theta - \tan^2\theta}\right]$ $= 4\sec\theta$. $y-$ coordinate of midpoint of LM $= \frac{1}{2}\left[\frac{5}{\sec\theta - \tan\theta} + \frac{-5}{\sec\theta + \tan\theta}\right]$ $=5\tan\theta$. $\therefore$ P is the midpoint if LM. $\therefore$ $LP=PM$	Correctly finding x and y coordinates of midpoint Or using distance formula & conclusion ➔2 Marks
(b) (i) $I_n = \int_1^2 (\ln x)^n dx$	
$= \left[x(\ln x)^n\right]_1^2 - \int_1^2 \frac{x \cdot n(\ln x)_1^2}{x} dx$	➔2 Marks
$= 2(\ln 2)^n - nI_{n-1}$	
(ii) $I_3 = 2(\ln 2)^3 - 3I_2$	
$= 2(\ln 2)^3 - 3[2(\ln 2)^2 - 2I_1]$	➔1 Mark
Now $I_1 = \int_1^2 \ln x dx$ $= \left[x\ln x - x\right]_1^2$ $= 2\ln 2 - 2 - 0 + 1$	
$= 2\ln 2 - 1$	➔1 Mark
$\therefore$ $I_3 = 2(\ln 2)^3 - 6(\ln 2)^2 + 12\ln 2 - 6$.	➔1 Mark

Question 3	
(a) $\int \frac{x+1}{x^2+1}dx = \int\left(\frac{x}{x^2+1}+\frac{1}{x^2+1}\right)dx$	➔1 Mark
$= \frac{1}{2}\ln(x^2+1)+\tan^{-1}x+c$	➔1 Mark
(b) See next page	(i) ➔1 Mark (ii) ➔2 Mark
(c) (i) Let $f(x) = x\cos x$ then $f(-x) = -x\cos(-x) = -f(x)$ $\therefore$ it's an odd function. $\therefore \int_{-\pi}^{\pi} x\cos x\,dx = 0$	Showing it's an odd function ➔1 Mark Answer = 0 ➔1 Mark
(ii) $\int_0^{\frac{\pi}{2}} \cos^5\theta\,d\theta = \int_0^{\frac{\pi}{2}}\cos\theta(\cos^4\theta)\,d\theta$ $= \int_0^{\frac{\pi}{2}}\cos\theta(1-\sin^2\theta)^2\,d\theta$ $= \int_0^{\frac{\pi}{2}}\cos\theta - 2\cos\theta\sin^2\theta+\cos\theta\sin^4\theta\,d\theta$	➔1 Mark
$= \left[\sin\theta-\frac{2}{3}\sin^3\theta+\frac{1}{5}\sin^5\theta\right]_0^{\frac{\pi}{2}}$	Correct integration➔1 Mark
$= 1-\frac{2}{3}+\frac{1}{5}=\frac{8}{15}$	Correct answer ➔1 Mark
(d) (i) $PF = \sqrt{(x-4)^2+y^2}$ $PM = \lvert x-1\rvert$ $\therefore PF^2 = 4PM^2$ $\therefore x^2-8x+16+y^2 = 4x^2-8x+4$ $\therefore 3x^2-y^2 = 12$ ➔ $\frac{x^2}{4}-\frac{y^2}{12}=1$	➔1 Mark **each** for *PF* and *PM* (by definition) ➔ ½ for squaring both sides & expanding ➔ ½ for correct locus.
(ii) asymptotes $y = \pm\sqrt{3}x$	Correct equations of asymptotes➔1 Mark
$\therefore$ line $y=\sqrt{3}x$ makes an angle of $\frac{\pi}{3}$ rads with positive x – axis and line $y=-\sqrt{3}x$ makes an angle of $\frac{2\pi}{3}$ rads. $\therefore$ acute angle is $\frac{\pi}{3}$ rads.	Correct answer with some explanation ➔1 Mark

Q3 (b) (i) $y = |f(x)|$ **1**



(ii) $y = \dfrac{1}{f(x)}$ **2**



Question 4	
(a) $\int_0^{\frac{\pi}{2}} \frac{dx}{2+\sin x}$	
[when $x = 0$, $t = 0$ and $x = \frac{\pi}{2}$ then $t = 1$]	
$= \int_0^1 \frac{\frac{2}{1+t^2}}{2+\frac{2t}{1+t^2}}\, dt$	Change of variables & correct t substitution ➔2 Marks
$= \int_0^1 \frac{2\, dt}{2+2t^2+2t} = \int_0^1 \frac{dt}{t^2+t+1}$	
$= \int_0^1 \frac{dt}{\left(t+\frac{1}{2}\right)^2+\frac{3}{4}}$	➔1 Mark
$= \frac{2}{\sqrt{3}}\left[\tan^{-1}\left(\frac{t+\frac{1}{2}}{\frac{\sqrt{3}}{2}}\right)\right]_0^1 = \frac{2}{\sqrt{3}}\left[\tan^{-1}\left(\frac{2t+1}{\sqrt{3}}\right)\right]_0^1$	Correct integration & substitution➔1 Mark
$= \frac{2}{\sqrt{3}}\left[\tan^{-1}\sqrt{3} - \tan^{-1}\frac{1}{\sqrt{3}}\right] = \frac{\pi}{3\sqrt{3}}$	
(b) (i) $\int_0^{2a} f(x)dx = \int_0^a f(x)dx + \int_a^{2a} f(x)dx$	➔1 Mark
Now $\int_a^{2a} f(x)dx = \int_a^0 f(2a-u)\cdot -du$	Let $u = 2a - x$ then $dx = -du$ & $x = a$ ➔ u = a & x = $2a$ ➔ u = 0. ➔1 Mark
$= \int_0^a f\, 2a-u)du = \int_0^a f(2a-x)dx$	
Since the definite integral is independent of the variable.	➔1 Mark
Hence $\int_0^{2a} f(x)dx = \int_0^a f(x)dx + \int_0^a f(2a-x)dx$	
$= \int_0^a \left[f(x) + f(2a-x)\right]dx$	

(ii) $\int_0^{\pi} \frac{x\,dx}{2+\sin x}$	
$= \int_0^{\frac{\pi}{2}} \left[\frac{x}{2+\sin x} + \frac{\pi - x}{2+\sin(\pi - x)} \right] dx$	Correct use of part (i) ➔1 Mark
$= \int_0^{\frac{\pi}{2}} \left[\frac{\pi}{2+\sin x} \right] dx$ since $\sin x = \sin(\pi - x)$	➔1 Mark
$= \pi \cdot \frac{\pi}{3\sqrt{3}}$ from part (a)	
$= \frac{\pi^2}{3\sqrt{3}}$	➔1 Mark
(c) (i) $\int_0^{\frac{\pi}{4}} \sec\theta\,d\theta = \left[\ln(\sec\theta + \tan\theta)\right]_0^{\frac{\pi}{4}}$	Correct integration➔1 Mark
$= \ln\left(\sec\frac{\pi}{4} + \tan\frac{\pi}{4}\right) - \ln(\sec 0 + \tan 0)$	Correct answer➔1 Mark
$= \ln\left(\sqrt{2}+1\right)$	
(ii) $\int_0^{\frac{\pi}{4}} \sec^3\theta\,d\theta = \int \sec\theta \cdot \frac{d}{d\theta}\tan\theta\,d\theta$	
$= \left[\sec\theta\tan\theta\right]_0^{\frac{\pi}{4}} - \int_0^{\frac{\pi}{4}} \tan\theta\sec\theta\tan\theta\,d\theta$	➔1 Mark
$= \sqrt{2} - \int_0^{\frac{\pi}{4}} \sec\theta\tan^2\theta\,d\theta$	
$= \sqrt{2} - \int_0^{\frac{\pi}{4}} \left(\sec^3\theta - \sec\theta\right) d\theta$	➔1 Mark
$\therefore\ 2\int_0^{\frac{\pi}{4}} \sec^3\theta\,d\theta = \sqrt{2} + \int_0^{\frac{\pi}{4}} \sec\theta d\theta$	
$= \sqrt{2} + \ln\left(\sqrt{2}+1\right)$	➔1 Mark using part (i)
$\therefore\ \int_0^{\frac{\pi}{4}} \sec^3\theta\,d\theta = \frac{1}{2}\left[\sqrt{2} + \ln\left(\sqrt{2}+1\right)\right]$	