Year 12 Extension II Term 1 2009

Question 1: **Marks**

(a) Find (i) $\int \frac{x-2}{x^2+1}\,dx$ **2**

(ii) $\int \frac{\sin^{-1} x}{\sqrt{1-x^2}}\,dx$ **1**

(b) Find $\int \frac{dx}{\sqrt{6-x-x^2}}$ **3**

(c) Evaluate $\int_0^{\frac{\pi}{4}} \tan^3\theta\,d\theta$ **3**

(d) If $I_n = \int x^n \sin x\,dx$, show that

(i) $I_n + n(n-1)I_{n-2} = x^{n-1}(n\sin x - x\cos x)$ **3**

(ii) Hence, evaluate $\int_0^{\pi} x^2 \sin x\,dx$ **3**

Question 2: **(START A NEW PAGE)**

(a) Using the separate graphs of $y = f(x)$ provided at the end of the examination paper, sketch the graphs of the following on **separate** diagrams. Clearly label the coordinates of any intercepts with the coordinate axes and the position of any asymptotes.

(i) $y = \frac{1}{f(x)}$ **2**

(ii) $y = f(2-x)$ **3**

(iii) $y^2 = f(x)$ **2**

(b) Given that $\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$, show that $\int_0^{\frac{\pi}{2}} \frac{\sin x}{\sin x + \cos x}dx = \frac{\pi}{4}$ **3**

(c) Find the centre of the circle that passes through the foci of the conics

$4x^2 + 9y^2 = 1$ and $y = 2x^2$. **5**

Question 3: **(START A NEW PAGE)** **Marks**

(a) A function $y = f(x)$ has the following properties:

(b) $f(0)=13$, $f(x) \to 0$ as $x \to -\infty$ and $f'(x) = e^x\left(x^2 - 5x + 6\right)$ **5**

Draw a sketch of a function with these properties, clearly showing the positions of any stationary points, inflexion points and asymptotes.

(c) If $x^2 + 4y^2 = 24$ defines the equation of an ellipse

(i) Find the coordinates of the foci and intercepts with the coordinate axes. **4**

(ii) Find the equations of the directrices. **1**

(iii) Draw a neat sketch of the ellipse showing the above features. **3**

(d) Derive the equation of the tangent to the hyperbola $\frac{x^2}{2} - \frac{y^2}{3} = -4$ at the point $P(4,6)$. **2**

Question 4: **(START A NEW PAGE)**

(a) The line $y = 2x - 4$ meets the hyperbola $\frac{x^2}{3} - \frac{y^2}{2} = 1$ at points *M* and *N*.

(i) Find the size of the acute angle between the asymptotes of the given hyperbola. **2**

(ii) Find the coordinates of the point of intersection of the tangent drawn from *M* and *N*. **3**

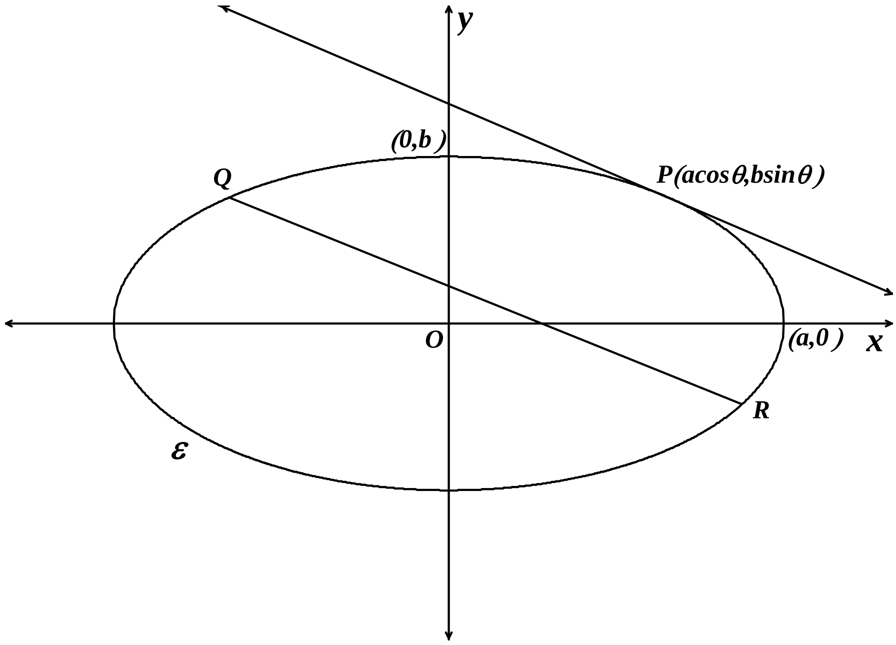
(b) (i) Find values for *A*, *B* and *C* so that $\frac{32}{16 - x^4} \equiv \frac{A}{4 + x^2} + \frac{B}{2 + x} + \frac{C}{2 - x}$ **3**

(ii) Hence, evaluate $\int_0^1 \frac{dx}{16 - x^4}$ **3**

QUESTION 4(c) is continued on the next page

QUESTION 4 (continued)

(c) Consider the ellipse ε with equation $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, and the points $P(a\cos\theta, b\sin\theta)$, $Q(a\cos(\theta+\phi), b\sin(\theta+\phi))$ and $R(a\cos(\theta-\phi), b\sin(\theta-\phi))$ on ε.



(i) Prove that the equation of the tangent to ε at the point P is $\frac{x\cos\theta}{a}+\frac{y\sin\theta}{b}=1$. **2**

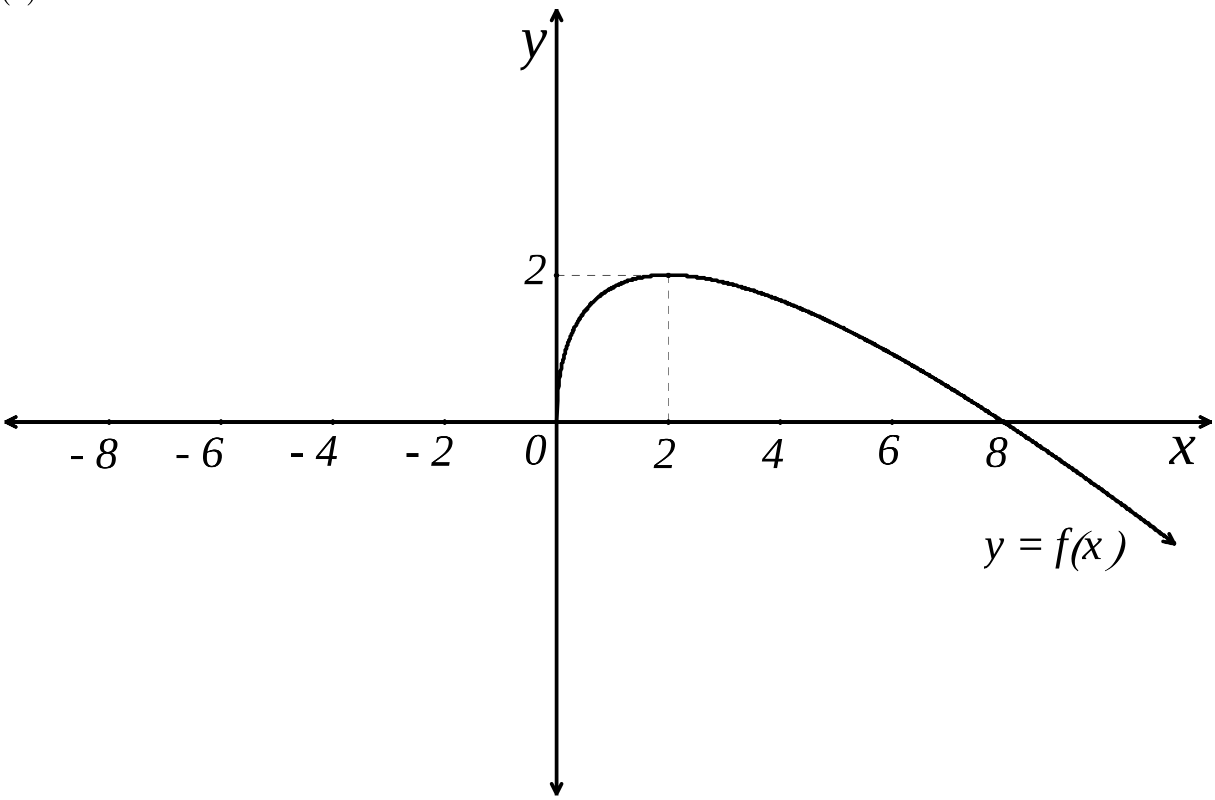
(ii) Show that the chord QR is parallel to the tangent at P. **2**

THIS IS THE END OF THE EXAMINATION PAPER

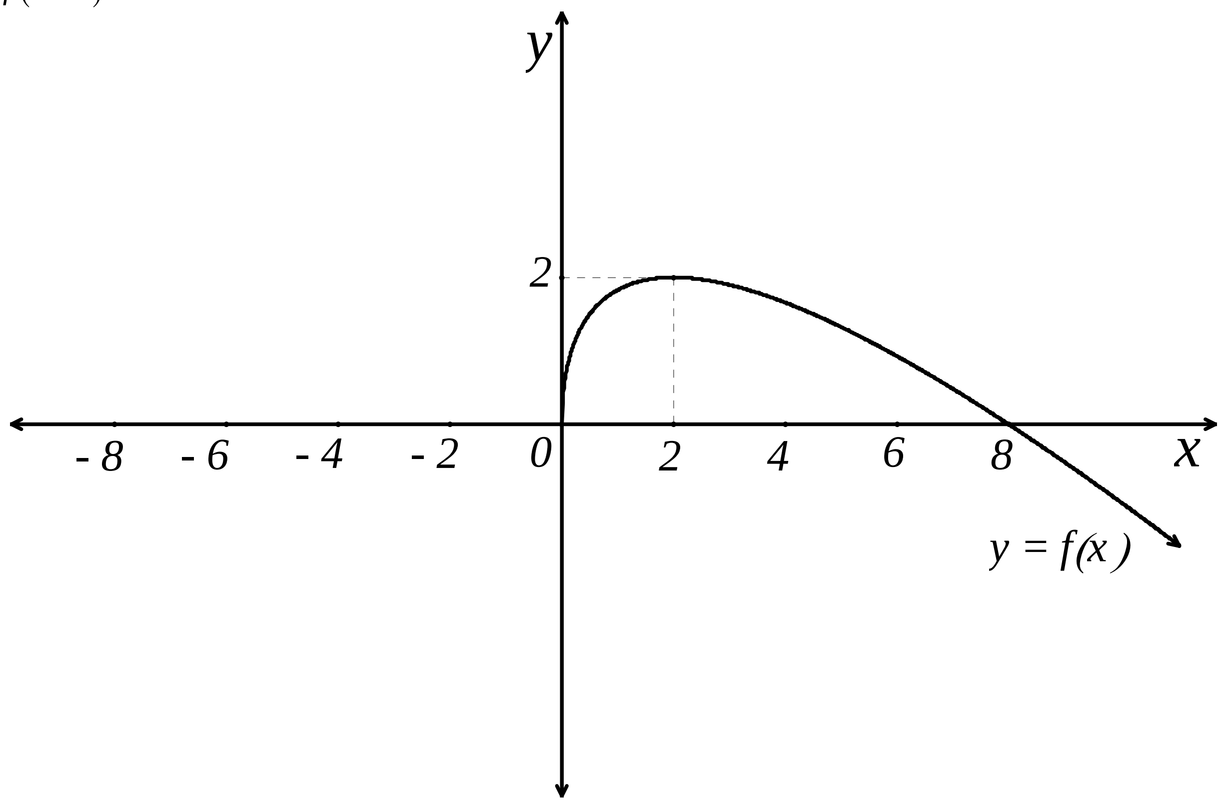
QUESTION 2(a)(i), (ii) STUDENT NUMBER: ………………………

INCLUDE THESE SHEETS WITH YOUR ANSWERS TO Q2

(i) $y = \dfrac{1}{f(x)}$



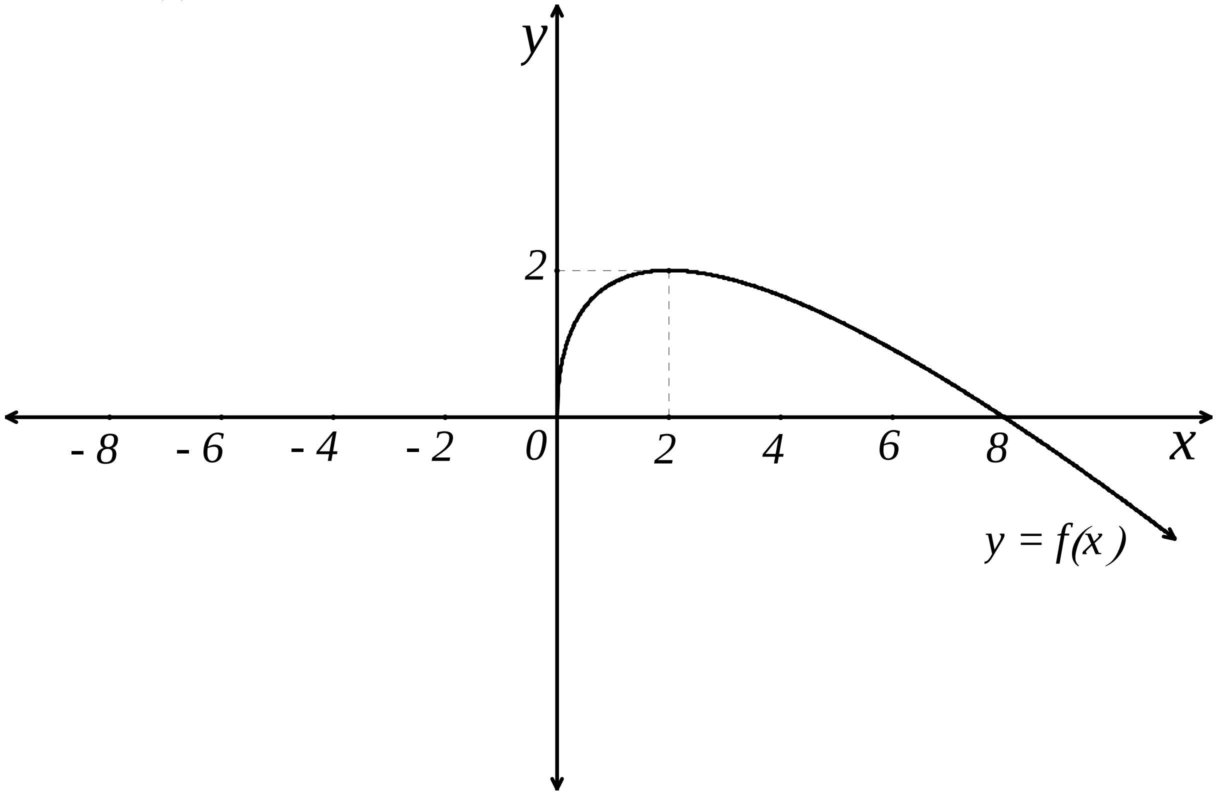
(ii) $y = f(2-x)$



QUESTION 2(a)(iii) STUDENT NUMBER: ………………………

INCLUDE THESE SHEETS WITH YOUR ANSWERS TO Q2

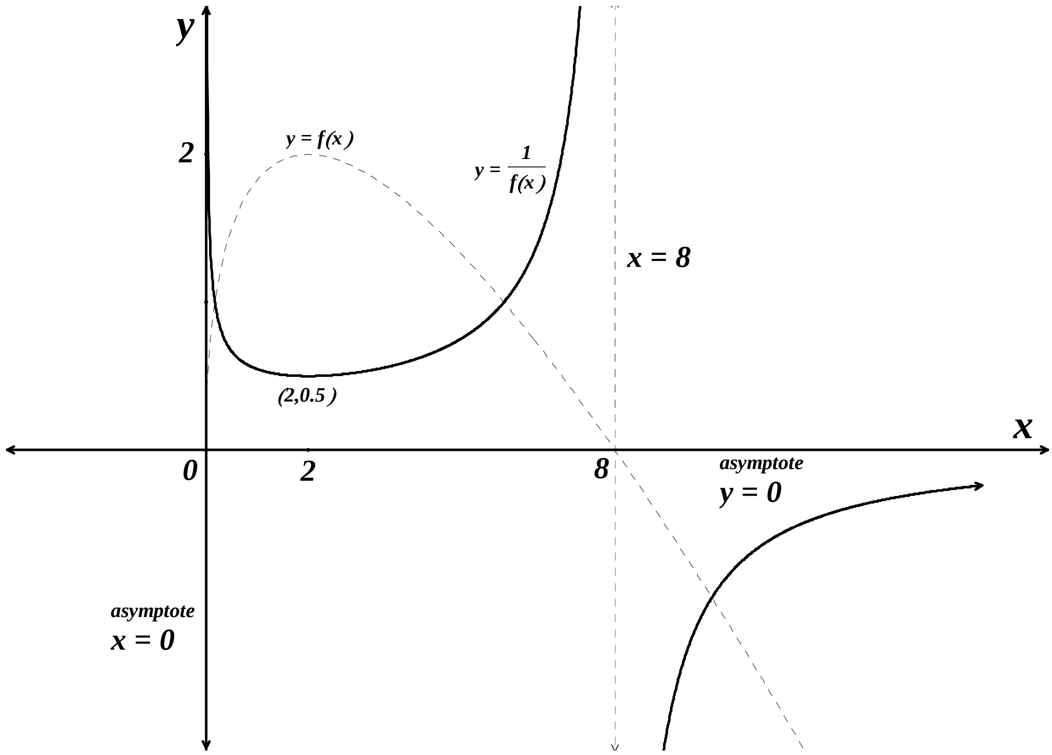
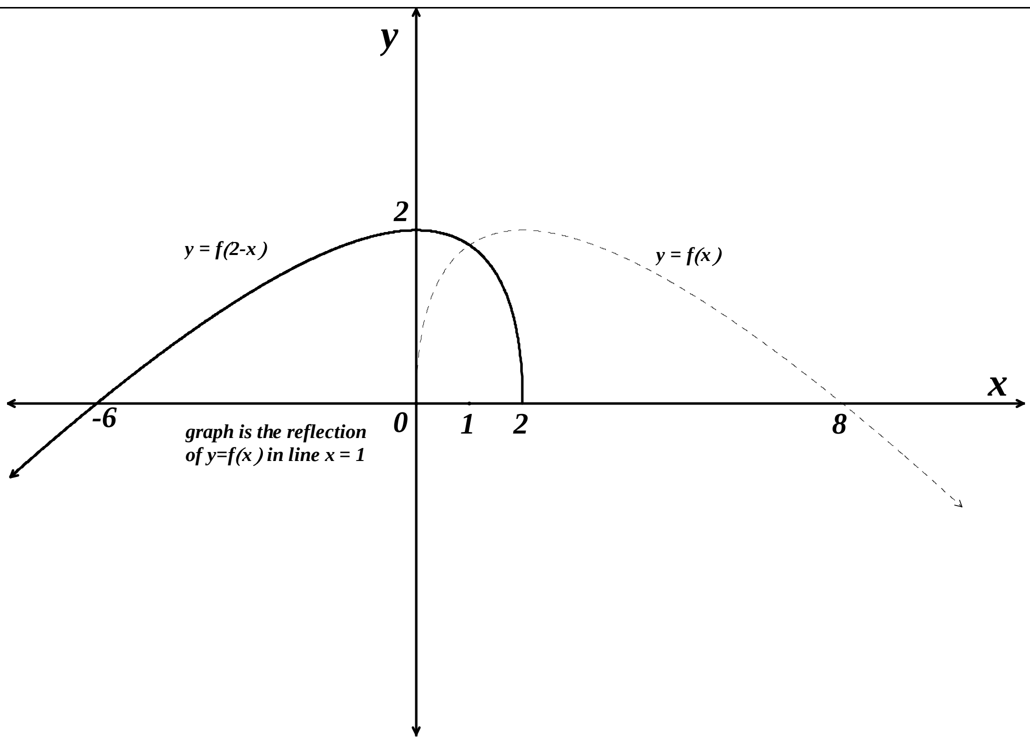
(iii) $y^2 = f(x)$



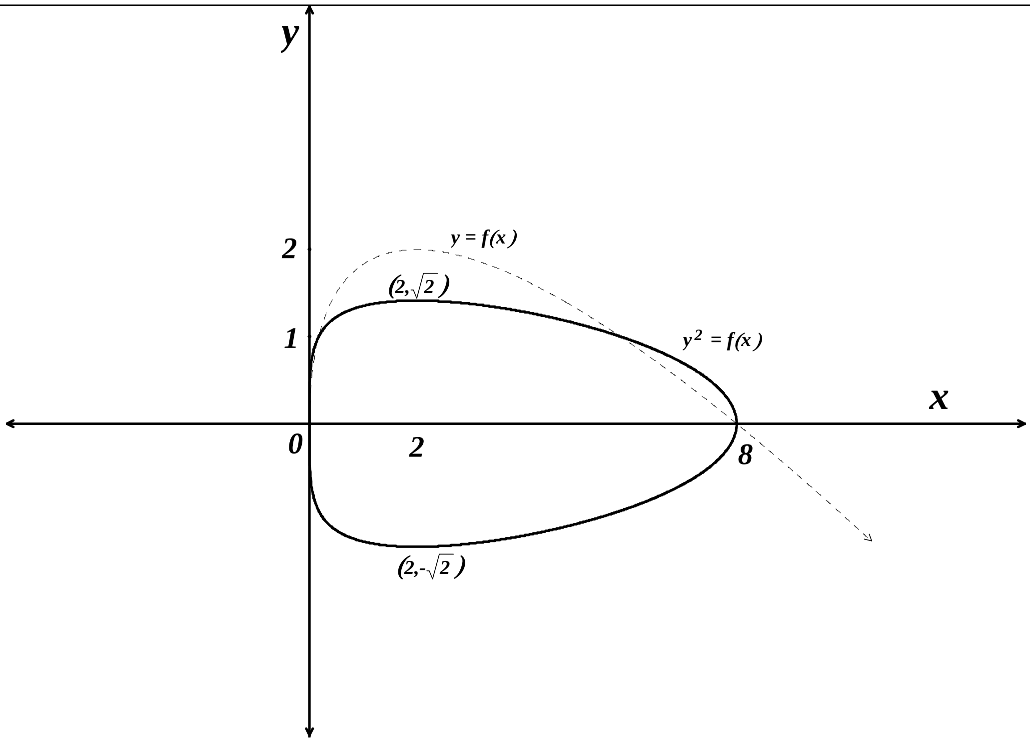
QUESTION 1			**MARKS**
(a)	(i)	$\int \frac{x-2}{x^2+1}dx = \int (\frac{x}{x^2+1} - \frac{2}{x^2+1})dx$ $= \frac{1}{2}\ln(x^2+1) - 2Tan^{-1}x + c$	
	(ii)	$\int \frac{\sin^{-1}x}{\sqrt{1-x^2}}\,dx = \frac{1}{2}\left(\sin^{-1}x\right)^2 + c$	
(b)	$\int \frac{dx}{\sqrt{6-x-x^2}} = \int \frac{dx}{\sqrt{(\frac{5}{2})^2 - (x+\frac{1}{2})^2}}$		**1**
	$= Sin^{-1}(\frac{x+\frac{1}{2}}{\frac{5}{2}}) + c$		**1**
	$= Sin^{-1}(\frac{2x+1}{5}) + c$		**1**
(c)	$\int_0^{\frac{\pi}{4}} \tan^3\theta\, d\theta = \int_0^{\frac{\pi}{4}} \tan\theta\left(\sec^2\theta - 1\right)d\theta$ $= \int_0^{\frac{\pi}{4}} \left(\sec^2\theta\tan\theta - \tan\theta\right)d\theta$ $= \left[\frac{1}{2}\tan^2\theta + \ln\left(\cos\theta\right)\right]_0^{\frac{\pi}{4}}$ $= \left(\frac{1}{2}\tan^2\left(\frac{\pi}{4}\right) + \ln\left(\cos\left(\frac{\pi}{4}\right)\right)\right) - \left(\frac{1}{2}\tan^2(0) + \ln(\cos(0))\right)$ $= \frac{1}{2} + \ln\left(\frac{1}{\sqrt{2}}\right) \quad \left[or \quad = \frac{1}{2}\left(1 - \ln 2\right)\right]$ Or		

	$\int_0^{\frac{\pi}{4}} \tan^3\theta\, d\theta = \int_0^{\frac{\pi}{4}} \tan\theta(\sec^2\theta - 1)\, d\theta$ $= \int_0^{\frac{\pi}{4}} (\sec^2\theta\tan\theta - \tan\theta)\, d\theta$ $= \int_0^{\frac{\pi}{4}} \{(\sec\theta\tan\theta)\sec\theta - \tan\theta\}\, d\theta$ $= \left[\frac{1}{2}\sec^2\theta + \ln(\cos\theta)\right]_0^{\frac{\pi}{4}}$ $= \left(\frac{1}{2}\sec^2\left(\frac{\pi}{4}\right) + \ln\left(\cos\left(\frac{\pi}{4}\right)\right)\right) - \left(\frac{1}{2}\sec^2(0) + \ln(\cos(0))\right)$ $= 1 + \ln\left(\frac{1}{\sqrt{2}}\right) - \frac{1}{2}$ $= \frac{1}{2} - \frac{1}{2}\ln 2 \qquad \left[or \quad = \frac{1}{2}(1 - \ln 2)\right]$	

(d)	(i)	$I_n = \int x^n Sinx\, dx$ $= \int x^n . \frac{d}{dx}\left(-\cos x\right) dx$ $= \left[x^n(-Cosx\right] - \int (-Cosx).nx^{n-1}dx$ $= -x^n Cosx + n\int x^{n-1}\frac{d}{dx}(Sinx)dx$ $= -x^n Cosx + n\left[x^{n-1}Sinx - \int (n-1)x^{n-2}Sinxdx\right]$ $= -x^n Cosx + nx^{n-1}Sinx - n(n-1)\int x^{n-2}Sinxdx$ $= -x^n Cosx + nx^{n-1}Sinx - n(n-1)I_{n-2}$ $\therefore \quad I_n + n(n-1)I_{n-2} = x^{n-1}(nSinx - xCosx)$	 **1** **1** **1**
	(ii)	$\int_0^{\pi} x^2 \sin x\, dx = I_2$ $I_n = \left[x^{n-1}\left(n\sin x - x\cos x\right)\right]_0^{\pi} - n\left(n-1\right)I_{n-2}$ $I_2 = \left[x\left(2\sin x - x\cos x\right)\right]_0^{\pi} - 2I_0$ $I_0 = \int_0^{\pi} \sin x\, dx$ $= \left[-\cos x\right]_0^{\pi}$ $= -\cos\pi + \cos 0$ $= 2$ $I_2 = \left[x\left(2\sin x - x\cos x\right)\right]_0^{\pi} - 2I_0$ $= \left\{\pi\left(2\sin\pi - \pi\cos\pi\right) - 0\right\} - 2\left\{2\right\}$ $= \pi^2 - 4$	

QUESTION 2			MARKS
(a)	(i)		
	(ii)		

Q2 (a) (iii)



(b)

$$\int_0^{\frac{\pi}{2}} \frac{\sin x}{\sin x + \cos x}\,dx = \int_0^{\frac{\pi}{2}} \frac{\sin\left(\frac{\pi}{2} - x\right)}{\sin\left(\frac{\pi}{2} - x\right) + \cos\left(\frac{\pi}{2} - x\right)}\,dx$$

$$= \int_0^{\frac{\pi}{2}} \frac{\cos x}{\cos x + \sin x}\,dx$$

Let $I = \int_0^{\frac{\pi}{2}} \frac{\sin x}{\sin x + \cos x}\,dx$

$$\therefore \quad 2I = \int_0^{\frac{\pi}{2}} \frac{\sin x}{\sin x + \cos x}\,dx + \int_0^{\frac{\pi}{2}} \frac{\cos x}{\cos x + \sin x}\,dx$$

$$= \int_0^{\frac{\pi}{2}} \frac{\sin x + \cos x}{\sin x + \cos x}\,dx$$

$$= \int_0^{\frac{\pi}{2}} 1\,dx$$

$$= \frac{\pi}{2}$$

$$I = \frac{\pi}{4}$$

(c)

$$\frac{x^2}{\frac{1}{4}} + \frac{y^2}{\frac{1}{9}} = 1 \quad \text{where } a = \frac{1}{2}, \quad b = \frac{1}{3}$$

$$b^2 = a^2(1 - e^2)$$

$$\frac{1}{9} = \frac{1}{4}(1 - e^2)$$

$$4 = 9(1 - e^2)$$

$$4 = 9 - 9e^2$$

$$9e^2 = 5$$

$$e^2 = \frac{5}{9}$$

$$e = \frac{\sqrt{5}}{3}$$

Foci $\quad (\pm ae, 0) = (\pm \frac{1}{2} . \frac{\sqrt{5}}{3}, 0)$

$$= (\pm \frac{\sqrt{5}}{6}, 0)$$

Parabola $\quad x^2 = \frac{1}{2} y$

$$= 4(\frac{1}{8}) y$$

$\therefore \quad focus \quad (0, \frac{1}{8})$

Let coordinates of centre be: *C(0, -k)*
Now *SC* = *BC* (both radii)

$$\frac{1}{8} + k = \sqrt{\left(\frac{\sqrt{5}}{6}\right)^2 + k^2}$$

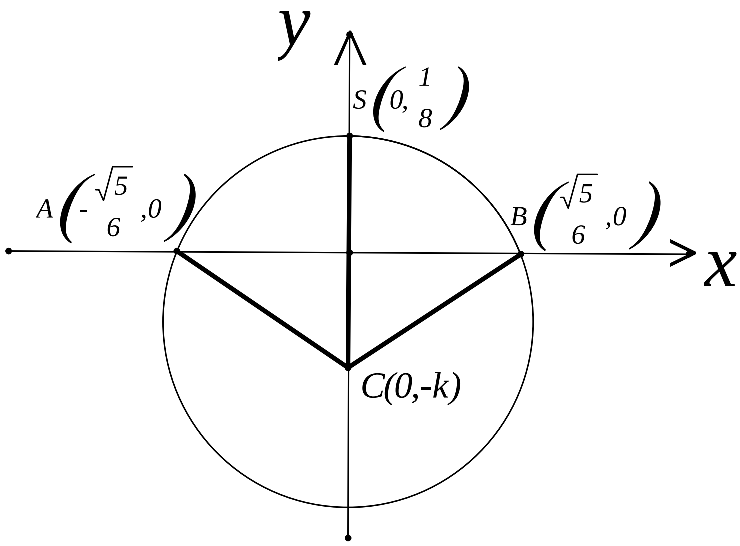
$$\left(\frac{1}{8} + k\right)^2 = \frac{5}{36} + k^2$$

$$\frac{1}{64} + \frac{1}{4} k + k^2 = \frac{5}{36} + k^2$$

$$\frac{1}{4} k = \frac{71}{576}$$

$$k = \frac{71}{144}$$

$\therefore$ coordinates of centre is $\left(0, -\frac{71}{144}\right)$



QUESTION 3		
(a)	at $x = 0$, $f^{/}(x) = 6$ as $x \to \infty$ $f'(x) \to \infty$ $\therefore$ gets steeper as $x \to -\infty$ $f'(x) \to 0$ $\therefore$ graph flattens out for $x < 2$ *or* $x > 3$, $x^2 - 5x + 6 > 0$, $\therefore$ $f'(x) > 0$ for $2 < x < 3$, $x^2 - 5x + 6 < 0$, $\therefore$ $f'(x) < 0$ for $x = 2$ or 3, $f^{'}(x) = 0$, $\therefore$ stat. pts.	

$f''(x) = e^x\left(x^2 - 3x + 1\right)$

For possible inflexion points $f''(x) = 0$

$e^x\left(x^2 - 3x + 1\right) = 0$

$x = \dfrac{3 \pm \sqrt{5}}{2}$ $\left(e^x \neq 0\right)$

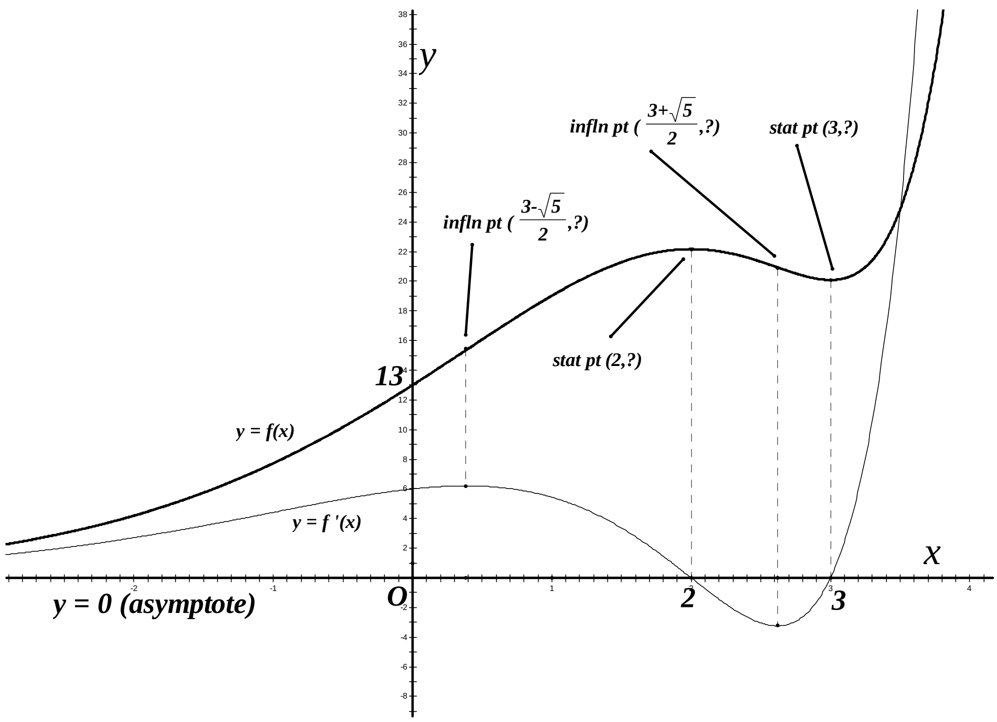
Test concavity change

x	0.3	$\frac{3-\sqrt{5}}{2} \approx 0.38$	0.4
$f''(x)$	≈ 0.26 > 0	0	≈ -0.06 < 0

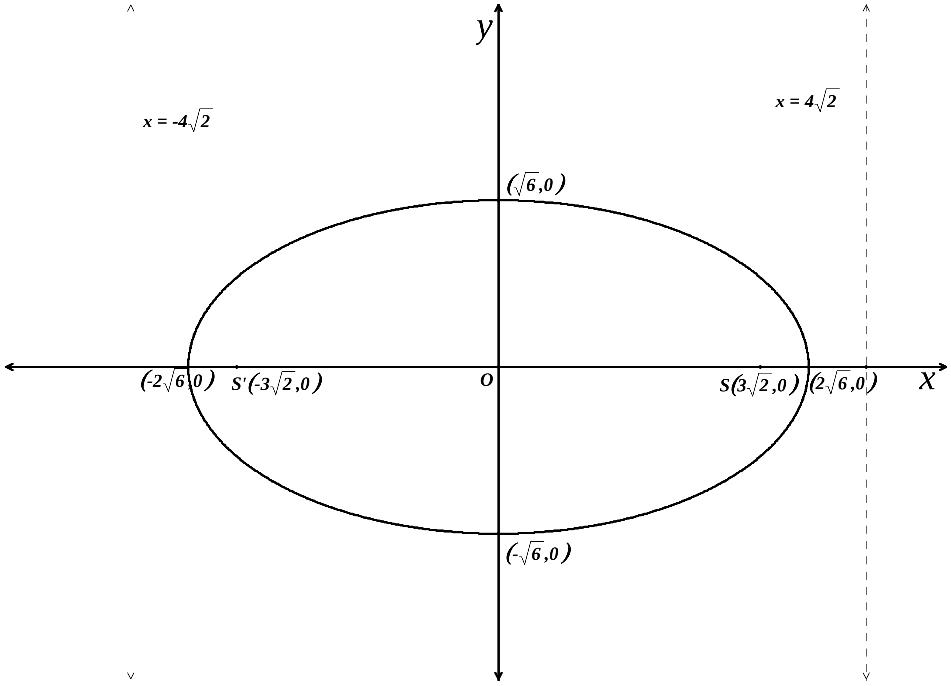
change in concavity, $\therefore$ inflexion point

x	2.5	$\frac{3+\sqrt{5}}{2} \approx 2.62$	2.7
$f''(x)$	≈ -3.05 < 0	0	≈ 2.83 > 0

change in concavity, $\therefore$ inflexion point



(b)	(i)	$\frac{x^2}{24}+\frac{y^2}{6}=1 \quad \Rightarrow \quad a=2\sqrt{6}$ and $b=\sqrt{6}$ From $b^2=a^2(1-e^2)$, we get $e=\sqrt{1-\frac{b^2}{a^2}}$ and hence $e=\sqrt{1-\frac{6}{24}}=\frac{\sqrt{3}}{2}$ $\therefore$ Foci given by $(\pm ae,0)=(\pm 3\sqrt{2},0)$ and coordinates of vertices/intercepts with axes $(\pm 2\sqrt{6},0)$ and $(0,\pm\sqrt{6})$	
	(ii)	$x=\pm\frac{a}{e}$ $=\pm\frac{2\sqrt{6}}{\frac{1}{2}\sqrt{3}}$ $=\pm 4\sqrt{2}$	

Q3 (b)	(iii)		
(c)	gradient: $\frac{2x}{2}-\frac{2y}{3}\frac{dy}{dx}=0$ $\therefore \frac{dy}{dx}=\frac{3x}{2y}$ $at\ (4,6)\quad \frac{dy}{dx}=\frac{3\times 4}{2\times 6}$ $=1$ Equation of tangent: $y-6=1(x-4)$ i.e. $x-y+2=0$ is the required tangent.		

QUESTION 4			
(a)	(i)	$\frac{x^2}{3}+\frac{y^2}{2}=1 \Rightarrow a=\sqrt{3}$ and $b=\sqrt{2}$ Equation of asymptotes: $y=\pm\frac{b}{a}x$ $y=\pm\frac{\sqrt{2}}{\sqrt{3}}x$ Hence slopes of these asymptotes are $m_1=\frac{\sqrt{2}}{\sqrt{3}}$ and $m_2=-\frac{\sqrt{2}}{\sqrt{3}}$ Let angle between asymptote $y=\frac{\sqrt{2}}{\sqrt{3}}x$ and x-axis be θ then $\tan\theta=\sqrt{\frac{2}{3}}$ and required angle = 2θ $\text{angle}=2\tan^{-1}\left(\sqrt{\frac{2}{3}}\right)$ $=78°$ (to nearest degree)	
	(ii)	Let $P(x_0, y_0)$ be an external point to the point of contact of tangents at M and N. Then the chord of contact is of the form: $\frac{x_0 x}{3}-\frac{y_0 y}{2}=1$ i.e. $2x_0 x-3y_0 y=6$ Comparing $2x-y=4$ and $2x_0 x-3y_0 y=6$ $\frac{2x_0}{2}=\frac{6}{4}$ *and* $\frac{3y_0}{1}=\frac{6}{4}$ $x_0=1\frac{1}{2}$ *and* $y_0=\frac{1}{2}$ Point is $\left(1\frac{1}{2},\frac{1}{2}\right)$	

(b)	(i)	$\frac{32}{16-x^4}=\frac{A(2+x)(2-x)+B(4+x)(2-x)+C(4+x^2)(2+x)}{(4+x^2)(2-x)(2+x)}$ $32\equiv A(2+x)(2-x)+B(4+x^2)(2-x)+C(4+x^2)(2+x)$ when $x=2$, $32=32C$ $C=1$ when $x=-2$, $32=32B$ $B=1$ when $x=0$, $32=4A+8B+8C$ $A=4$ $\therefore\frac{32}{16-x^4}=\frac{4}{(4+x^2)}+\frac{1}{(2-x)}+\frac{1}{(2+x)}$	
	(ii)	$\int_0^1\frac{dx}{16-x^4}=\frac{1}{32}\int_0^1\left(\frac{4}{4+x^2}+\frac{1}{2-x}+\frac{1}{2+x}\right)dx$ $=\frac{1}{32}\left[2\tan^{-1}\left(\frac{x}{2}\right)-\ln(2-x)+\ln(2+x)\right]_0^1$ $=\frac{1}{32}\left\{\left(2\tan^{-1}\left(\frac{1}{2}\right)-\ln(1)+\ln(3)\right)-\left(2\tan^{-1}(0)-\ln(2)+\ln(2)\right)\right\}$ $=\frac{1}{32}\left(2\tan^{-1}\left(\frac{1}{2}\right)+\ln(3)\right)$	
(c)	(i)	$\frac{2x}{a^2}+\frac{2y}{b^2}\frac{dy}{dx}=0$ $\frac{dy}{dx}=-\frac{b^2x}{a^2y}$ at $P(a\cos\theta,b\sin\theta)$, $\frac{dy}{dx}=-\frac{b^2(a\cos\theta)}{a^2(b\sin\theta)}$ $=-\frac{b\cos\theta}{a\sin\theta}$ Tangent is $y-b\sin\theta=-\frac{b\cos\theta}{a\sin\theta}(x-a\cos\theta)$ $(a\sin\theta)y-ab\sin^2\theta=-(b\cos\theta)x+ab\cos^2\theta$ $(b\cos\theta)x+(a\sin\theta)y=ab(\sin^2\theta+\cos^2\theta)$ $(b\cos\theta)x+(a\sin\theta)y=ab$ $\frac{(b\cos\theta)}{ab}x+\frac{(a\sin\theta)}{ab}y=\frac{ab}{ab}$ $\frac{x\cos\theta}{a}+\frac{y\sin\theta}{b}=1$	
	(ii)	slope of tangent $=-\frac{b\cos\theta}{a\sin\theta}$ $=-\frac{b}{a}\cot\theta$	

		slope of chord $QR = \dfrac{b\sin(\theta+\phi)-b\sin(\theta-\phi)}{a\cos(\theta+\phi)-a\cos(\theta-\phi)}$ $= \dfrac{b}{a}\dfrac{(\sin\theta\cos\phi+\cos\theta\sin\phi)-(\sin\theta\cos\phi-\cos\theta\sin\phi)}{(\cos\theta\cos\phi-\sin\theta\sin\phi)-(\cos\theta\cos\phi+\sin\theta\sin\phi)}$ $= \dfrac{b}{a}\left(\dfrac{2\cos\theta\sin\phi}{-2\sin\theta\sin\phi}\right)$ $= -\dfrac{b}{a}\cot\theta$ $\therefore$ tangent is parallel to chord (equal slopes)	