Year 12 - Term 1 - Mathematics - Extension 2 – 2011

QUESTION 1 (15 Marks) **Marks**

(a) (i) Find: $\int \tan^2 4\theta \, d\theta$. **2**

(ii) Find $\int \frac{dx}{x^2+2x+4}$. **3**

(iii) Evaluate: $\int_0^1 \frac{x}{\sqrt{2-x}} \, dx$. **3**

(iv) Evaluate: $\int_1^3 \frac{4}{x^2-4x} \, dx$ **3**

(b) Draw a neat sketch of the hyperbola $3x^2 - y^2 = 12$ showing

(i) the intercepts with the co-ordinate axes, **1**

(ii) the positions and co-ordinates of the foci, **1**

(iii) the positions and equations of the directrices. **1**

(iv) the positions and equations of the asymptotes. **1**

QUESTION 2 (15 Marks) (START A NEW PAGE) **Marks**

(a) (i) Evaluate: $\int_1^e x \ln x \, dx$. **3**

(ii) Evaluate: $\int_0^{\frac{\pi}{4}} \sin^3 \theta \, d\theta$. **3**

(iii) Using the substitution $u = e^x$, find $\int \frac{e^{2x}}{e^x - 1} \, dx$. **3**

(b) (i) Draw a neat sketch of the ellipse $9x^2 + 25y^2 = 225$ showing

(α) the intercepts with the co-ordinate axes, **1**

(β) the positions and co-ordinates of the foci, **1**

(γ) the positions and equations of the directrices. **1**

(ii) Write down the equation of the tangent to the ellipse at the point $P(5\cos\theta, 3\sin\theta)$. **1**

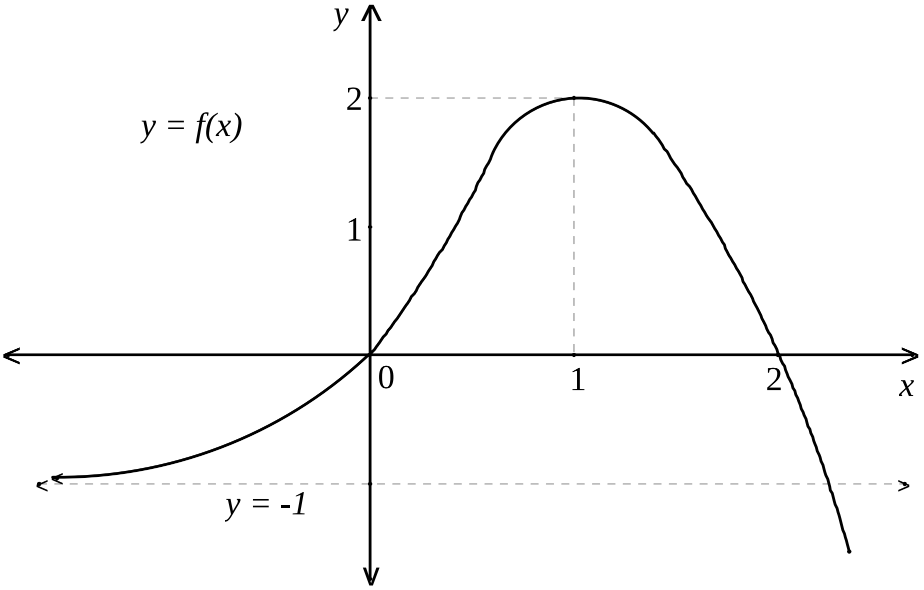
(iii) The tangent at *P* meets the tangents from the ends of the major axis at the points *Q* and *R*. Prove that the intervals *QS* and *RS* are perpendicular where *S* is the focus with the positive *x*-coordinate. **2**

QUESTION 3 (15 Marks) (START A NEW PAGE) **Marks**

(a) (i) Express $\frac{1}{1+x+x^2+x^3}$ in the form $\frac{A}{1+x}+\frac{Bx+C}{1+x^2}$, where A, B and C are rational numbers. **2**

(ii) Hence evaluate $\int_0^1 \frac{dx}{1+x+x^2+x^3}$. **3**

(b) The graph of $y = f(x)$ is illustrated. The line $y = -1$ is a horizontal asymptote.



Using the separate graphs provided, sketch each of the graphs below. In each case, clearly label any maxima or minima and the equations of any asymptotes.

(i) $y = \frac{1}{f(x)}$. **2**

(ii) $y = 2f(x+1)$. **2**

(c) (i) Given the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, prove that the line $y = mx + c$ is a tangent provided that $c^2 = a^2m^2 + b^2$. **2**

(ii) Two tangents are drawn from the point $T(p,q)$ to touch the above ellipse at points M and N. Show that the gradients of these tangents are the roots of the quadratic equation $(a^2 - p^2)m^2 + 2pqm + (b^2 - q^2) = 0$. **3**

(iii) If the above two tangents are perpendicular, show that $a^2 + b^2 = p^2 + q^2$. **1**

QUESTION 4 (15 Marks) (START A NEW PAGE) **Marks**

(a) (i) Given that $I_n = \int_0^{\frac{\pi}{2}} \theta \sin^n \theta \, d\theta$, prove that $I_n = \frac{n-1}{n} I_{n-2} + \frac{1}{n^2}$ for $n \geq 2$. **3**

(ii) Hence show that $\int_0^{\frac{\pi}{2}} \theta \sin^5 \theta \, d\theta = \frac{149}{225}$. **3**

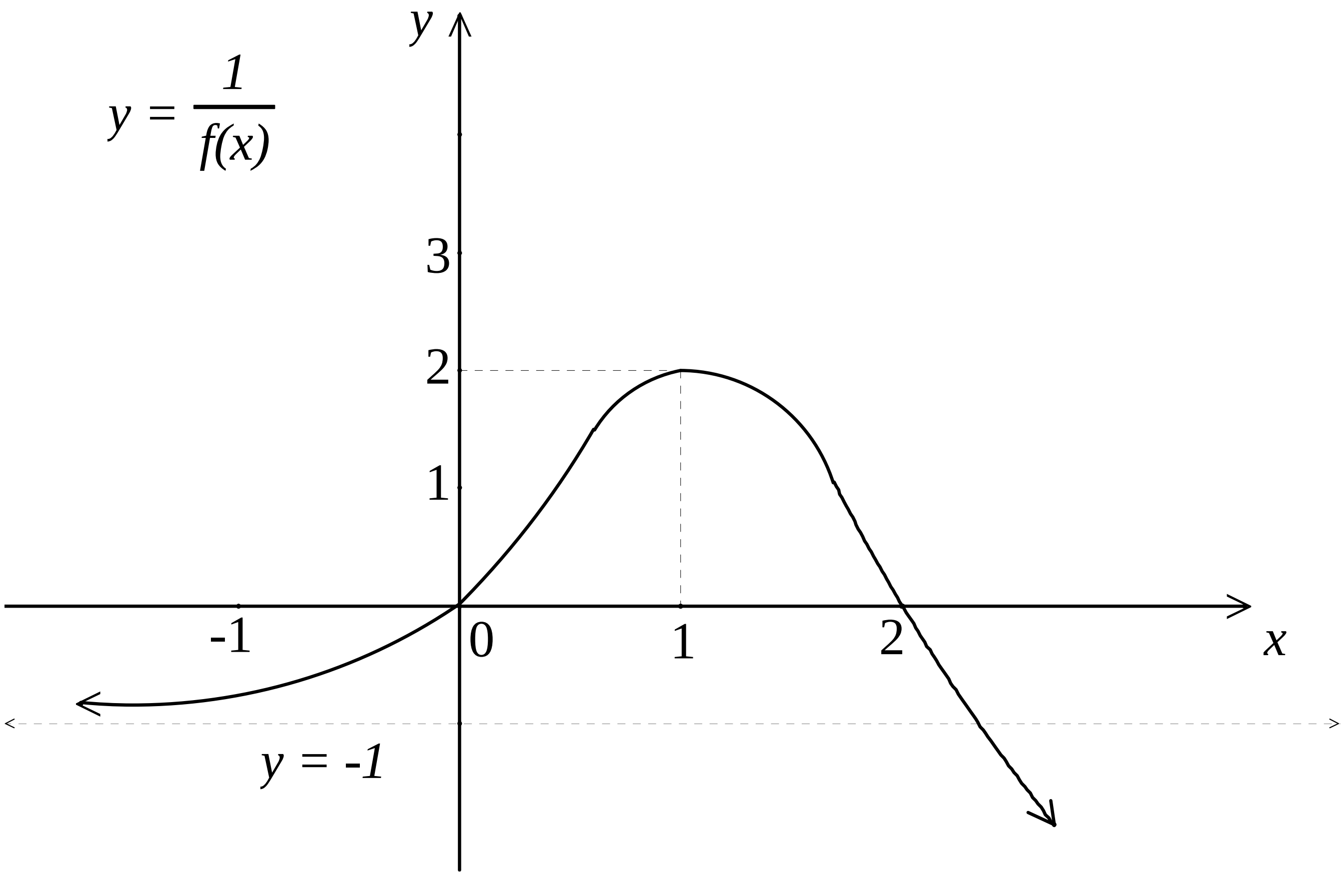
(b) (i) Use the substitution $t = \tan \frac{x}{2}$ to prove that $\int_0^{\frac{\pi}{2}} \frac{dx}{2+\sin x} = \frac{\pi}{3\sqrt{3}}$. **3**

(ii) Show that $\int_0^{2a} f(x)\, dx = \int_0^{a} \{ f(x) + f(2a-x) \} dx$. **3**

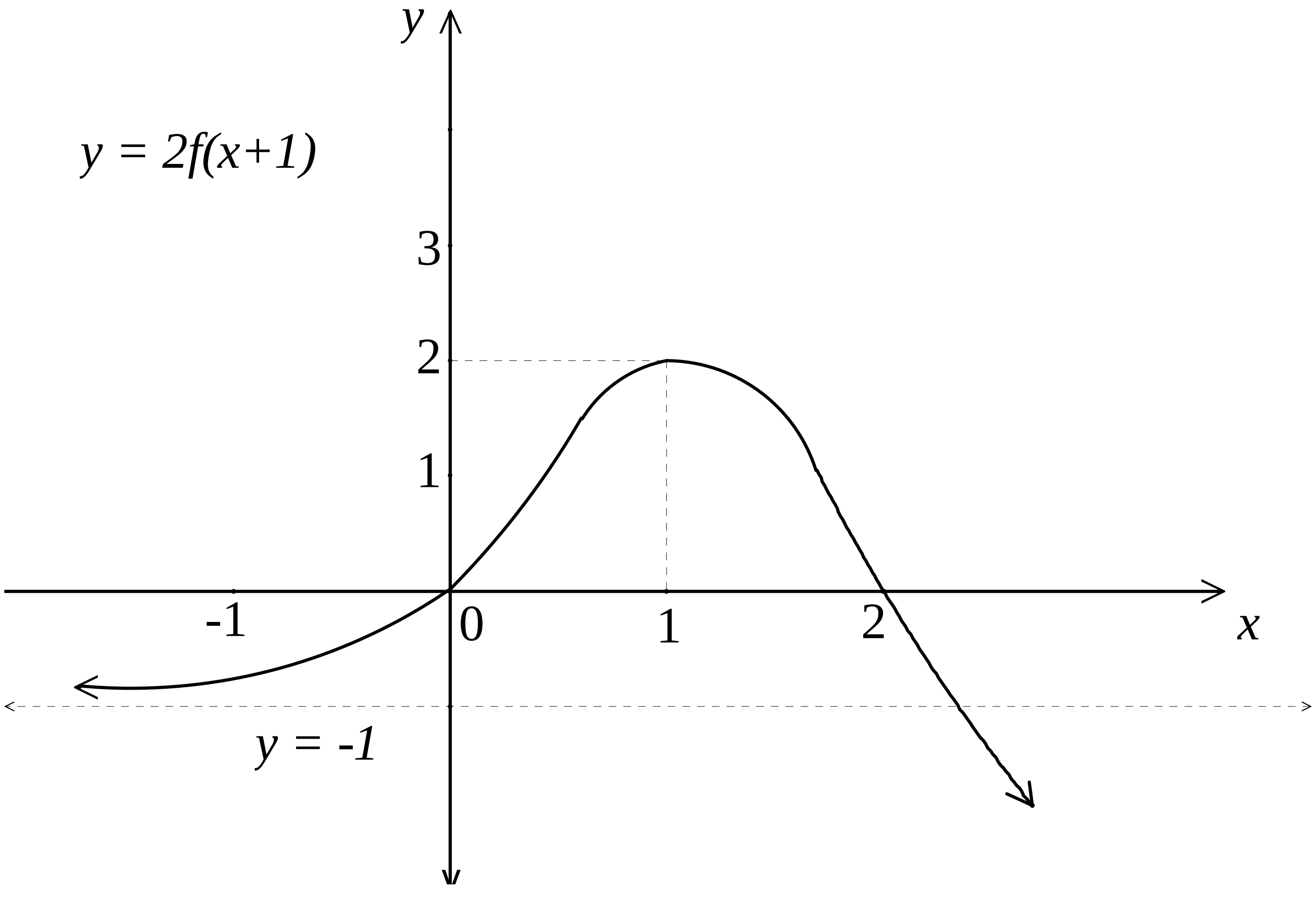
(iii) Hence evaluate $\int_0^{\pi} \frac{x}{2+\sin x} dx$. **3**

⏩ ⏩ THIS IS THE END OF THE EXAMINATION PAPER ⏪ ⏪

Question 3(b)(i) *Student #: ....*



Question 3(b)(ii) *Student #: ....*

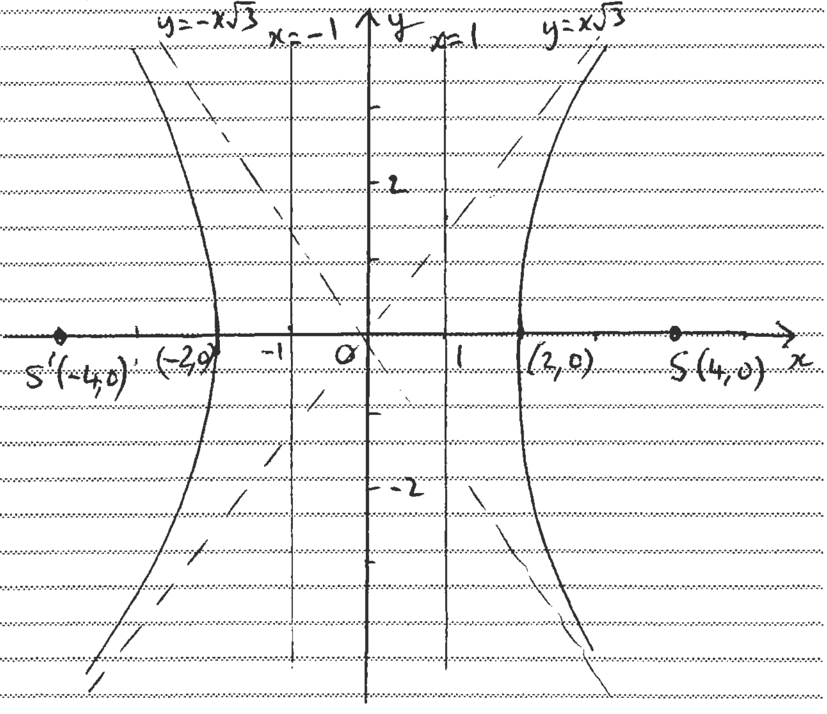


2011 - TERM 1 (ASST 2)

MATHEMATICS Extension 1 : Question ...1...

Suggested Solutions	Marks	Marker's Comments
a) i) $\int \tan^2 4\theta \, d\theta = \int (\sec^2 4\theta - 1)\, d\theta$	1	
$= \frac{\tan 4\theta}{4} - \theta + c$	1	
ii) $\int \frac{dx}{(x+1)^2+3} = \frac{1}{\sqrt{3}} \int \frac{\sqrt{3}\, dx}{(x+1)^2+(\sqrt{3})^2}$	1, 1	If c was omitted, ½ mark was deducted once in paper.
$= \frac{1}{\sqrt{3}} \tan^{-1}\left(\frac{x+1}{\sqrt{3}}\right) + c$	1	
iii) $I = \int_0^1 \frac{x\, dx}{\sqrt{2-x}}$ Let $u = 2-x$ "$du = -dx$"		
When $x=1, u=1$ $x=0, u=2$ $\therefore I = \int_2^1 \frac{2-u(-du)}{\sqrt{u}}$	1	
$= \int_1^2 \left(\frac{2}{\sqrt{u}} - \sqrt{u}\right) du$		
$= \left[4\left(u^{1/2}\right) - \frac{2}{3}\left(u^{3/2}\right)\right]_1^2$	1	
$= 4\sqrt{2} - \frac{2}{3}.2\sqrt{2} - 4 + \frac{2}{3}$		
$= \frac{8\sqrt{2}}{3} - \frac{10}{3}$	1	½ mark docked if terms not gathered.
iv) Let $\frac{4}{x(x-4)} = \frac{A}{x} + \frac{B}{x-4}$		
$\therefore A(x-4) + Bx \equiv 4$		
Substitute $x = 4 \Rightarrow B = 1$ Substitute $x = 0 \Rightarrow A = -1$		
$\therefore \int_1^3 \frac{4\, dx}{x^2 - 4x} = \int_1^3 \left(-\frac{1}{x} + \frac{1}{x-4}\right) dx$	1	

MATHEMATICS Extension 1 : Question ..1..... (p2 of 2)

Suggested Solutions	Marks	Marker's Comments
$= -\int_1^3 \frac{dx}{x} + \int_1^3 \left(\frac{-dx}{4-x}\right)$ (Ensuring correct domain)		Many used | | signs.
$= \left[-\ln x + \ln(4-x)\right]_1^3$	1	Any mention of ln(-1) or ln(-3) lost a mark.
$= -\ln 3 + \ln 1 + \ln 1 - \ln 3$		
$= \underline{\underline{-2\ln 3}}$	1	
b) $\frac{x^2}{4} - \frac{y^2}{12} = 1$ $\therefore a^2 = 4,\ b^2 = 12$		
$b^2 = a^2(e^2-1) \Rightarrow e^2 = 4$		
$\Rightarrow \underline{\underline{e = 2}}\ (e>0)$		
$\therefore$ Foci : $(\pm ae, 0) = \underline{\underline{(\pm 4, 0)}}$		Generally well done.
Directrices : $x = \pm \frac{a}{e} \rightarrow \underline{\underline{x = \pm 1}}$		
Intercepts : $(y=0)$ $x = \pm 2 \rightarrow \underline{\underline{(\pm 2, 0)}}$		½ mark deducted if $y = x\sqrt{3}$ was not above 45°
$(x=0)$ No such points		
Asymptotes $\underline{\underline{y = \pm x\sqrt{3}}}$		
	4	

Y12 MATH EXT2 ASSESSMEN TASK 2
TERM 1, 2011

MATHEMATICS Extension 2: Question 2

Suggested Solutions	Marks	Marker's Comments
Q 2(a) (i) $I=\int_1^e x\ln x\,dx$ $u=\ln x$ $dv=x\,dx$ $du=\frac{1}{x}dx$ $v=\frac{1}{2}x^2$ $I=\left[\frac{1}{2}x^2\ln x\right]_1^e-\frac{1}{2}\int_1^e x\,dx$ ✓ $=\left[\frac{1}{2}e^2-0\right]$ ✓ $-\frac{1}{2}\times\frac{1}{2}x^2\Big]_1^e$ $=\frac{1}{2}e^2-\frac{1}{4}(e^2-1)$ ✓ $=\frac{1}{2}e^2-\frac{1}{4}e^2+\frac{1}{4}$ $=\underline{\frac{1}{4}(e^2+1)}$	$\frac{1}{2}$ 1	1 For correct IBPs $\frac{1}{2}$ for $\frac{1}{2}e^2-0$ $\boxed{3}$
(ii) $I=\int_0^{\pi/4}\sin^3\theta\,d\theta$ $=\int_0^{\frac{\pi}{4}}\sin\theta\times\sin^2\theta\,d\theta=$ $=\int_0^{\pi/4}\sin\theta(1-\cos^2\theta)\,d\theta$ ✓ $=\int_0^{\frac{\pi}{4}}\sin\theta-\sin\theta.\cos^2\theta\,d\theta$ $=\left[-\cos\theta+\frac{1}{3}\cos^3\theta\right]_0^{\frac{\pi}{4}}$ ✓ $\equiv\left[\frac{1}{3}\cos^3\theta-\cos\theta\right]_0^{\frac{\pi}{4}}$ $=\left(\frac{1}{3}.\frac{1}{2\sqrt{2}}-\frac{1}{\sqrt{2}}\right)-\left(\frac{1}{3}-1\right)$ $=\frac{1}{6\sqrt{2}}-\frac{1}{\sqrt{2}}-\left(-\frac{2}{3}\right)$ $=\underline{\frac{2}{3}-\frac{5}{6\sqrt{2}}}=\underline{\frac{4\sqrt{2}-5}{6\sqrt{2}}}=\underline{\frac{8-5\sqrt{2}}{12}}$	$\frac{1}{2}$ 1 $\frac{1}{2}$, $\frac{1}{2}$ $\frac{1}{2}$	Method 2 Let $u=\cos\theta$ $du=-\sin\theta\,d\theta$ $\sin\theta\,d\theta=-du$ θ u $\pi/4$ $\frac{1}{\sqrt{2}}$ 0 1 $I=+\int_{\frac{1}{\sqrt{2}}}^{1}(u^2-1)\,du$ etc — — — — — — — — Method 3 IBPs $u=\sin^2\theta$ $dV=\sin\theta\,d\theta$ $du=2\sin\theta\cos\theta$ $V=-\cos\theta$ $I=\left[-\cos\theta\sin^2\theta\right]_0^{\pi/4}+2\int\sin\theta\cos^2\theta\,d\theta$ $=-\frac{1}{2\sqrt{2}}-0-\frac{2}{3}\cos^3\theta\Big]_0^{\pi/4}$ $\boxed{3}$
(iii) $I=\int\frac{e^{2x}}{e^x-1}\,dx$ $u=e^x$ $[x=\ln u]$. $du=e^x\,dx=u\,dx$ $\therefore dx=\frac{du}{u}$ $I=\int\frac{u^2}{u-1}\times\frac{du}{u}=\int\frac{u\,du}{u-1}$ $\begin{array}{r}1\\ u-1\,\overline{)\,u}\\ \underline{u-1}\\ 0+1\end{array}$ $=\int 1+\frac{1}{u-1}\,du$ $=u+\ln\lvert u-1\rvert+c$ $\therefore I=\underline{e^x+\ln\lvert e^x-1\rvert+C}$	✓ ✓ ✓	$\boxed{3}$ $-\frac{1}{2}$ if no Abs. brackets

MATHEMATICS Extension 2: Question 2

Suggested Solutions	Marks	Marker's Comments

Q2(b) $9x^2 + 25y^2 = 225 \Rightarrow \frac{x^2}{25} + \frac{y^2}{9} = 1$

$a^2 = 25 \Rightarrow a = 5$

$b^2 = 9 \Rightarrow b = 3$

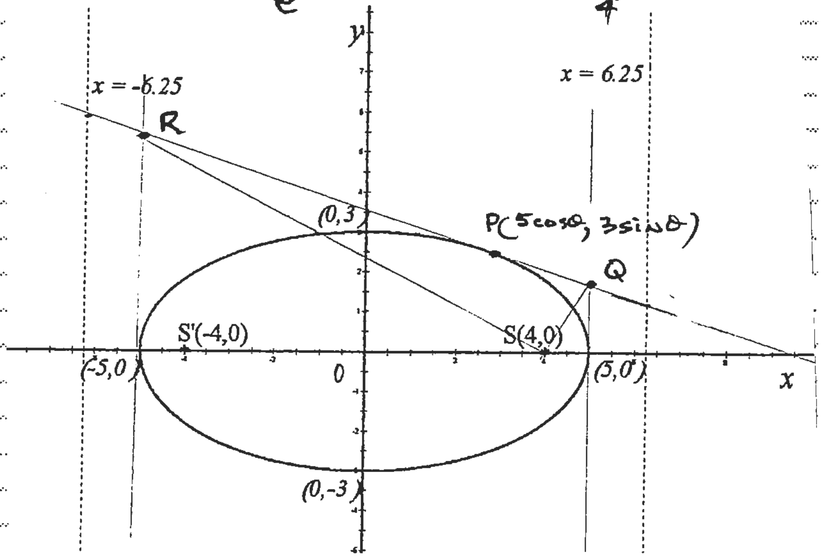
$e^2 = 1 - \frac{b^2}{a^2} = 1 - \frac{9}{25} = \frac{16}{25}$

$\therefore e = \frac{4}{5}$ ($0 < e < 1$ for Ellipse)

Intercepts : $(0, \pm 3)$ $(\pm 5, 0)$ VERTICES

Foci : $(\pm ae, 0) \Rightarrow S(4,0)$; $S'(-4,0)$

Directrices : $x = \pm \frac{a}{e} \Rightarrow x = \pm \frac{25}{4} = \pm 6\frac{1}{4}$



$\frac{x^2}{25} + \frac{y^2}{9} = 1$

$\boxed{3}$

(ii) Equ. of tangent at $P(x_1, y_1)$: $\frac{x_1 x}{a^2} + \frac{y_1 y}{b^2} = 1$

Equ. of Tangent at $P(5\cos\theta, 3\sin\theta)$: $\frac{\cos\theta}{5}x + \frac{\sin\theta}{3}y = 1$ ✓ OR EQUIV. $\boxed{1}$

(iii) Using (ii) WHEN $x = 5$ $y = \frac{3(1-\cos\theta)}{\sin\theta}$ $\therefore$ Q could be $\left(5, \frac{3(1-\cos\theta)}{\sin\theta}\right)$ $\left(\frac{1}{2}\right)$

WHEN $x = -5$ $y = \frac{3(1+\cos\theta)}{\sin\theta}$ $\therefore$ R could be $\left(-5, \frac{3(1+\cos\theta)}{\sin\theta}\right)$

As $S = (4,0)$

$m_{SQ} = \frac{3(1-\cos\theta)}{\sin\theta} - 0 \Big/ (5-4) = \frac{3(1-\cos\theta)}{\sin\theta}$ $\left(\frac{1}{2}\right)$

$m_{RS} = \frac{3(1+\cos\theta)}{\sin\theta} - 0 \Big/ (-5-4) = \frac{1+\cos\theta}{-3\sin\theta} = -\frac{(1+\cos\theta)}{3\sin\theta}$

So $m_{RS} \times m_{SQ} = -\frac{(1+\cos\theta)}{3\sin\theta} \times \frac{3(1-\cos\theta)}{\sin\theta}$

$= -\frac{(1-\cos^2\theta)}{\sin^2\theta} = -\frac{\sin^2\theta}{\sin^2\theta}$ ✓

$= -1$, $\theta \neq 0$ or π

1 To show fully why -1 $\boxed{2}$

$\therefore$ $RS \perp SQ$ (as product of gradients is -1)

Solutions to Ext 2 T1 Q3

a i) $A = \frac{1}{2}$ — $\frac{1}{2}$m

$B = -\frac{1}{2}$ — $\frac{1}{2}$m

$C = \frac{1}{2}$ — $\frac{1}{2}$m

$\frac{1}{2(1+x)} + \frac{-x+1}{2(1+x^2)}$ — $\frac{1}{2}$m Some forgot this line

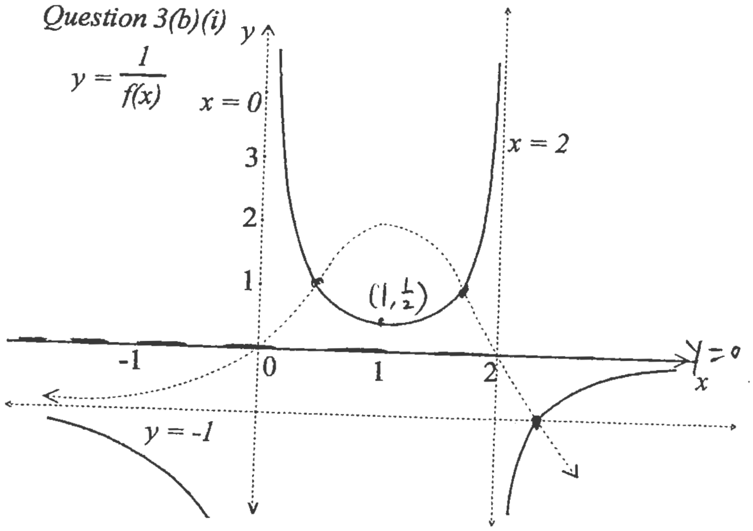
ii) $\int_0^1 \frac{dx}{2(1+x)} - \int_0^1 \frac{x\,dx}{2(1+x^2)} + \int_0^1 \frac{dx}{2(1+x^2)}$

$= \frac{1}{2}\Big[\ln(1+x)\Big]_0^1 - \frac{1}{4}\Big[\ln(1+x^2)\Big]_0^1 + \frac{1}{2}\Big[\tan^{-1}x\Big]_0^1$ — $\frac{1}{2}$m + $\frac{1}{2}$m + $\frac{1}{2}$m

$= \frac{1}{2}\ln 2 - \frac{1}{4}\ln 2 - \frac{1}{2}\frac{\pi}{4}$ — 1 m

$= \frac{\ln 2}{4} + \frac{\pi}{8}$ — $\frac{1}{2}$ m

b)

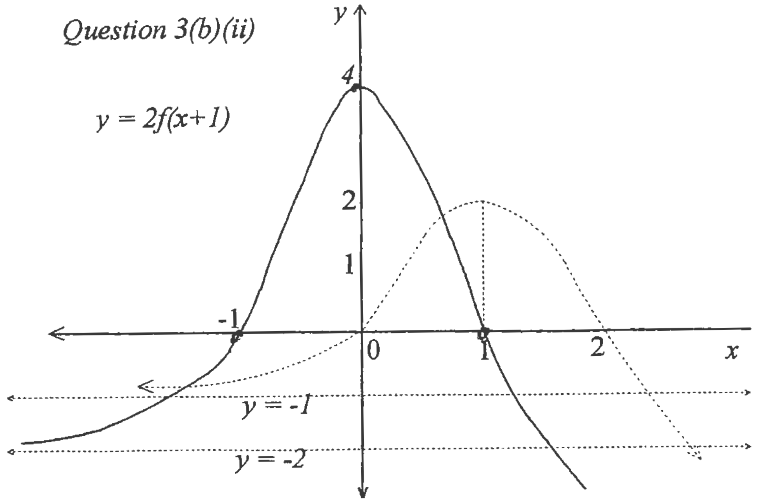


Asy $x=0$, $x=2$, $y=0$, $y=1$ $\frac{1}{2}$m

Each branch $\frac{1}{2}$m @ $\Rightarrow 1\frac{1}{2}$m

must meet $y = 1, -1$

must show min $(1, \frac{1}{2})$



Asy $y = -2$ $\frac{1}{2}$m

x, y intercepts $\frac{1}{2}$m

Each side graph $\frac{1}{2}$m $\Rightarrow 1$m

2011 Ext 2 T1 Q3

(i) $\frac{x^2}{a^2} + \frac{(mx+c)^2}{b^2} = 1$ $\frac{1}{2}$m

$(b^2+a^2m^2)x^2 + 2a^2mcx + a^2(c^2-b^2) = 0$ $\frac{1}{2}$m

tangent if $\Delta = 0$

$4a^2m^2c^2 - 4(b^2+a^2m^2) \times a^2(c^2-b^2) = 0$ $\frac{1}{2}$m

$4a^2b^2(a^2m^2 - c^2 + b^2) = 0$ $\frac{1}{2}$m

$a \neq 0$, $b \neq 0$

$c^2 = a^2m^2 + b^2$

Some of Students write $c = \sqrt{a^2m^2+b^2}$ -1m

(ii) $q = mp + c$ 1 m

$c^2 = a^2m^2 + b^2$

$(q-mp)^2 = a^2m^2 + b^2$ 1 m

$q^2 - 2mpq + m^2p^2 = a^2m^2 + b^2$

$(a^2-p^2)m^2 + 2pqm + (b^2-q^2) = 0$ 1 m

(iii) $m_1 \times m_2 = -1$ for perpendicular lines

$\frac{b^2-q^2}{a^2-p^2} = -1$ product of roots is c/a 1 m

$a^2+b^2 = p^2+q^2$

QUESTION 4 (15 Marks) (START A NEW PAGE) **Mark**

(a) (i) Given that $I_n = \int_0^{\frac{\pi}{2}} \theta \sin^n \theta \, d\theta$, prove that $I_n = \frac{n-1}{n} I_{n-2} + \frac{1}{n^2}$ for $n \geq 2$. **3**

$I_n = \int \theta \sin^{n-1} \theta \times \sin \theta \, d\theta$

$u = \theta \sin^{n-1} \theta$ $\qquad dv = \sin \theta \, d\theta$

$du = \sin^{n-1} \theta + (n-1) \theta \sin^{n-2} \theta \times \cos \theta \, d\theta$

$v = -\cos \theta$

Solution:

$$I_n = \int_0^{\frac{\pi}{2}} \left\{ \theta \sin^{n-1} \theta \times \frac{d}{d\theta}(-\cos\theta) \right\} d\theta$$

$= uv - \int v \, du$

$$= \left[-\theta \cos\theta \sin^{n-1}\theta \right]_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} \left\{ (-\cos\theta) \times \left(\sin^{n-1}\theta + \theta(n-1)\cos\theta \sin^{n-2}\theta \right) \right\} d\theta$$

$$= 0 + \int_0^{\frac{\pi}{2}} \left(\cos\theta \sin^{n-1}\theta + (n-1)\theta \cos^2\theta \sin^{n-2}\theta \right) d\theta$$

$$= \int_0^{\frac{\pi}{2}} \left(\cos\theta \sin^{n-1}\theta \right) d\theta + (n-1) \int_0^{\frac{\pi}{2}} \left(\theta \cos^2\theta \sin^{n-2}\theta \right) d\theta$$

$$= \left[\frac{1}{n} \sin^n \theta \right]_0^{\frac{\pi}{2}} + (n-1) \int_0^{\frac{\pi}{2}} \left\{ \theta \left(1 - \sin^2\theta\right) \sin^{n-2}\theta \right\} d\theta$$

$$= \frac{1}{n} + (n-1) \int_0^{\frac{\pi}{2}} \left\{ \theta \sin^{n-2}\theta - \theta \sin^n \theta \right\} d\theta$$

$$= \frac{1}{n} + (n-1) \int_0^{\frac{\pi}{2}} \left\{ \theta \sin^{n-2}\theta \right\} d\theta - (n-1) \int_0^{\frac{\pi}{2}} \left\{ \theta \sin^n \theta \right\} d\theta$$

$$= \frac{1}{n} + (n-1) I_{n-2} - (n-1) I_n$$

$$I_n + (n-1) I_n = \frac{1}{n} + (n-1) I_{n-2}$$

$$n I_n = \frac{1}{n} + (n-1) I_{n-2}$$

$$I_n = \frac{1}{n^2} + \frac{n-1}{n} I_{n-2}$$

(ii) Hence show that $\int_0^{\frac{\pi}{2}} \theta \sin^5 \theta \, d\theta = \frac{149}{225}$. **3**

Solution:

$$\int_0^{\frac{\pi}{2}} \theta \sin^5 \theta \, d\theta = I_5$$

$$I_5 = \frac{1}{25} + \frac{4}{5} I_3$$

$$I_3 = \frac{1}{9} + \frac{2}{3} I_1$$

$$I_1 = \int_0^{\frac{\pi}{2}} \theta \sin\theta \, d\theta$$

$$= \int_0^{\frac{\pi}{2}} \theta \frac{d}{d\theta}(-\cos\theta)\, d\theta$$

$$= [-\theta\cos\theta]_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} (-\cos\theta)\, d\theta$$

$$= 0 + \int_0^{\frac{\pi}{2}} \cos\theta \, d\theta$$

$$= [\sin\theta]_0^{\frac{\pi}{2}}$$

$$= 1$$

$$I_3 = \frac{1}{9} + \frac{2}{3}(1)$$

$$= \frac{7}{9}$$

$$I_5 = \frac{1}{25} + \frac{4}{5}\left(\frac{7}{9}\right)$$

$$= \frac{149}{225}$$

$$\int_0^{\frac{\pi}{2}} \theta \sin^5\theta \, d\theta = \frac{149}{225}$$

(b) (i) Use the substitution $t = \tan\frac{x}{2}$ to prove that $\int_0^{\frac{\pi}{2}} \frac{dx}{2+\sin x} = \frac{\pi}{3\sqrt{3}}$. **3**

Solution:

$$\int_0^{\frac{\pi}{2}} \frac{dx}{2+\sin x} = \int_0^1 \left(\frac{1}{2+\frac{2t}{1+t^2}}\right) \times \frac{2dt}{1+t^2}$$

$$= \int_0^1 \left(\frac{2dt}{2(1+t^2)+2t}\right)$$

$$= \int_0^1 \left(\frac{1}{t^2+t+1}\right) dt$$

$$= \int_0^1 \left(\frac{1}{\left(t+\frac{1}{2}\right)^2 + \frac{3}{4}}\right) dt$$

$$= \frac{2}{\sqrt{3}}\left[\tan^{-1}\left(\frac{t+\frac{1}{2}}{\frac{\sqrt{3}}{2}}\right)\right]_0^1$$

$$= \frac{2}{\sqrt{3}}\left[\tan^{-1}\left(\frac{2t+1}{\sqrt{3}}\right)\right]_0^1$$

$$= \frac{2}{\sqrt{3}}\left\{\left(\tan^{-1}\left(\frac{3}{\sqrt{3}}\right) - \tan^{-1}\left(\frac{1}{\sqrt{3}}\right)\right)\right\}$$

$$= \frac{2}{\sqrt{3}}\left(\frac{\pi}{3} - \frac{\pi}{6}\right)$$

$$= \frac{\pi}{3\sqrt{3}}$$

(ii) Show that $\int_0^{2a} f(x)\,dx = \int_0^a \{f(x) + f(2a-x)\}\,dx$. **3**

Solution:

$$\int_0^{2a} f(x)\,dx = \int_0^a f(x)\,dx + \int_a^{2a} f(x)\,dx$$

Consider

$\int_a^{2a} f(x)\,dx$, let $u = 2a - x$, $du = -dx$, $x = a \Rightarrow u = a$, $x = 2a \Rightarrow u = 0$

$$\int_a^{2a} f(x)\,dx = \int_a^0 f(2a-u)\,(-du)$$

$$= \int_0^a f(2a-u)\,du$$

$$= \int_0^a f(2a-x)\,dx \quad (let\ u = x)$$

$$\int_0^{2a} f(x)\,dx = \int_0^a f(x)\,dx + \int_0^a f(2a-x)\,dx$$

$$= \int_0^a \{f(x) + f(2a-x)\}\,dx$$

Alternative

(ii)

LHS $= \int_0^{2a} f(x)\,dx = F(x)\Big]_0^{2a} = F(2a) - F(0)$, where $F(x)$ is a primitive of $f(x)$.

RHS $= \int_0^a f(x) + f(2a-x)\,dx$

$= \Big[F(x) - F(2a-x)\Big]_0^a$

$= F(a) - F(a) - \big(F(0) - F(2a)\big)$

$= F(2a) - F(0)$

$\therefore$ LHS = RHS

$\therefore \int_0^{2a} f(x)\,dx = \int_0^a f(x) + f(2a-x)\,dx.$

(iii) Hence evaluate $\int_0^{\pi} \frac{x}{2+\sin x}\,dx$. **3**

Solution:

$$\int_0^{\pi} \frac{x}{2+\sin x}\,dx = \int_0^{\frac{\pi}{2}} \left(\frac{x}{2+\sin x} + \frac{\pi - x}{2+\sin(\pi - x)}\right)dx$$

$$= \int_0^{\frac{\pi}{2}} \left(\frac{x}{2+\sin x} + \frac{\pi - x}{2+\sin x}\right)dx$$

$$= \int_0^{\frac{\pi}{2}} \left(\frac{x+\pi-x}{2+\sin x}\right)dx$$

$$= \int_0^{\frac{\pi}{2}} \frac{\pi\,dx}{2+\sin x}$$

$$= \pi\int_0^{\frac{\pi}{2}} \frac{dx}{2+\sin x}$$

$$= \pi\left(\frac{\pi}{3\sqrt{3}}\right)$$

$$= \frac{\pi^2}{3\sqrt{3}}$$