QUESTION 1

Marks

(a) Evaluate: $\displaystyle\int_0^4 \frac{x}{\sqrt{9+x^2}}\,dx$. **2**

(b) Find:

(i) $\displaystyle\int \frac{1}{1+e^{-x}}\,dx$. **2**

(ii) $\displaystyle\int \sec x \tan^3 x\,dx$. **2**

(c) An ellipse has the equation $\dfrac{x^2}{4}+y^2=1$.

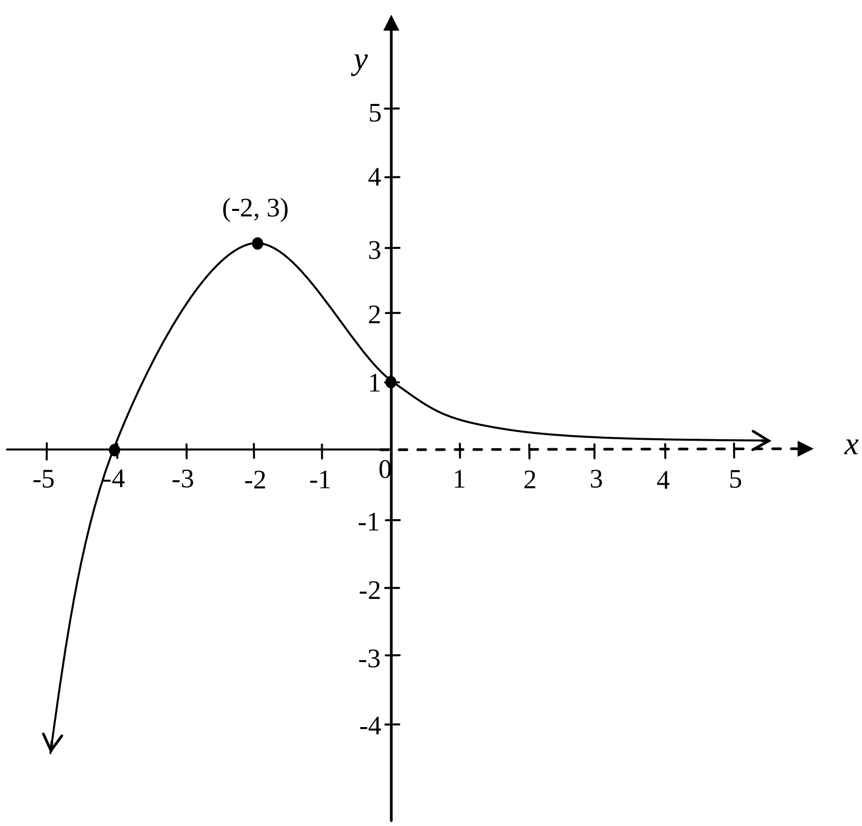
(i) Calculate the eccentricity for this ellipse. **2**

(ii) Draw a neat sketch of the ellipse, clearly labelling the foci, directrices and intercepts with the coordinate axes. **3**

(d) Find the coordinates of the points on the graph of $x^3+y^3=3xy$ at which the tangent lines are parallel to the x-axis. **4**

QUESTION 2 START A NEW PAGE

(a) The diagram below shows the graph of $y=f(x)$. The graph has a horizontal asymptote at $y=0$.



Draw, on separate sets of axes, sketches of the following graphs.

(i) $y=f(|x|)$ **1**

(ii) $y=2^{f(x)}$ **3**

Question 2 continued

Marks

(b) Given $I_n = \int_0^1 \frac{x^n}{x^2+1}\,dx$ for n = 1, 2, 3, ...

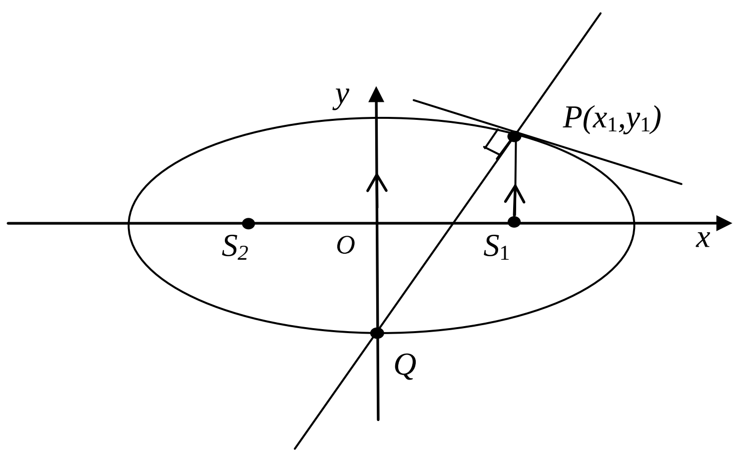
(i) Show that: **2**

$$I_n = \frac{1}{n-1} - I_{n-2} \text{ for } n \geq 2.$$

(ii) Hence, evaluate: **2**

$$\int_0^1 \frac{x^5}{x^2+1}\,dx.$$

(c) $P(x_1, y_1)$ is a point on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ with foci S_1 and S_2, so that PS_1 is parallel to the y-axis and $y_1 \geq 0$, as shown in the diagram below.



(i) Show that the y-coordinate of P can be given by $y_1 = a(1-e^2)$, where e is the eccentricity of the ellipse. **2**

(ii) Prove that the equation of the normal at P is $x - ey - ae^3 = 0$. **3**

(iii) For a particular ellipse the normal at P passes through point Q which is at the end of the minor axis, as shown.
Calculate the value of e^2 for this ellipse, expressing your answer as a surd in simplest form. **2**

QUESTION 3 START A NEW PAGE

Marks

(a) A hyperbola has foci at $S_1(0, 6)$ and $S_2(0, -6)$. One of the vertices is at $A(0, -2)$.

(i) Find the equation of the hyperbola. **3**

(ii) Find the equations of the directrices and the asymptotes of the hyperbola. **2**

(b) Consider the function $f(x) = e^{-x}\sin x$.

(i) Show that the graph of $y = f(x)$

(α) intersects the x-axis at $x = n\pi$, where n is an integer, **1**

(β) has stationary points at $x = \dfrac{(4n+1)\pi}{4}$. **1**

(ii) Sketch the graph of $y = f(x)$ for $-\pi \le x \le \pi$. **2**

(iii) Show that $\int e^{-x}\sin x\, dx = -\frac{1}{2}e^{-x}(\sin x + \cos x) + c$, where c is a constant. **3**

(iv) If A_n is the magnitude of the area of the region bounded by the curve $y = e^{-x}\sin x$ and the x-axis for $(n-1)\pi \le x \le n\pi$ show that: **3**

$$\frac{A_1}{A_0} = e^{-\pi}.$$

QUESTION 4 START A NEW PAGE

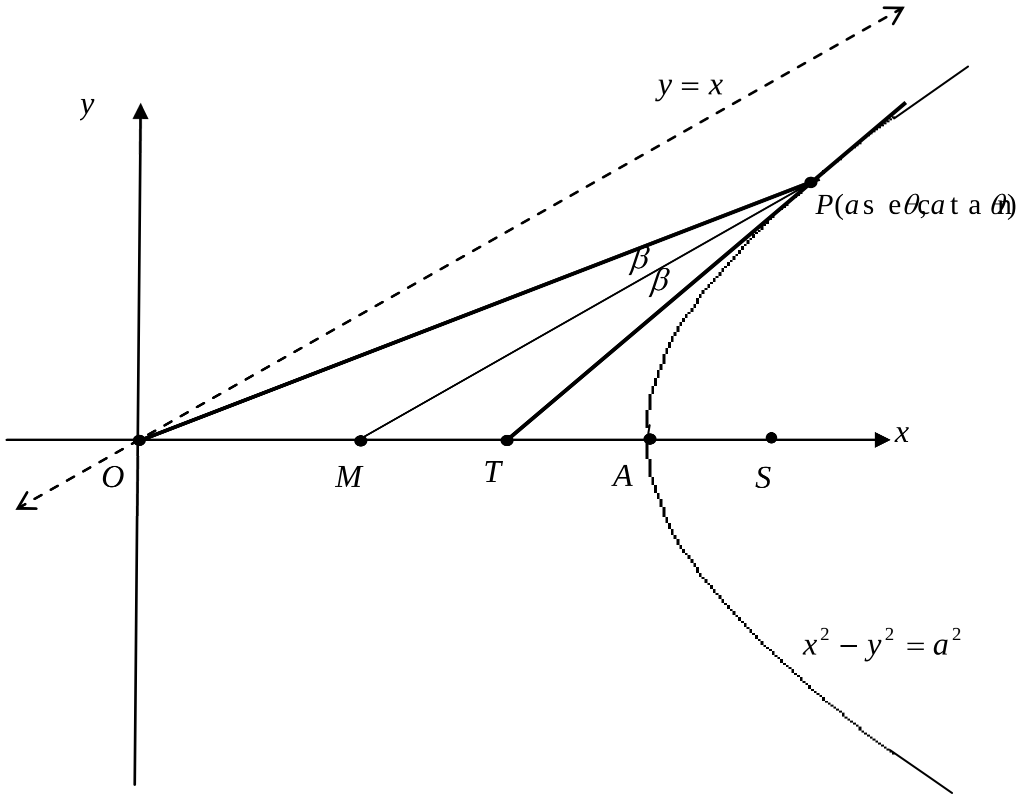
Marks

(a) (i) Prove that $\int_0^a f(x)dx = \int_0^a f(a-x)dx$. **2**

(ii) Hence, or otherwise, calculate the value of $\int_0^{\pi} \dfrac{x\sin x}{1+\cos^2 x}\, dx$. **4**

Question 4 continued **Marks**

(b) $P(a\sec\theta, a\tan\theta)$ is a point on the hyperbola $x^2 - y^2 = a^2$, where $0 < \theta < \dfrac{\pi}{2}$ as shown in the diagram.



The point of intersection of the tangent at P with the x-axis is point T. O is the origin.
Let $\angle OPT = 2\beta$,

(i) Show that $\tan 2\beta = \dfrac{\cos^2\theta}{2\sin\theta}$. **3**

(ii) By using the formula $\tan 2\beta = \dfrac{2\tan\beta}{1-\tan^2\beta}$, show that $\tan\beta = \left[\dfrac{1-\sin\theta}{\cos\theta}\right]^2$. **3**

(iii) Given that MP is the bisector of $\angle OPT$, as shown in the diagram above, prove that MP is always parallel to one asymptote of the hyperbola. **3**

END OF EXAMINATION

MATHEMATICS Extension 2: Question 1

Suggested Solutions	Marks	Marker's Comments
a) $\int_0^4 \frac{x}{\sqrt{9+x^2}}\,dx$ $= \frac{1}{2}\int_0^4 2x(9+x^2)^{-\frac{1}{2}}\,dx$ $= \frac{1}{2}\left[\frac{(9+x^2)^{\frac{1}{2}}}{\frac{1}{2}}\right]_0^4$ $= \left[\sqrt{9+x^2}\right]_0^4 = \sqrt{9+16} - \sqrt{9+0}$ $= 5 - 3$ $= 2$	2	1 Integration 1 Answer
b) (i) $\int \frac{1}{1+e^{-x}}\,dx$ $= \int \frac{1}{1+e^{-x}} \times \frac{e^x}{e^x}\,dx$ $= \int \frac{e^x}{e^x+1}\,dx$ $= \ln(e^x+1) + C$	2	1 change of integral 1 answer
(ii) $\int \sec x \tan^3 x\,dx$ $= \int \sec x(\tan^2 x)\tan x\,dx$ $= \int \sec x(\sec^2 x - 1)\tan x\,dx$ $= \int \sec^3 x \tan x - \sec x \tan x\,dx$ $= \int \sec x \tan x(\sec^2 x) - \sec x \tan x\,dx$ $= \frac{\sec^3 x}{3} - \sec x + C$	2	1 use of identity and re organisation of integral 1 answer
c) i) $\frac{x^2}{4} + y^2 = 1 \quad a = 2 \quad y = 1$ $b^2 = a^2(1-e^2)$ $1 = 4(1-e^2)$ $\frac{1}{4} = 1 - e^2$ $e^2 = 1 - \frac{1}{4} = \frac{3}{4} \quad \therefore e = \frac{\sqrt{3}}{2}$	2	1 correct sub into formula 1 answer

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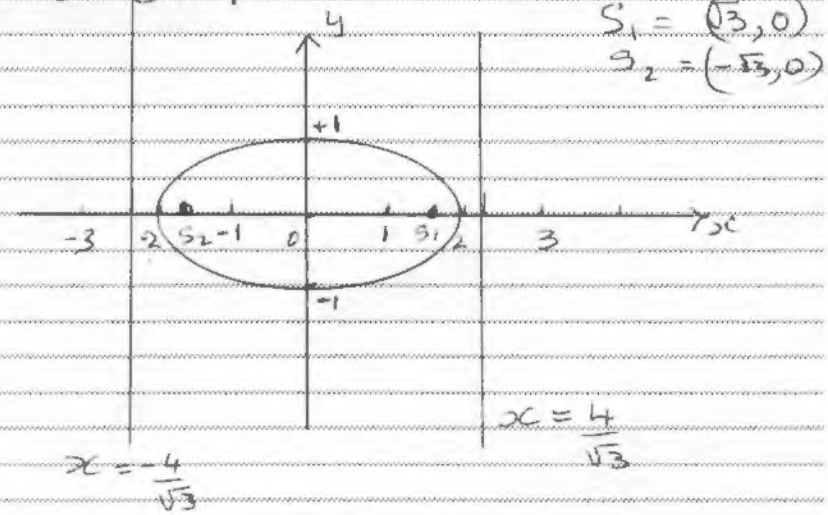
MATHEMATICS Extension 2: Question 1

Suggested Solutions	Marks	Marker's Comments
c) (ii) Foci : $x = \pm ae \quad y = 0$		
$ae = 2 \times \frac{\sqrt{3}}{2} = \sqrt{3} \sim 1{\cdot}7$	3	1 Foci
Directrices $x = \pm \frac{a}{e} = \frac{2}{\sqrt{3}/2}$		1 directrices
$= \pm \frac{4}{\sqrt{3}} \sim \pm 2{\cdot}3$		
$a = 2 \quad b = 1$		1 scale & shape, axes etc
		
d) $x^3 + y^3 = 3xy \quad (i)$	4	
$3x^2 + 3y^2 \frac{dy}{dx} = 3x \frac{dy}{dx} + 3y$		1 differentiation
for horizontal tangent $\frac{dy}{dx} = 0$		
$\therefore y = x^2$		1 $y = x^2$
sub into (i) $x^3 + (x^2)^3 = 3x(x^2)$		
$\therefore x^3 + x^6 = 3x^3$		
$\therefore x^6 - 2x^3 = 0$		
$x^3(x^3 - 2) = 0$		
$\therefore x = 0 \quad or \quad x = \sqrt[3]{2}$		1 x values
$y = 0 \qquad y = \sqrt[3]{4}$		
Points $(0, 0)$, $(\sqrt[3]{2}, \sqrt[3]{4})$		1 y values.
		(or 1 for each point)

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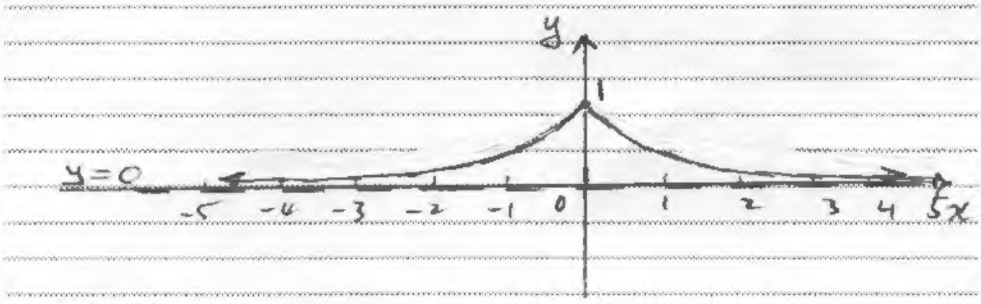
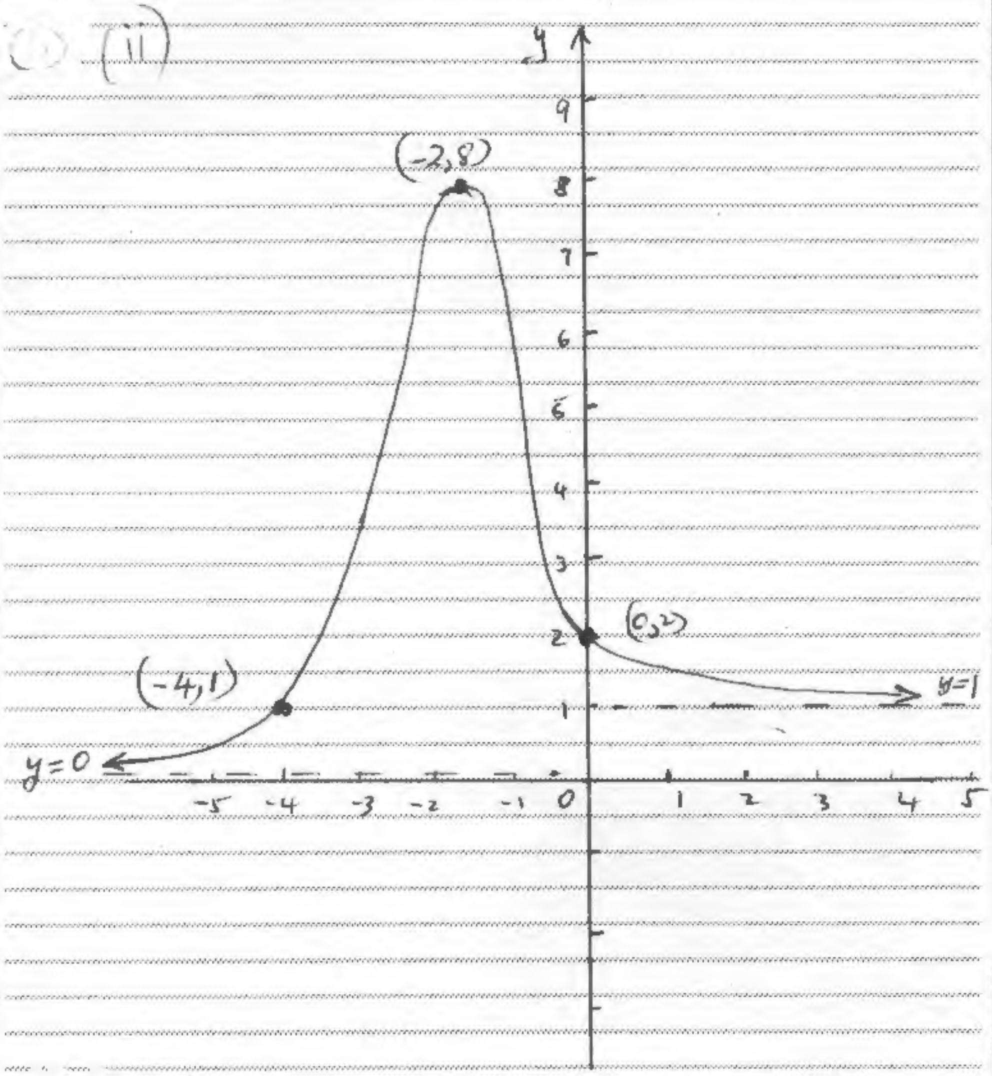
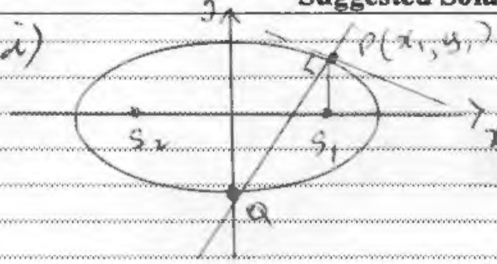
MATHEMATICS Extension 2: Question 2

Suggested Solutions	Marks	Marker's Comments
(a) (i) 	1	must show asymptote $y=0$ reflection about $x=0$
(ii) 	3	1 Asymptotes $y=0$ $y=1$ 1 y intercept $(0,2)$ Point $(-4,1)$ 1 Max at $(-2,8)$ & shape

MATHEMATICS Extension 2: Question 2 continued		
Suggested Solutions	**Marks**	**Marker's Comments**
b) (i) $I_n = \int_0^1 \frac{x^n}{x^2+1}\,dx \quad n=1,2,3\ldots$	2	
$= \int_0^1 x^{n-2} \times \frac{x^2}{(x^2+1)}\,dx$		
$= \int_0^1 x^{n-2} \times \left[1 - \frac{1}{x^2+1}\right] dx$		① rearranging integral.
$= \int_0^1 x^{n-2}\,dx - \int_0^1 \frac{x^{n-2}}{x^2+1}\,dx$		
$= \left[\frac{x^{n-1}}{n-1}\right]_0^1 - I_{n-2}$		① integration
$= \frac{1}{n-1} - I_{n-2}$		
$\int_0^1 \frac{x^5}{x^2+1}\,dx = I_5$	2	
$I_5 = \frac{1}{5-1} - I_3$		
$= \frac{1}{4} - \left[\frac{1}{3-1} - I_1\right]$		
$= \frac{1}{4} - \left[\frac{1}{2} - \int_0^1 \frac{x}{x^2+1}\,dx\right]$		① correct sub to I_1
$= \frac{1}{4} - \frac{1}{2} + \left[\frac{1}{2}\ln(x+1)\right]_0^1$		
$= -\frac{1}{4} + \frac{1}{2}\ln 2 - \frac{1}{2}\ln 1$		
$= \frac{1}{2}\ln 2 - \frac{1}{4}$		① answer

MATHEMATICS Extension 2: Question 2 (continued)

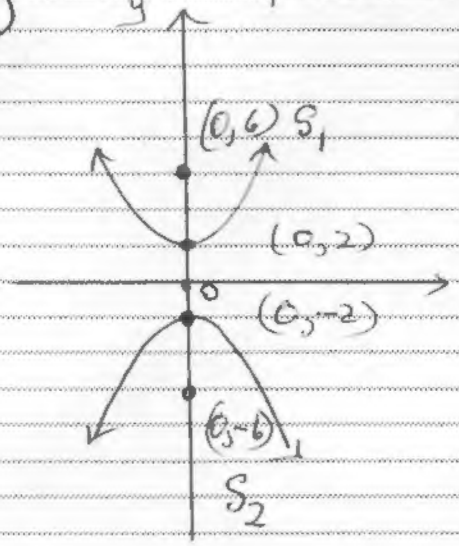
Suggested Solutions	Marks	Marker's Comments
(c) i)  $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$	2	
$x_1 = ae$		
$b^2 = a^2(1-e^2)$		} 1
$\frac{(ae)^2}{a^2}+\frac{y^2}{a^2(1-e^2)}=1$		
$e^2+\frac{y^2}{a^2(1-e^2)}=1$		
$y^2 = (1-e^2)\times a^2(1-e^2)$		1 showing result.
$y_1 = a(1-e^2)$ as $y_1 \geq 0$		
(ii) $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$	3	
$\frac{2x}{a^2}+\frac{2y}{b^2}\frac{dy}{dx}=0$		
$\therefore \frac{dy}{dx} = -\frac{2x}{a^2}\times\frac{b^2}{2y}$		1 $\frac{dy}{dx}$ formula
$x=ae \quad y=a(1-e^2)$		
$m_T = -\frac{2(ae)}{a^2}\times\frac{a^2(1-e^2)}{2a(1-e^2)}$		
$= -e$		1 grad of tangent.
$m_N = \frac{1}{e}$		
Equation of normal.		
$y-y_1 = m(x-x_1)$		
$y-a(1-e^2) = \frac{1}{e}(x-ae)$		1 equation
$y-a+ae^2 = \frac{x}{e}-a$		
$y+ae^2 = \frac{x}{e}$		
$0 = x-ey-ae^3$		
Normal is $x-ey-ae^3=0$		

MATHEMATICS Extension 2: Question 2 continued

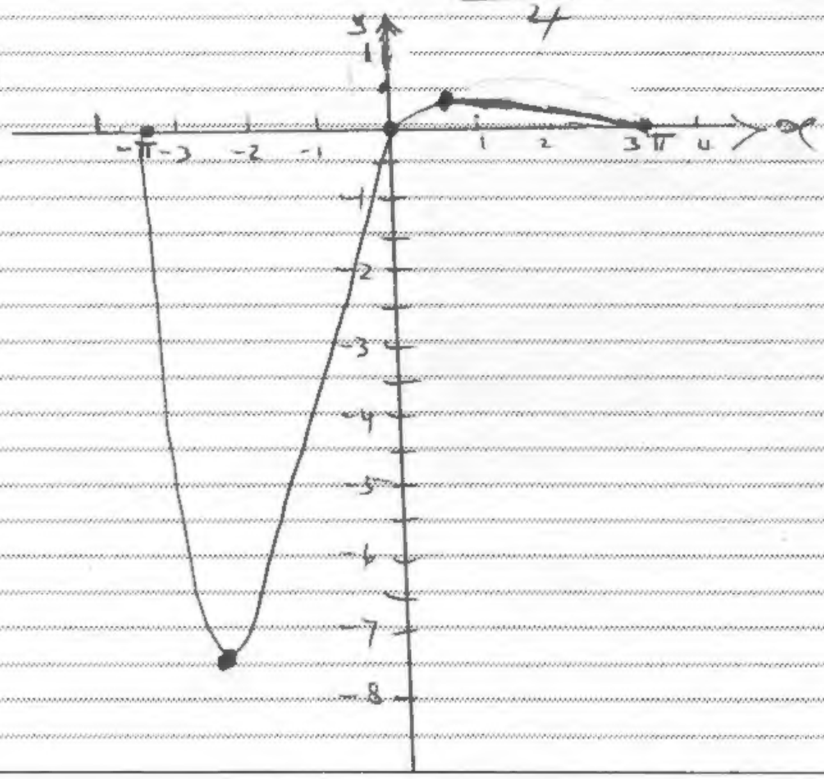
Suggested Solutions	Marks	Marker's Comments
(c) (iii) $Q = (0, -b)$ $b^2 = a^2(1-e^2)$	2	
$\therefore x = 0$ $y = -a\sqrt{1-e^2}$		
$x - ey - ae^3 = 0$		
$+ea\sqrt{1-e^2} = ae^3$		
$\sqrt{1-e^2} = e^2$		1 an equation in e only
$1 - e^2 = e^4$		
$e^4 + e^2 - 1 = 0$		
$\therefore e^2 = \frac{-1 \pm \sqrt{1+4}}{2}$		
but $e^2 > 0$		
$\therefore e^2 = \frac{-1+\sqrt{5}}{2}$		1 answer

MATHEMATICS Extension 2: Question 3

Suggested Solutions	Marks	Marker's Comments
(a) (i) foci: $S_1 = (0,6)$ $S_2 (0,-6)$	3	
$\therefore$ centre is $(0,0)$		
Vertex $= (0,-2)$		
$\therefore$ Hyperbola is of the form		
$\frac{y^2}{b^2} - \frac{x^2}{a^2} = 1$		① basic equation of hyperbola
$b = 2$		
$be = 6$		
$2e = 6$		
$e = 3$		① value of e
$a^2 = b^2(e^2-1)$ $b^2 = 4$		
$= 4(9-1)$		
$= 32$		
Equation of Hyperbola is $\frac{y^2}{4} - \frac{x^2}{32} = 1$		① answer
(ii) Directrices		
$y = \pm b/e$ $y = \pm \frac{2}{3}$	2	① answer
Asymptotes		
$x = \pm a/b\, y$		
$x = \pm \frac{\sqrt{32}}{2} y$		
$x = \pm 2\sqrt{2}\, y$		
OR $y = \pm \frac{1}{2\sqrt{2}} x$		① answer



MATHEMATICS Extension 2: Question 3

Suggested Solutions	Marks	Marker's Comments
b) i) $y = e^{-x}\sin x$	1	
α) x intercepts $y=0$		
$0 = e^{-x}\sin x \quad e^{-x} \neq 0$		must include $e^{-x} \neq 0$
$\therefore \sin x = 0$		1
$x = n\pi \quad n \in \mathbb{Z}$		
β) stationary points $\frac{dy}{dx} = 0$	1	
$\frac{dy}{dx} = e^{-x}\cos x - e^{-x}\sin x$		
$0 = e^{-x}(\cos x - \sin x)$		no loss of mark if not write $e^{-x} \neq 0$
$\cos x = \sin x$ as $e^{-x} \neq 0$		
$\tan x = 1$		1
$x = n\pi + \frac{\pi}{4}$		
$x = \frac{(4n+1)\pi}{4}$		
(ii) 	2	Accurate graph not required. 1 Must show • intercepts at $x = \pm\pi$ and 0 1 • turning points at $x = -3\pi/4$ and $x = \pi/4$ Approx values are: $(0.8, 0.3)$ $(2.4, -7.5)$

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MATHEMATICS Extension 2: Question 3

Suggested Solutions	Marks	Marker's Comments
(iii) $I = \int e^{-x} \sin x \, dx$	3	
$= \left[-e^{-x} \sin x\right] + \int e^{-x} \cos x \, dx$		1 first Integration by parts
$= \left[-e^{-x} \sin x\right] + \left[-e^{-x} \cos x - \int e^{-x} \sin x \, dx\right]$		1 2nd integration by part
$= -e^{-x}(\sin x + \cos x) - \int e^{-x} \sin x \, dx$		
$= -e^{-x}(\sin x + \cos x) - I + C$		
$\therefore 2I = -e^{-x}(\sin x + \cos x) + C$		1 showing answer.
$I = -\frac{1}{2} e^{-x}(\sin x + \cos x) + C$		
(iv) $A_1 = \left[-\frac{1}{2} e^{-x}(\sin x + \cos x)\right]_0^{\pi}$	3	
$= \left\{-\frac{1}{2} e^{-\pi}(0-1)\right\} - \left\{-\frac{1}{2} e^{0}(0+1)\right\}$		
$= \frac{1}{2} e^{-\pi} + \frac{1}{2}$		1
$A_0 = \left\lvert\left[-\frac{1}{2} e^{-x}(\sin x + \cos x)\right]_{-\pi}^{0}\right\rvert$		
$= \left\lvert\left\{-\frac{1}{2} e^{0}(0+1)\right\} - \left\{-\frac{1}{2} e^{\pi}(0-1)\right\}\right\rvert$		No loss of mark here for not using absolute
$= \left\lvert -\frac{1}{2} - \frac{1}{2} e^{\pi}\right\rvert = \frac{1}{2} + \frac{1}{2} e^{\pi}$		1
$\frac{A_1}{A_0} = \frac{\cancel{\frac{1}{2}}(e^{-\pi}+1)}{\cancel{\frac{1}{2}}(1+e^{\pi})} = \frac{e^{-\pi}\cancel{(1+e^{\pi})}}{\cancel{(1+e^{\pi})}}$		1 loss of mark if absolute not used for A_0
$= e^{-\pi}$		

MATHEMATICS Extension 2: Question 4

Suggested Solutions	Marks	Marker's Comments
(a) (i) LHS $= \int_0^a f(x)\,dx = -\int_a^0 f(a-u)\,du$	2	1
$u = a - x$ $\quad = \int_0^a f(a-u)\,du$		1 change of variable values.
$x=0 \quad u=a$ $\quad = RHS$ $x=a \quad u=0$		
(ii) $\int_0^{\pi} \frac{x\sin x}{1+\cos^2 x}\,dx = \int_0^{\pi} \frac{(\pi-x)\sin(\pi-x)}{1+\cos^2(\pi-x)}\,dx$	4	
$= \int_0^{\pi} \frac{(\pi-x)\sin x}{1+\cos^2 x}\,dx$		
$= \int_0^{\pi} \frac{\pi\sin x}{1+\cos^2 x}\,dx - \int_0^{\pi} \frac{x\sin x}{1+\cos^2 x}\,dx$		Change variable 1 and split integral.
$\therefore 2\int_0^{\pi} \frac{x\sin x}{1+\cos^2 x}\,dx = \pi\int_0^{\pi} \frac{\sin x}{1+\cos^2 x}\,dx$		1 Move part integral to LHS
Use substitution $y = \cos x$ $\frac{dy}{dx} = -\sin x$		
$x=0 \quad y=1$ $x=\pi \quad y=-1$		
$\therefore 2I = \pi\int_1^{-1} \frac{-1}{1+y^2}\,dy$		1 substitution
$\therefore I = \frac{-\pi}{2}\left[\tan^{-1} y\right]_1^{-1}$		
$= \frac{-\pi}{2}\left[\tan^{-1}(-1) - \tan^{-1}(1)\right]$		
$= \frac{-\pi}{2} \times \left[-\frac{\pi}{4} - \frac{\pi}{4}\right] = \frac{\pi^2}{4}$		1 answer

MATHEMATICS Extension 2: Question 4 continued

Suggested Solutions	Marks	Marker's Comments
b) (i) $x^2 - y^2 = a^2$	3	
$2x - 2y\frac{dy}{dx} = 0$		
$\frac{dy}{dx} = \frac{-2x}{-2y}$		
$= \frac{x}{y}$		
gradient of PT $= \frac{a\sec\theta}{a\tan\theta}$		
$= \frac{1}{\cos\theta} \times \frac{\cos\theta}{\sin\theta} = \frac{1}{\sin\theta}$		1 grad PT
gradient of OP $= \frac{a\tan\theta}{a\sec\theta} = \sin\theta$		1 grad OP
Angle between 2 lines is acute		
$\tan 2\beta = \left\lvert\frac{m_1 - m_2}{1 + m_1 m_2}\right\rvert$		
$= \frac{\frac{1}{\sin\theta} - \sin\theta}{1 + \frac{1}{\sin\theta} \times \sin\theta}$		
$= \frac{1 - \sin^2\theta}{\sin\theta\,[1+1]}$		1 use of angle between two lines formula
$= \frac{\cos^2\theta}{2\sin\theta}$		

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MATHEMATICS Extension 2: Question 4 continued

Suggested Solutions	Marks	Marker's Comments
b) (ii) Let $t = \tan\beta$	3	

$$\tan 2\beta = \frac{2t}{1-t^2}$$

$$\frac{\cos^2\theta}{2\sin\theta} = \frac{2t}{1-t^2}$$

$$\cos^2\theta - t^2\cos^2\theta = 4t\sin\theta$$

$$0 = t^2\cos^2\theta + 4t\sin\theta - \cos^2\theta$$

① Quadratic in $\tan\beta$

$$t = \frac{-4\sin\theta \pm \sqrt{16\sin^2\theta + 4\cos^4\theta}}{2\cos^2\theta}$$

$$= \frac{-4\sin\theta \pm 2\sqrt{4\sin^2\theta + (1-\sin^2\theta)^2}}{2\cos^2\theta}$$

① change of $\cos^4\theta$ to $(1-\sin^2\theta)^2$

$$= \frac{-4\sin\theta \pm 2\sqrt{4\sin^2\theta + 1 - 2\sin^2\theta + \sin^4\theta}}{2\cos^2\theta}$$

$$= \frac{-4\sin\theta \pm 2\sqrt{1 + 2\sin^2\theta + \sin^4\theta}}{2\cos^2\theta}$$

$$= \frac{-4\sin\theta \pm 2\sqrt{(1+\sin^2\theta)^2}}{2\cos^2\theta}$$

$$= \frac{-4\sin\theta \pm 2(1+\sin^2\theta)}{2\cos^2\theta}$$

$$= \frac{-\cancel{4}^{2}\sin\theta \pm \cancel{2}(1+\sin^2\theta)}{\cancel{2}\cos^2\theta}$$

$$= \frac{-\sin^2\theta - 2\sin\theta - 1}{\cos^2\theta} \text{ or } \frac{1 - 2\sin\theta + \sin^2\theta}{\cos^2\theta}$$

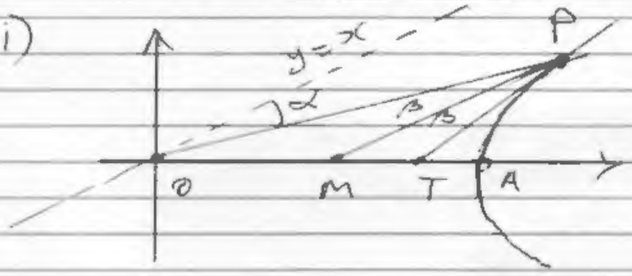
① full solution

$$= -\left[\frac{1+\sin\theta}{\cos\theta}\right]^2 \text{ or } \left[\frac{1-\sin^2\theta}{\cos\theta}\right]^2$$

But $0 < \beta < \pi/4$

$$\therefore \tan\beta = \left[\frac{1-\sin^2\theta}{\cos\theta}\right]^2$$

MATHEMATICS Extension 1 : Question……..

Suggested Solutions	Marks	Marker's Comments
(iii) 	3	
Let angle between OP and asymptote be α. Asymptote : $y = x$ gradient $= 1$ $\tan\alpha = \dfrac{\text{grad of asymptote} - m_{OP}}{1 - \text{grad asymptote} \times m_{OP}}$		
$= \dfrac{1 - \sin\theta}{1 + 1\times\sin\theta}$		1 sub into formula.
$= \dfrac{1-\sin\theta}{1+\sin\theta} \times \dfrac{1-\sin\theta}{1-\sin\theta}$ $= \dfrac{(1-\sin\theta)^2}{1-\sin^2\theta}$ $= \dfrac{(1-\sin\theta)^2}{\cos^2\theta}$ $= \left[\dfrac{1-\sin\theta}{\cos\theta}\right]^2$ $= \tan\beta$		1 showing $\tan\alpha = \tan\beta$
as α and β are acute $\therefore \alpha = \beta$ $\therefore$ MP is parallel to asymptote $y = x$ as alternate angles are equal.		1 explanation