Question 1 **Marks**

(a) i) Find $\int \sqrt{e^x}\ dx.$ **1**

ii) Find $\int \frac{\sqrt{x}}{x-1}\ dx.$ **4**

iii) Use the substitution $t = \tan\frac{x}{2}$, evaluate $\int_0^{\frac{2\pi}{3}} \frac{1}{5+4\cos x}\ dx.$ **4**

(b) Consider the hyperbola $\frac{x^2}{4} - \frac{y^2}{3} = 1.$

The point $P(4, 3)$ lies on the hyperbola. The normal at P to the hyperbola meets the x-axis at N. The vertical line through P meets the asymptote in the first quadrant at L.

i) Sketch the hyperbola, clearly showing all the above information. **2**

ii) Show that the equation of the normal to the hyperbola at P is $x + y = 7$. **2**

iii) Show that LN is perpendicular to OL. **2**

Question 2 (Start a new Page)

(a) Consider the ellipse $\frac{x^2}{16} + \frac{y^2}{12} = 1.$

i) Sketch the ellipse, clearly showing the directrices and foci. **3**

ii) The tangent and normal to the ellipse at P (2,3) cut the y-axis at A and B respectively. **2**
Given that the equation of the tangent to the ellipse at P is $x + 2y = 8$.
Find the coordinates of A and B.

iii) Show that AB subtends a right angle at the focus (S)of the ellipse. **2**

iv) Give the reason why the points A, P, S and B are concyclic. **1**

(b) The graph on the next page shows a function which has x-intercepts at $x = 0$ and $x = 2$. There is a vertical asymptote at $x = 1$ and a horizontal asymptote of $y = 1$.

Without using calculus, sketch the following graphs on the ANSWER sheet provided at the end of this question paper, clearly showing any asymptotes and intercepts.

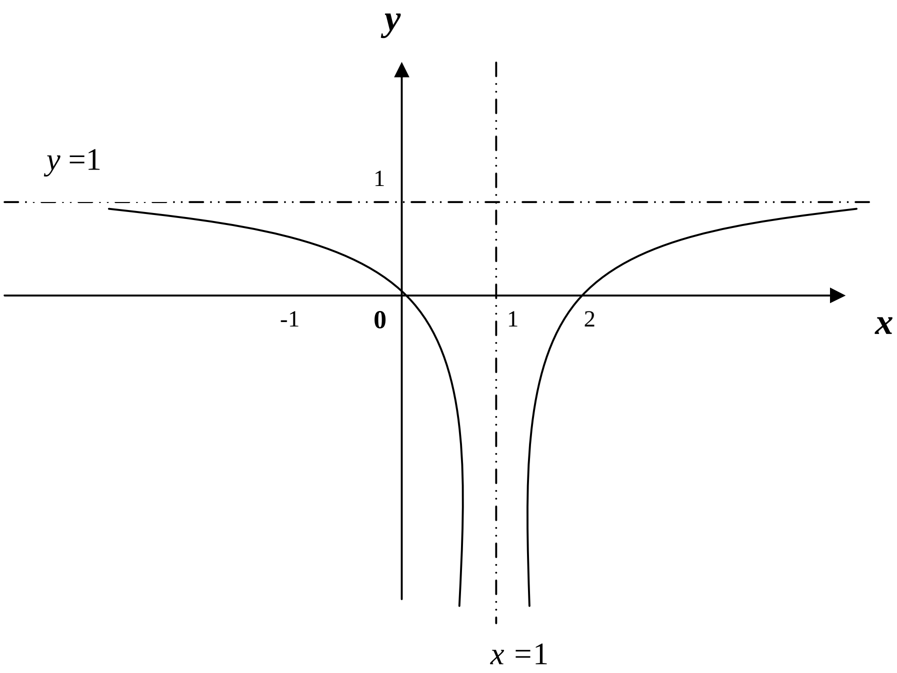
Question 2(b) cont'd **Marks**

i) $y = |f(x)|$ **1**

ii) $y = f(x-1)$ **1**

iii) $y^2 = f(x)$ **2**

iv) $y = \tan^{-1} f(x)$ **3**



Question 3 (Start a new Page)

(a) i) Show that $\sqrt{\dfrac{8-x}{x}} = \dfrac{4-x}{\sqrt{8x-x^2}} + \dfrac{4}{\sqrt{8x-x^2}}$. **1**

ii) Hence evaluate $\int_0^2 \sqrt{\dfrac{8-x}{x}}\, dx.$ **5**

Question 3 cont'd **Marks**

(b) $P(a\cos\theta, b\sin\theta)$ is a point in the first quadrant on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ and $Q\,(a\sec\theta,\ b\tan\theta)$ is a point on the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$, where $a>b>0$.

i) Sketch the ellipse , the hyperbola and their common auxillary circles $x^2+y^2=a^2$ and $x^2+y^2=b^2$ on the same diagram, showing the angle θ and the related points P and Q. Show clearly how the positions of P and Q are determined by the value of θ, where $0<\theta<\frac{\pi}{2}$. **3**

ii) Prove that the tangent to the ellipse at P is $\frac{x\cos\theta}{a}+\frac{y\sin\theta}{b}=1$. **2**

iii) The tangent to the hyperbola at Q and the tangent to the ellipse at P meet at T. **4**
Given that the equation of the tangent to the hyperbola at Q is $\frac{x\sec\theta}{a}-\frac{y\tan\theta}{b}=1$.
Find the coordinates of T.

Question 4 (Start a new Page)

(a) i) Given $f(x) = f\,(a-x)$ and using the substitution $u = a - x$, prove that **3**

$$\int_0^a xf(x)dx=\frac{a}{2}\int_0^a f(x)dx.$$

ii) Given that $f(x) = \frac{\sin x}{1+\cos^2 x}$, prove that $f(x) = f(\pi - x)$. **1**

iii) Hence evaluate $\int_0^\pi \frac{x\sin x}{1+\cos^2 x}dx$. **3**

(b) Let $I_n = \int_0^1 x(1-x^5)^n\,dx$, where $n\geq 0$ is an integer.

i) Show that $I_n=\frac{5n}{5n+2}I_{n-1}$ for $n\geq 1$. **4**

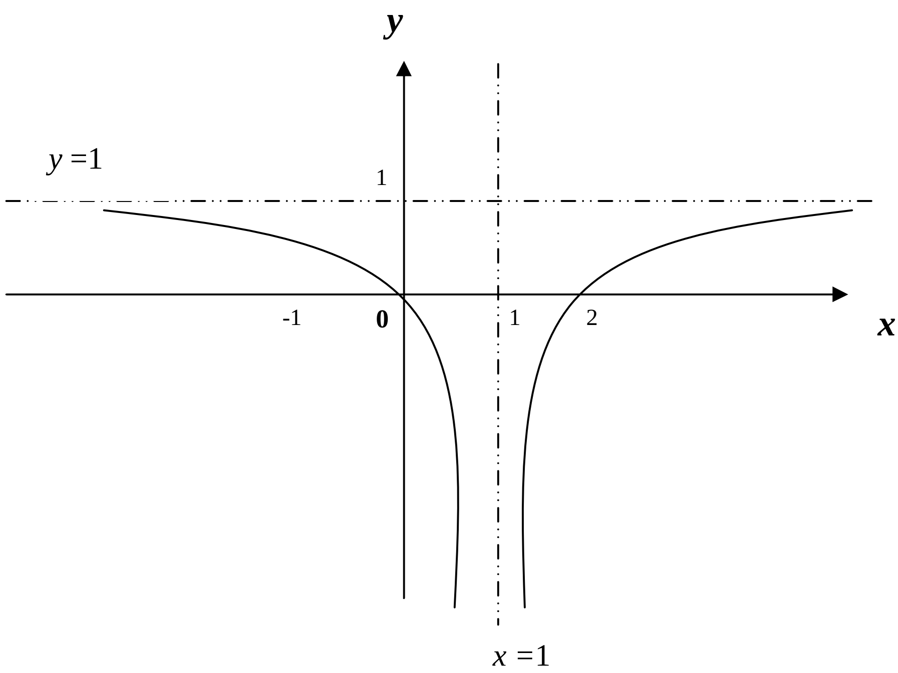
ii) Show that $I_n=\frac{5^n n!}{2\times 7\times 12\times\cdots\times(5n+2)}$ for $n\geq 1$. **3**

iii) Hence evaluate I_4. **1**

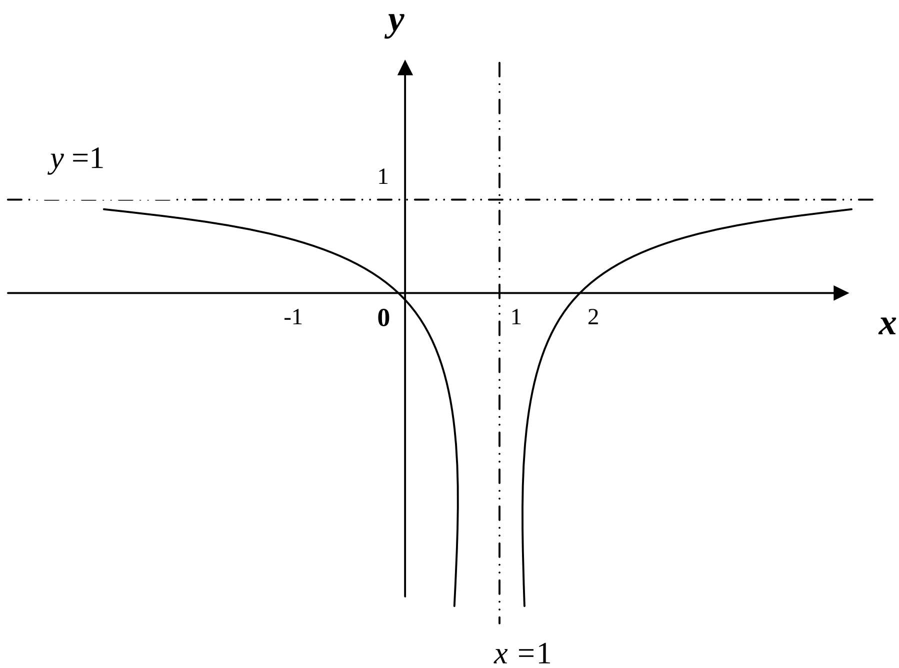
END

Answer sheet for Q2(b) **Student No:__________________**

(i) Sketch $y = |f(x)|$



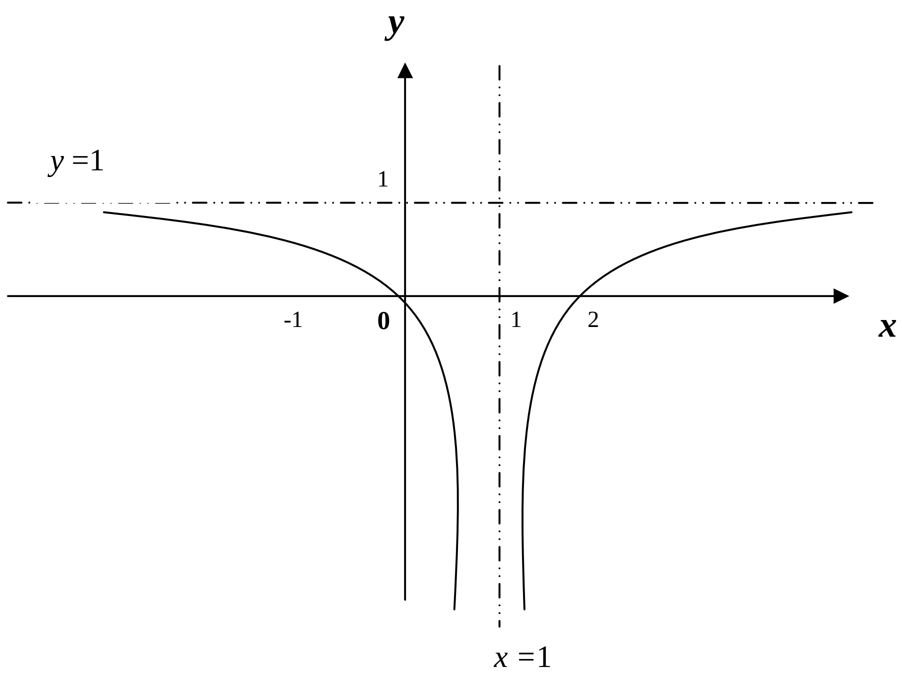
(ii) Sketch $y = f(x - 1)$



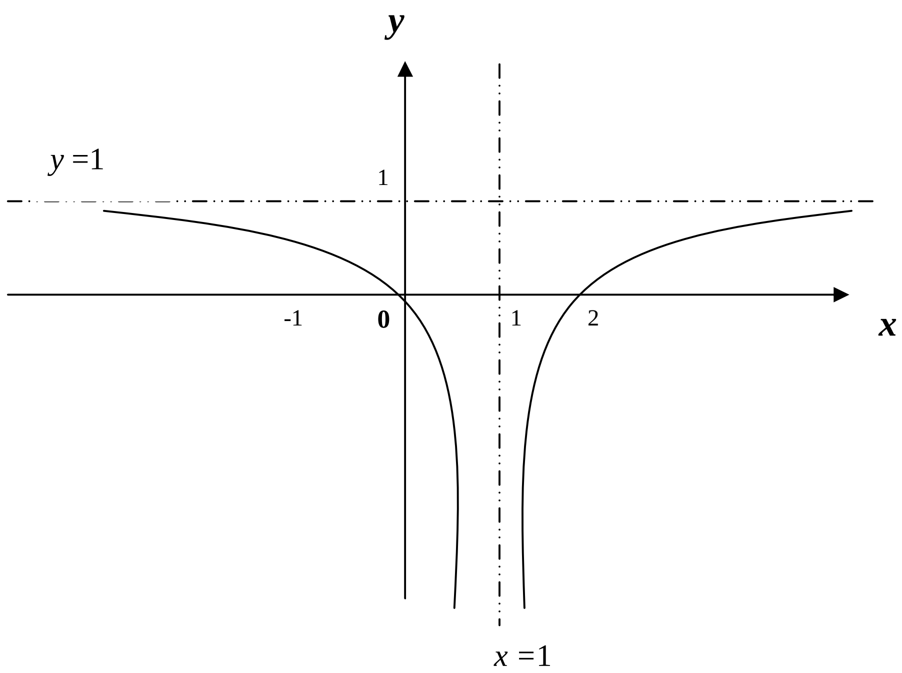
Turn over for parts(iii) and (iv)

Answer sheet for Q2(b) **Student No.________________**

(iii) Sketch $y^2 = f(x)$



(iv) Sketch $y = \tan^{-1} f(x)$



Now attach this sheet with your answer booklet for Question 2.

1 a. i) $\int e^{\frac{x}{2}}\,dx = 2e^{\frac{x}{2}} + c$ # 1

ii) $2\int \frac{u^2\,du}{u^2-1}$ 1 $\sqrt{x}=u$

$x = u^2$

$= 2\int \frac{u^2-1+1\,du}{(u^2-1)}$ $dx = 2u\,du$

$= 2\int du + 2\int \frac{du}{u^2-1}$ 1

$= 2u + \int \frac{du}{u-1} - \int \frac{du}{u+1}$ 1

$= 2u + \ln\left(\frac{u-1}{u+1}\right) + c$

$= 2\sqrt{x} + \ln\left(\frac{\sqrt{x}-1}{\sqrt{x}+1}\right) + c$ 1 #

iii) $\int_0^{\frac{2\pi}{3}} \frac{dx}{5+4\cos x}$ $t = \tan\frac{x}{2}$

$dx = \frac{2}{1+t^2}$

$= \int_0^{\sqrt{3}} \frac{\frac{2}{1+t^2}\,dt}{5+4\frac{(1-t^2)}{1+t^2}}$ 1 $\sqrt{1+t^2}$, t, $\frac{x}{2}$, 1

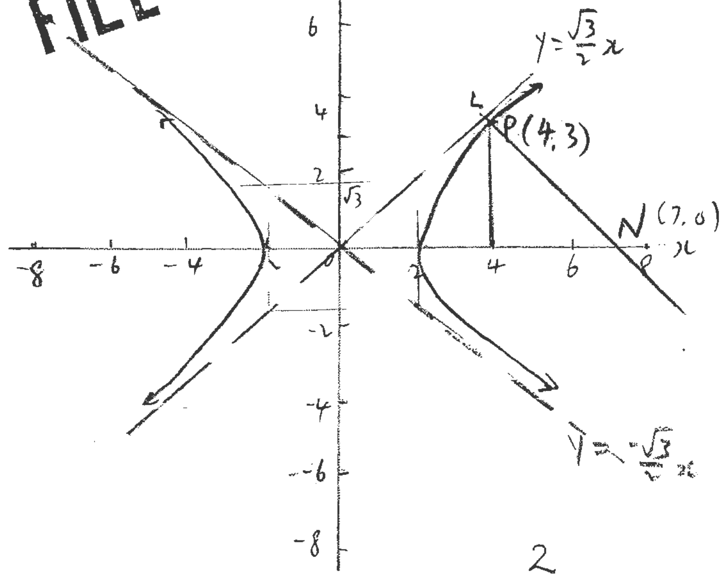
$= \int_0^{\sqrt{3}} \frac{2\,dt}{5+5t^2+4-4t^2}$

$= \int_0^{\sqrt{3}} \frac{2\,dt}{3^2+t^2}$ 1

$= \frac{2}{3}\left[\tan^{-1}\frac{t}{3}\right]_0^{\sqrt{3}}$ 1

$= \frac{2}{3}\tan^{-1}\frac{1}{\sqrt{3}} = \frac{2}{3}\times\frac{\pi}{6} = \frac{\pi}{9}$ 1

1 b i)



2

ii) $\frac{x^2}{4} - \frac{y^2}{3} = 1$

$\frac{\cancel{2}x}{4} - \frac{\cancel{2}yy'}{3} = 0$

$\frac{x}{4} = \frac{yy'}{3}$

$y' = \frac{x}{4}\cdot\frac{3}{y}\Big|_{(4,3)} = 1$ 1

$m^{\perp} = -1$

Eq of Normal at P: $y-3 = -(x-4)$

$x+y=7$ # 1

iii) To find L : Eq of asympt. $y = \frac{\sqrt{3}}{2}x$

$x=4$, $y = \frac{\sqrt{3}}{2}\cdot 4 = 2\sqrt{3}$

$\therefore L = (4, 2\sqrt{3})$ 1

To find N : $y=0$, $x=7$ $N = (7,0)$

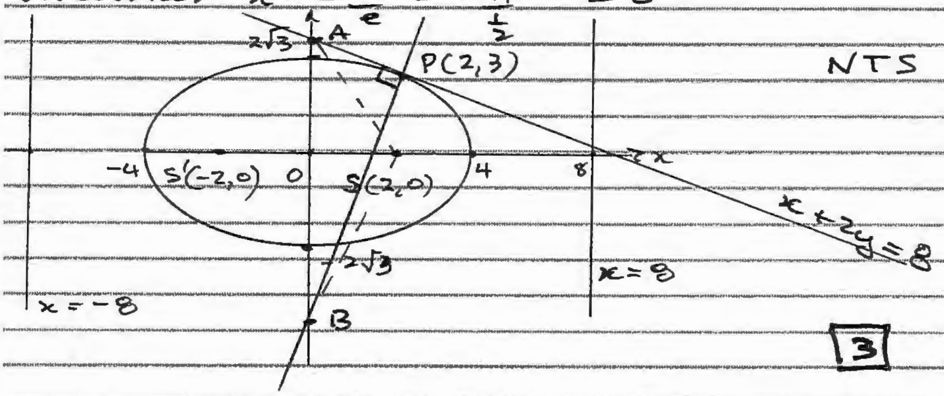
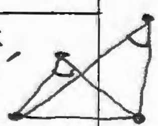
$m(LN) = \frac{2\sqrt{3}}{-3} = -\frac{2}{\sqrt{3}}$, $m(OL) = \frac{2\sqrt{3}}{4} = \frac{\sqrt{3}}{2}$

$\therefore m(LN)\times m(OL) = -1$ 1

$LN \perp OL$ #

Y12 ASSESSMENT TASK 2, TERM 1 2010
MATH. EXT 2.

MATHEMATICS Extension 2: Question 2

Suggested Solutions	Marks	Marker's Comments
Q2(a)(i) $\frac{x^2}{16}+\frac{y^2}{12}=1$		
(i) $a^2=16 \therefore a=4$, $b^2=12$, $b=2\sqrt{3}$ $\therefore e^2=1-\frac{b^2}{a^2}=1-\frac{12}{16}=\frac{1}{4}$		
$\therefore e=\frac{1}{2}$ $(0<e<1)$		1 For $e=\frac{1}{2}$
$\therefore$ focii $(\pm ae, 0)=(\pm 2, 0)$		
Directrices $x=\pm\frac{a}{e}=\pm\frac{4}{\frac{1}{2}}=\pm 8$		
		½ For S(2,0) ½ For x = 8 1 For Ellipse shape
[3]		
(ii) $x+2y=8$		
For A: $x=0 \therefore y=4$ A = (0, 4)	½	½ For (0,4)
Gradient of tangent $m_T=-\frac{1}{2}$ " " normal $m_N=+2$	½	½ For $m_N=2$
$\therefore$ Equ. of normal $y-3=2(x-2)$ $y=2x-1$ [2]		
For B: $x=0 \therefore y=-1$ B = (0, −1)	1	1 For (0, −1)
(iii) A(0, −4) S(2, 0) B(0, −1)		
$m_{AS}=\frac{4-0}{0-2}=-2$		½ For $m_{AS}=-2$
$m_{SB}=\frac{-1-0}{0-2}=+\frac{1}{2}$		½ For $m_{SB}=\frac{1}{2}$
$\therefore m_{AS}\times m_{SB}=-2\times\frac{1}{2}=-1$		½ For showing −1
$\therefore AS\perp SB$ $\therefore \angle ASB=90°$ ie. AB subtends a right angle at S.	[2]	½ For conclusion
(iv) • $\angle APB=90°$ (Tangent and normal at 90°) $\angle ASB=90°$ (shown above) $\therefore$ A, P, S and B are concyclic points as interval AB subtends equal angles at 2 points on the same side of it, then endpoints and the 2 points are concyclic	✓	1 For 'correct reason(s)'  [1]

• could reason using semicircles, diameter + reason.

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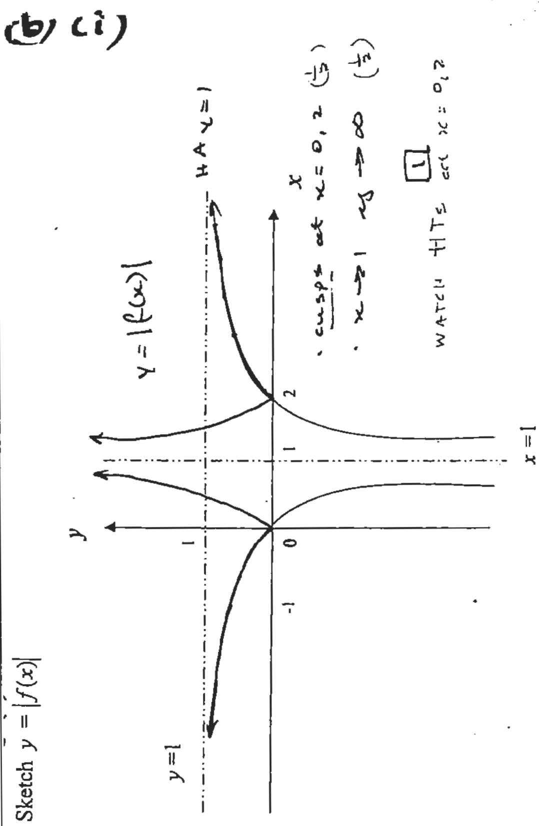
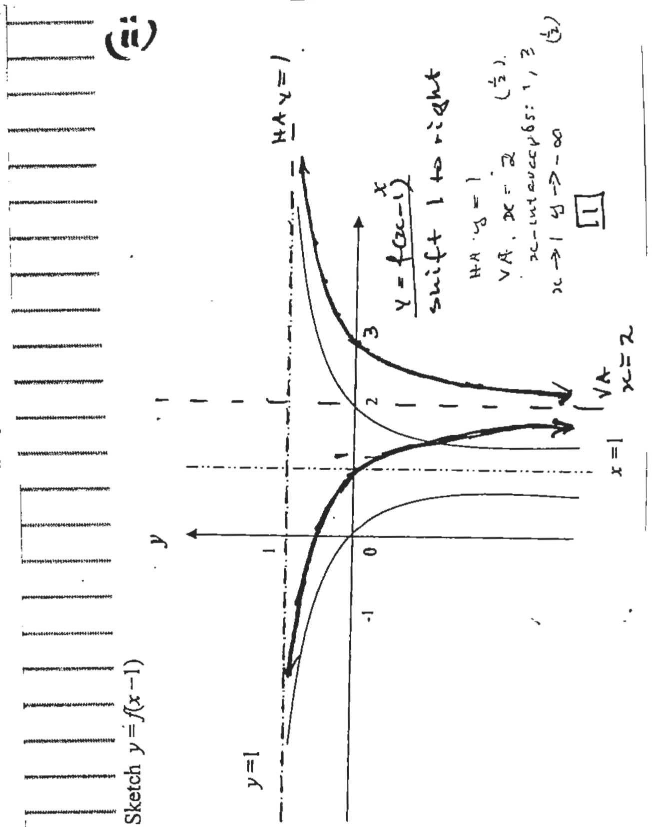
Suggested Solutions	Marks	Marker's Comments

(b) (i)

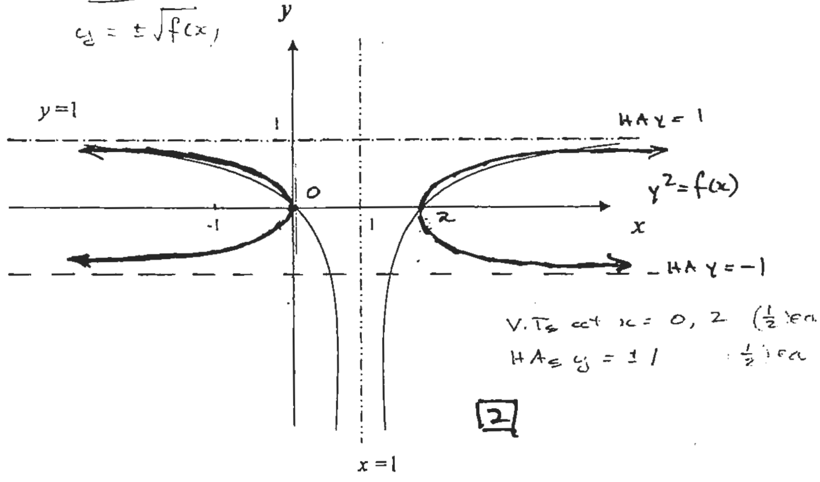


(ii)



(iii) Sketch $y^2 = f(x)$

$y = \pm\sqrt{f(x)}$

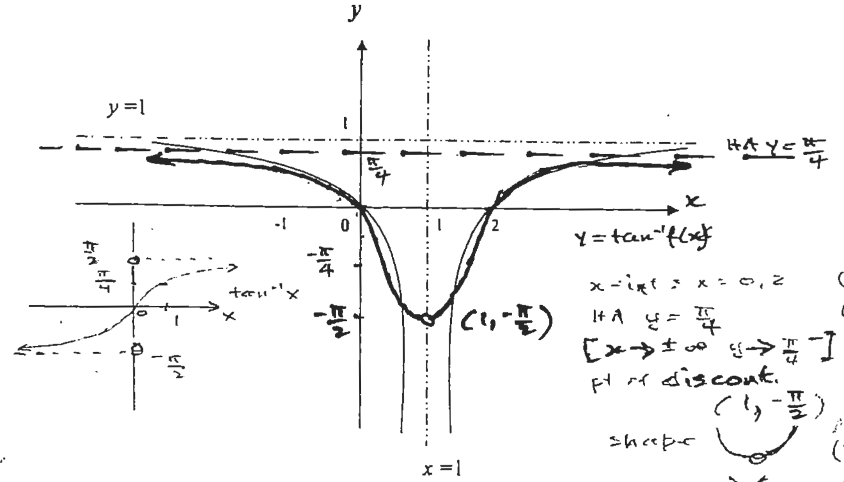


V.Ts at $x = 0, 2$ ($\frac{1}{2}$) ea

HAs $y = \pm 1$ ($\frac{1}{2}$) ea

[2]

(iv) Sketch $y = \tan^{-1} f(x)$



x-int: $x = 0, 2$ ($\frac{1}{2}$)

HA $y = \frac{\pi}{4}$ (1)

[$x \to \pm\infty$, $y \to \frac{\pi}{4}^-$]

Pt of discont. $(1, -\frac{\pi}{2})$ ($\frac{1}{2}$)

shape ($\frac{1}{2}$)

∅

∅

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3 a i) $\sqrt{\frac{8-x}{x}} = \sqrt{\frac{8-x}{x}} \cdot \sqrt{\frac{8-x}{x}} = \frac{8-x}{\sqrt{(8-x)x}}$

$= \frac{4-x}{\sqrt{8x-x^2}} + \frac{4}{\sqrt{8x-x^2}}$ / #

ii) $\int_0^2 \sqrt{\frac{8-x}{x}}\,dx = \int \frac{4-x\,dx}{\sqrt{8x-x^2}} + 4\int \frac{dx}{\sqrt{8x-x^2}}$ /

$\frac{1}{2}\int_0^2 \frac{2(4-x)dx}{\sqrt{8x-x^2}} = \frac{1}{2}\left[\frac{(\sqrt{8x-x^2})}{\frac{1}{2}}\right]_0^2 = \sqrt{16-4}$

$= \sqrt{12} = 2\sqrt{3}$ # /

$4\int_0^2 \frac{dx}{\sqrt{8x-x^2}} = 4\int_0^2 \frac{dx}{\sqrt{-(x^2-8x+16-16)}}$

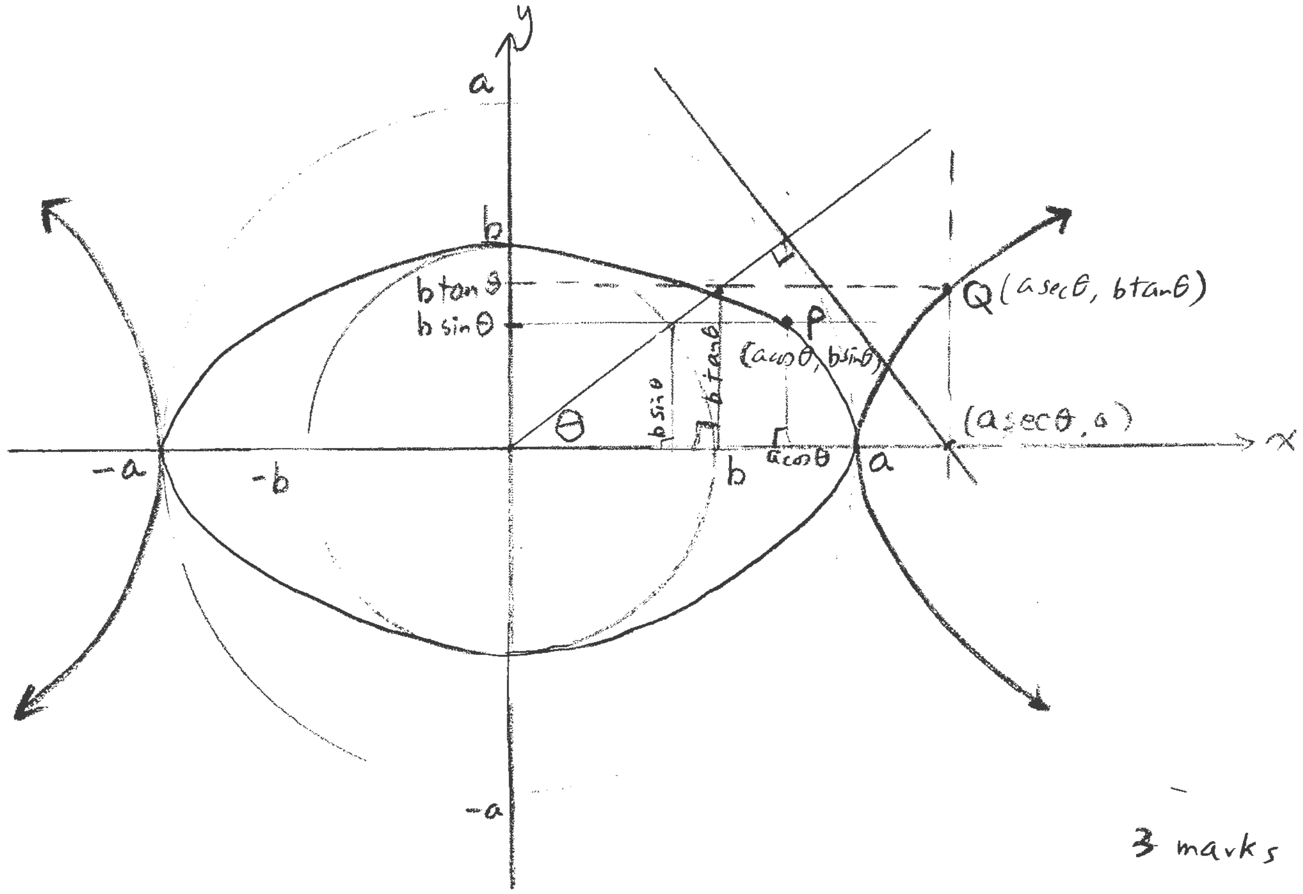
$= 4\int \frac{dx}{16-(x-4)^2}$ / $= 4\left[\sin^{-1}\left(\frac{x-4}{4}\right)\right]_0^2$ /

$= 4\sin^{-1}(-\frac{1}{2}) - \sin^{-1}(-1)$

$= 4(-\frac{\pi}{6} + \frac{\pi}{2}) = \frac{4\pi}{3}$ # /

Ans $2\sqrt{3} + \frac{4\pi}{3}$ #

i)



3 marks

ii) $\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = -\frac{b\cos\theta}{a\sin\theta}$

Eq of tangent at P: $y - b\sin\theta = -\frac{b\cos\theta}{a\sin\theta}(x-a)$

$xb\cos\theta + ya\sin\theta = ab(\cos^2\theta + \sin^2\theta)$

$\frac{x\cos\theta}{a} + \frac{y\sin\theta}{b} = 1$

iii) $\frac{x\cos\theta}{a} + \frac{y\sin\theta}{b} = 1$ ① $\times \sec\theta$ $\frac{x}{a} + \frac{y\tan\theta}{b} = \sec\theta$ ③

$\frac{x\sec\theta}{a} - \frac{y\tan\theta}{b} = 1$ ②

② + ③ $\frac{x}{a}(1+\sec\theta) = 1+\sec\theta$ $\therefore x = a$ #

Sub $x = a$ into ① $\cos\theta + \frac{y\sin\theta}{b} = 1$

$\frac{y\sin\theta}{b} = 1 - \cos\theta$

$y = \frac{b}{\sin\theta}(1-\cos\theta)$ #

MATHEMATICS Extension 2: Question ...4...

Suggested Solutions	Marks	Marker's Comments
a)(i) $f(x) = f(a-x)$ $u = a - x$, $x = a - u$, $x = a$ $u = 0$, $x = 0$ $u = a$, $dx = -du$		
$\int_0^a x f(x)\,dx$		
$= \int_a^0 (a-u) f(a-u)(-du)$		① correct substitution
$= \int_0^a (a-u) f(u)\,du$		
$= a\int_0^a f(u)\,du - \int_0^a u f(u)\,du$		
change variable		① change of variable
$\int_0^a x f(x)\,dx = a\int_0^a f(x)\,dx - \int_0^a x f(x)\,dx$		split integral.
$2\int_0^a x f(x)\,dx = a\int_0^a f(x)\,dx$		① Change to $f(x)$ to give answer.
$\int_0^a x f(x)\,dx = \frac{a}{2}\int_0^a f(x)\,dx$		
(ii) $\dfrac{\sin(\pi - x)}{1 + [\cos(\pi - x)]^2} = \dfrac{\sin x}{1 + (-\cos x)^2}$		
because $\sin(\pi - x) = \sin x$, $\cos(\pi - x) = -\cos x$.		① must show $\sin(\pi - x) = \sin x$, $\cos(\pi - x) = -\cos x$
$\therefore f(\pi - x) = \dfrac{\sin x}{1 + \cos^2 x} = f(x)$		
(iii) $\int_0^\pi \dfrac{x\sin x}{1+\cos^2 x}\,dx = \dfrac{\pi}{2}\int_0^\pi \dfrac{\sin x}{1+\cos^2 x}\,dx$		① change integral.
as $f(\pi - x) = f(x)$ $= -\dfrac{\pi}{2}\Big[\tan^{-1}(\cos x)\Big]_0^\pi$		① integrate
$= -\dfrac{\pi}{2}\tan^{-1}(\cos\pi - \cos 0)$		
$= -\dfrac{\pi}{2}\left(\tan^{-1}(-1) - \tan^{-1}(1)\right)$		
$= -\dfrac{\pi}{2} \times \left(-\dfrac{\pi}{4} - \dfrac{\pi}{4}\right)$		
$= \dfrac{\pi^2}{4}$		① answer.

MATHEMATICS Extension 2: Question 4

Suggested Solutions | **Marks** | **Marker's Comments**

b i)

$$I_n = \int_0^1 x(1-x^5)^n\,dx$$

$$= \left[\frac{x^2}{2}(1-x^5)\right]_0^1 - \int_0^1 \frac{x^2}{2}\, n(1-x^5)^{n-1}(-5x^4)\,dx$$

$$= (0-0) + \frac{5n}{2}\int_0^1 x^6(1-x^5)^{n-1}\,dx$$

$$= \frac{5n}{2}\int_0^1 x(x^5-1+1)(1-x^5)^{n-1}\,dx$$

$$= \frac{5n}{2}\int x(1-x^5)^{n-1}\,dx - \frac{5n}{2}\int x(1-x^5)^n\,dx$$

$$I_n = \frac{5n}{2}I_{n-1} - \frac{5n}{2}I_n$$

$$I_n\left[1+\frac{5n}{2}\right] = \frac{5n}{2}I_{n-1}$$

$$I_n = \left(\frac{5n}{2+5n}\right)I_{n-1}$$

(ii)

$$I_n = \frac{5n}{2+5n}I_{n-1} \qquad n \geq 1$$

$$I_1 = \frac{5}{7}I_0 \qquad I_0 = \int_0^1 x(1-x^5)^0\,dx$$

$$I_2 = \frac{10}{12}I_1 \qquad = \int_0^1 x\,dx$$

$$I_3 = \frac{15}{17}I_2 \qquad = \left[\frac{x^2}{2}\right]_0^1$$

$$= \frac{1}{2}$$

$$I_n = \frac{5n}{5n+2}\cdot\frac{5(n-1)}{5(n-1)+2}\cdot\frac{5(n-2)}{5(n-2)+2}\cdots\frac{5\times(3)}{5(3)+2}\times\frac{5(2)}{5(2)+2}\times I_0$$

$$= \frac{5^n(n)(n-1)(n-2)\cdots 3\times 2\times 1}{(5n+2)(5n-3)(5n-8)\cdots 17\times 12\times 2}$$

$$= \frac{5^n\, n!}{2\times 7\times 12\times\cdots(5n+2)}$$

$$I_4 = \frac{5^4\times 4!}{2\times 7\times 12\times 17\times 22} = \frac{15000}{62832}$$

$$= \frac{625}{2618}$$

Marker's Comments

(1) Integrate by parts

(1) simplify

(1) separate integrals

(1) complete solution

(2) must show series at both ends including I_0

(1) Calculate I_0

($\frac{1}{2}$) ($\frac{1}{2}$) must show 5 outside each term to make 5^n and $n(n-1)(n-2)\cdots 3\cdot 2\cdot 1$

(1) correct answer only.