Question 1. **Marks**

(a) Find $\int_0^{\frac{\pi}{2}} (\cos^3 x).\sqrt{\sin x}\, dx$. **3**

(b) Find $\int \frac{dx}{1-e^x}$. **2**

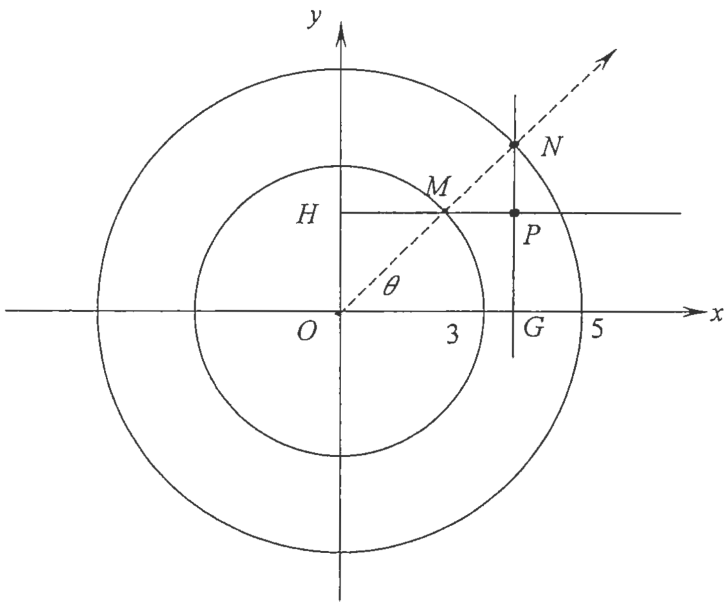
(c) (i) Find the real numbers a and b such that: **2**

$$\frac{x-1}{(x-2)(x^2-4x+5)} = \frac{1}{x-2} + \frac{ax+b}{x^2-4x+5}.$$

(ii) Hence, find $\int \frac{x-1}{(x-2)(x^2-4x+5)}\,dx$. **3**

(d) Using the substitution $t = \tan\frac{x}{2}$, find $\int \frac{dx}{3+2\cos x}$. **3**

(e) **2**



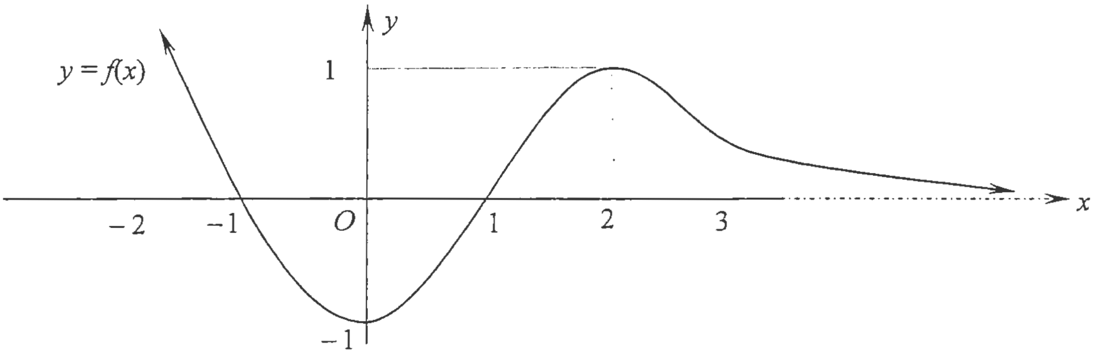
Not to scale

The circles above have centres at O and with radii 3 units and 5 units respectively. A ray from O makes an angle of θ with the positive x-axis and cuts the circles at points M and N as shown.
NG is drawn parallel to the y-axis. NG and MH intersect at point P.

Find the Cartesian equation for the locus of P as θ varies.

Question 2. **[Start a New Page]** **Marks**

(a) The sketch of the graph of the function $y = f(x)$ is shown below.



On *separate number planes*, sketch the graphs of the following, showing all essential features.

(i) $y = f(x-1)$. **2**

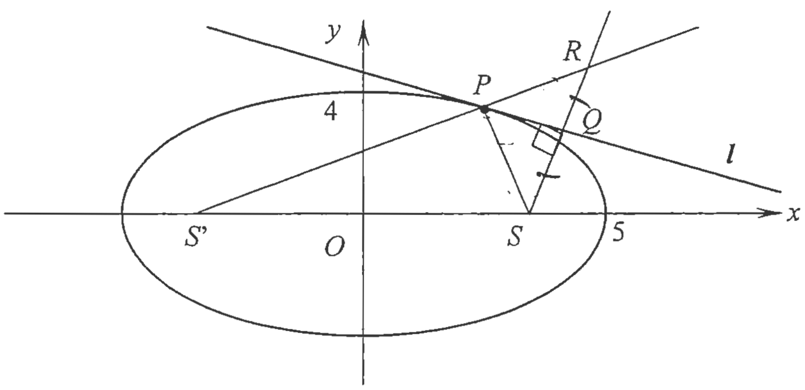
(ii) $y = \dfrac{1}{f(x)}$. **2**

(iii) $y = \ln[f(x)]$. **2**

(iv) $y^2 = f(x)$. **2**

(v) $y = xf(x)$. **2**

(b) The diagram below shows the ellipse $\boldsymbol{E}$: $\dfrac{x^2}{25} + \dfrac{y^2}{16} = 1$.
The line $\boldsymbol{l}$ is tangent to the ellipse $\boldsymbol{E}$ at the point P.
The foci of the ellipse are S and S'. The perpendicular to $\boldsymbol{l}$ through S meets $\boldsymbol{l}$ at the point Q. The lines SQ and $S'P$ meet at the point R.



(i) Copying the diagram onto your paper and by using the reflection property of the ellipse at P, prove that $SQ = RQ$. **2**

(ii) Explain why $S'R = 10$. . **1**

(iii) Hence, or otherwise, prove that Q lies on the circle $x^2 + y^2 = 25$. **2**

Question 3. **[Start a New Page]** **Marks**

(a) Let $I = \int_1^3 \frac{\sin^2\left(\frac{\pi x}{8}\right)}{x(4-x)}\,dx$,

(i) By using the substitution $t = 4 - x$, show that: $I = \int_1^3 \frac{\cos^2\left(\frac{\pi t}{8}\right)}{t(4-t)}\,dt$. **2**

(ii) Hence, find the value of I. **3**

(b) The hyperbola $\boldsymbol{H}$ has the Cartesian equation $\frac{x^2}{4} - \frac{y^2}{5} = 1$, where $P\left(2\sec\theta, \sqrt{5}\tan\theta\right)$ is a point on $\boldsymbol{H}$.

(i) Find the eccentricity e. **1**

(ii) Find the coordinates of the foci. **1**

(iii) State the equation of the asymptotes. **1**

(iv) Sketch the hyperbola $\boldsymbol{H}$. **2**

(v) Show that the tangent to $\boldsymbol{H}$ at P has the equation: **2**

$$\frac{x\sec\theta}{2} - \frac{y\tan\theta}{\sqrt{5}} = 1.$$

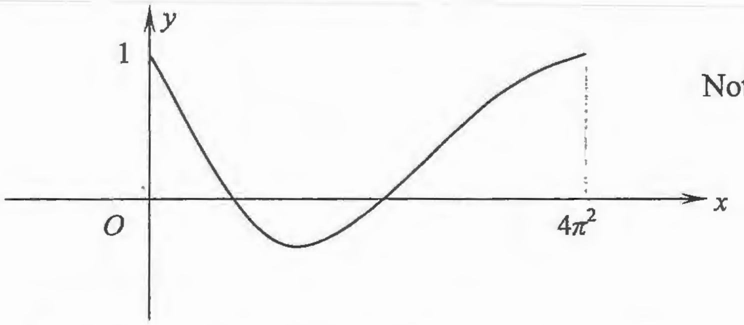
(vi) If this tangent cuts the asymptotes at L and M, prove that $LP = PM$. **2**

(c) For the general ellipse: $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, for $a > b$, describe the effect on the ellipse as the eccentricity $e \to 0^+$. **1**

Question 4. **[Start a New Page]** **Marks**

(a) Show that the equation of the normal to the ellipse: $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$, where the eccentricity $e=\dfrac{1}{\sqrt{3}}$ at point $P(a\cos\theta, b\sin\theta)$ is $\dfrac{ax}{\cos\theta}-\dfrac{by}{\sin\theta}=\dfrac{a^2}{3}$. **3**

(b) (i) Given the sketch of the curve $y=\cos\sqrt{x}$, for $0\le x\le 4\pi^2$, **1**



Not to scale

Show that the area, A square units, bounded by the curve $y=\cos\left(\sqrt{|x|}\right)$ and the x-axis for $-\dfrac{\pi^2}{4}\le x\le\dfrac{\pi^2}{4}$, can be expressed as

$$A=2\int_0^{\frac{\pi^2}{4}}\cos\sqrt{x}\,dx.$$

(ii) Hence, find the area A. **4**

(c) Given $I_n=\displaystyle\int_1^e(\ln x)^n\,dx$, for $n=0, 1, 2, \ldots$

(i) Show that $I_n=e-nI_{n-1}$, for $n=1, 2, 3, \ldots$ **2**

(ii) Let $J_n=\dfrac{I_n}{n!}$, show that: $\dfrac{1}{e}(1+J_{10})=1-\dfrac{1}{1!}+\dfrac{1}{2!}-\dfrac{1}{3!}+\ldots+\dfrac{1}{10!}$. **3**

(iii) Given that: $\displaystyle\sum_{r=0}^{n}\frac{(-1)^r}{r!}=\frac{1}{e}\left[1+(-1)^nJ_n\right]$ for $n=0, 1, 2, \ldots$ **2**

Deduce the sum to infinity of the series:

$\displaystyle\sum_{r=0}^{\infty}\frac{(-1)^r}{r!}=1-\frac{1}{1!}+\frac{1}{2!}-\frac{1}{3!}+\ldots$, justify your answer.

THE END

MATHEMATICS Extension 2 : Question 1

Suggested Solutions	Marks	Marker's Comments
(a) $\int_0^{\frac{\pi}{2}} \cos^3 x \sqrt{\sin x}\, dx$		$\sin^{\frac{3}{2}} x$ $\sin^{\frac{7}{2}} x$
$= \int_0^{\frac{\pi}{2}} (1-\sin^2 x)\cos x\,(\sin x)^{\frac{1}{2}}\, dx$	1	$\sin^{\frac{5}{2}} x$ no marks deducted but not acceptable notation
$= \int_0^{\frac{\pi}{2}} \left[(\sin x)^{\frac{1}{2}} - (\sin x)^{\frac{5}{2}}\right] \cos x\, dx$		
$= \left[\frac{2}{3}(\sin x)^{\frac{3}{2}} - \frac{2}{7}[\sin x]^{\frac{7}{2}}\right]_0^{\frac{\pi}{2}}$	1	-1 each mistake.
$= \frac{2}{3} - \frac{2}{7} - 0 -$		some people could <u>NOT</u> do $\int x^{5/2}\, dx$ correctly.
$= \frac{8}{21}$	1	
(b) $\int \frac{dx}{1-e^x} = \int \frac{-e^{-x}}{1-e^{-x}}\, dx$	1	$-\frac{1}{2}$ not $\| \|$
$= -\ln\|1-e^{-x}\| + C$		$-\frac{1}{2}$ not simplifying
OR $x - \ln\|1-e^x\| + C$	1	$\ln(e^x) = x$.
(c) $\frac{x-1}{(x-2)(x^2-4x+5)} = \frac{1}{x-2} + \frac{ax+b}{x^2-4x+5}$		
$x-1 = x^2-4x+5 + (ax+b)(x-2)$	1	Many found a, b by equating coefficients OK.
$x=0$: $-1 = 5 - 2b$, $b = 3$		
$x=1$: $0 = 2 - (a+3)$, $a = -1$	1	
$\int \frac{(x-1)\, dx}{(x-2)(x^2-4x+5)} = \int \frac{1}{x-2} + \frac{3-x}{x^2-4x+5}\, dx$		
$= \int \left(\frac{1}{x-2} - \frac{1}{2}\left(\frac{2x-4}{x^2-4x+5}\right) + \frac{1}{x^2-4x+5}\right) dx$		1 each expression
$= \ln\|x-2\| - \frac{1}{2}\ln(x^2-4x+5) + \tan^{-1}(x-2) + C$	3	$-$ Difficulty found in balancing coefficients; $-\frac{1}{2}$ for sign error

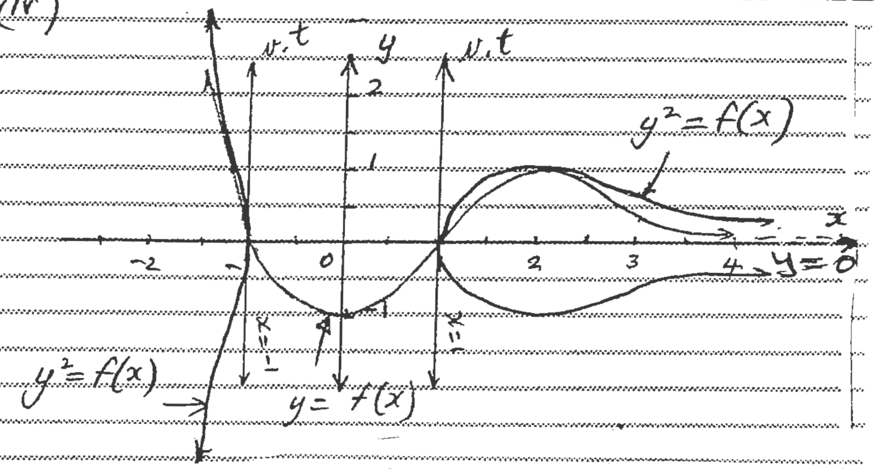
MATHEMATICS Extension 1 : Question.......

Suggested Solutions	Marks	Marker's Comments
(d) $\int \frac{dx}{3+2\cos x}$ $\quad t = \tan\frac{x}{2}$		-1 each mistake.
$= \int \frac{1}{3+2\left(\frac{1-t^2}{1+t^2}\right)} \frac{2\,dt}{1+t^2}$	1	-1 Not substituting $t = \tan\frac{x}{2}$
$= \int \frac{2\,dt}{5+t^2}$	1	some people found $\int \frac{2\,dt}{5-t^2}$.
$= \frac{2}{\sqrt{5}} \tan^{-1}\frac{t}{\sqrt{5}} + C$	1	$\frac{1}{2}$ for $\frac{2\,dt}{1+t^2}$.
$= \frac{2}{\sqrt{5}} \tan^{-1}\left(\frac{\tan\frac{x}{2}}{\sqrt{5}}\right) + C$		
e) $P(5\cos\theta, 3\sin\theta)$	1	
$x = 5\cos\theta \quad y = 3\sin\theta$	1	
$\cos^2\theta + \sin^2\theta = 1$		
$\frac{x^2}{25} + \frac{y^2}{9} = 1$		

MATHEMATICS Extension 2: Question 2

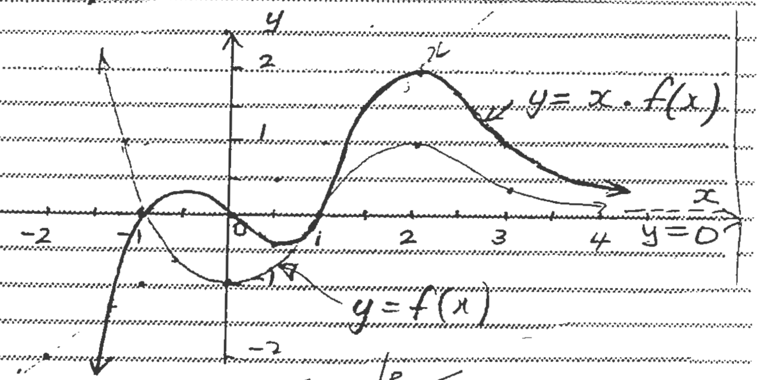
Suggested Solutions | **Marks** | **Marker's Comments**

(iv)



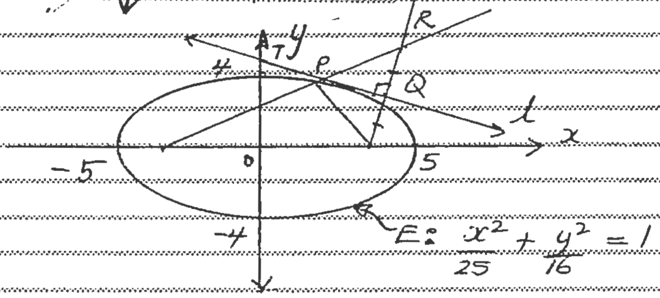
Marks: ②

(v)



Marks: ②

b)



(i) Prove: $SQ = RQ$

Proof: T is the intersection of l, with the y-axis

$T\hat{P}S' = Q\hat{P}S$ (reflection property of an ellipse)

$T\hat{P}S = Q\hat{P}R$ (vertically opposite angles are equal)

$\therefore Q\hat{P}S = Q\hat{P}R$

In ΔSPQ and ΔRPQ:-

$Q\hat{P}S = Q\hat{P}R$ (proven above)

$P\hat{Q}R = P\hat{Q}S = 90°$ (as $SQ \perp l$)

PQ is common

$\therefore \Delta SPQ \equiv \Delta RPQ$ (AAS)

$\therefore SQ = RQ$ (corresponding sides of congruent triangles are equal)

Marker's Comments:

½ reflection property

½ vertically opp angles

① congruent triangle proof

\\mercury\staffhome$\WOH\Admin_M Fac\Assessment info\Suggested Mk solns template_V3.doc

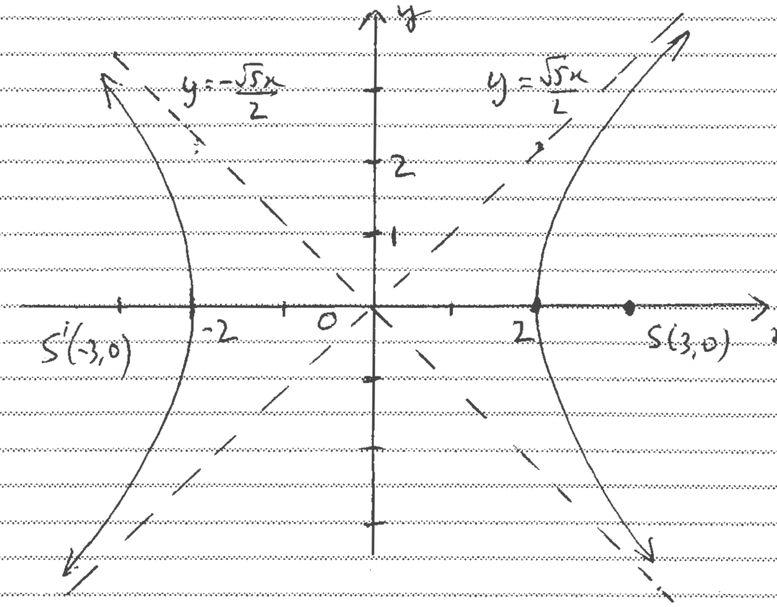
MATHEMATICS Extension 2: Question 2

Suggested Solutions	Marks	Marker's Comments
(ii) Explain why $S'R=10$ $(\frac{1}{2})$ $S'P+PS=2\times5=10$ (sum of distances from any point on the ellipse to the foci equals $2a$) From (i) $PR=PS$ (corresponding sides of congruent triangles are equal) $(\frac{1}{2})$ ie $S'P+PR=10$ Now, $S'R=S'P+PR=10$ #	1	$\frac{1}{2}$ must refer to definition of ellipse $\frac{1}{2}$ $PR=PS$ & reason
(iii) Several Methods: Easiest method. In $\Delta SRS'$, OQ' is the join of the midpoints of 2 sides of the triangle $\therefore$ OQ is parallel to and half of $S'R$ ie $OQ=5$ $\therefore$ Q lies on the circle, centre $(0,0)$, radius 5 units, ie $x^2+y^2=25$	2	$\frac{1}{2}$ refer to midpts $\frac{1}{2}$ midpoint theorem $\frac{1}{2}$ $OQ=\frac{1}{2}SR$ $\frac{1}{2}$ conclusion

2012 TERM 1 **MATHEMATICS Extension 2: Question 3**

Suggested Solutions	Marks	Marker's Comments
a) $I = \int_1^3 \frac{\sin^2(\pi x/8)}{x(4-x)}\,dx$ Let $t = 4-x$, "$dt = -dx$". When $x=3,\ t=1$; $x=1,\ t=3$ $\therefore I = \int_3^1 \frac{\sin^2\left(\frac{\pi}{8}(4-t)\right)(-dt)}{(4-t)\,t}$	1	Generally well done. A few careless with limits.
$= \int_1^3 \frac{\sin^2\left(\frac{\pi}{2}-\frac{\pi t}{8}\right)}{t(4-t)}\,dt$ $= \int_1^3 \frac{\cos^2(\pi t/8)}{t(4-t)}\,dt$ (since $\sin\left(\frac{\pi}{2}-\alpha\right)=\cos\alpha$)	1	
$2I = \int_1^3 \frac{\sin^2(\pi x/8)+\cos^2(\pi x/8)}{x(4-x)}\,dx$ (Dummy variable in (i))	1	Combining
$2I = \int_1^3 \frac{dx}{x(4-x)}$ Let $\frac{1}{x(4-x)} = \frac{A}{x}+\frac{B}{4-x}$; $\therefore A(4-x)+Bx \equiv 1$; Sub $x=4$: $B=1/4$; $x=0$: $A=1/4$ $2I = \frac{1}{4}\int_1^3 \frac{dx}{x} + \frac{1}{4}\int_1^3 \frac{dx}{4-x}$	1	Partial Fractions.
$2I = \left[\frac{1}{4}\ln x - \frac{1}{4}\ln(4-x)\right]_1^3$ $2I = \frac{1}{4}(\ln 3 - \ln 1 - \ln 1 + \ln 3)$ $2I = \frac{2\ln 3}{4}$ $\therefore I = \frac{\ln 3}{4}$	1	½ deducted if left as 2I.
b) i) In hyperbola $a=2,\ b=\sqrt{5}$ $\therefore e^2 = \frac{a^2+b^2}{a^2} = \frac{9}{4}$ $\therefore e = \frac{3}{2}$ ($e>1$ for hyperbola)	1	½ off if choice of sign unexplained
ii) Foci are $(\pm ae, 0) = (\pm 3, 0)$	1	
iii) Asymptotes $\frac{x^2}{4} = \frac{y^2}{5} \Rightarrow y = \pm\frac{x\sqrt{5}}{2}$	1	

2012 TERM 1 **MATHEMATICS Extension 2 : Question 3**

Suggested Solutions	Marks	Marker's Comments
iv) 	2	Barest minimum shown for full marks Many did not put scale on y axis.
v) $\frac{dy}{dx}=\frac{dy}{d\theta}\Big/\frac{dx}{d\theta}=\frac{\sqrt5\sec^2\theta}{2\sec\theta\tan\theta}=\frac{\sqrt5}{2}\operatorname{cosec}\theta$		
$\therefore$ Tangent is $(y-\sqrt5\tan\theta)=\frac{\sqrt5}{2}\operatorname{cosec}\theta(x-2\sec\theta)$	1	
$\frac{y}{\sqrt5}-\tan\theta=\frac{x}{2\sin\theta}-\frac{1}{\sin\theta\cos\theta}$		Many made too large a last step without further explanation
$\frac{x}{2\sin\theta}\tan\theta-\frac{y\tan\theta}{\sqrt5}=\tan^2\theta-\sec^2\theta$		
$\frac{x\sec\theta}{2}-\frac{y\tan\theta}{\sqrt5}=1 \quad (\tan^2\theta-\sec^2\theta=1)$	1	
vi) At L, substitute $y=x\sqrt5/2$		
$\frac{x\sec\theta}{2}-\frac{x\tan\theta\sqrt5}{\sqrt5.2}=1$		
$x=\frac{2}{\sec\theta-\tan\theta}=2(\sec\theta+\tan\theta)$		1 for L and M
Substitute for $y \Rightarrow L=(2(\sec\theta+\tan\theta),\sqrt5(\sec\theta+\tan\theta))$	1	1 for midpoint.
Similarly M is $(2(\sec\theta-\tan\theta),\sqrt5(\sec\theta-\tan\theta))$		Fudging was penalised.
Midpoint $ML=(2\sec\theta,\sqrt5\tan\theta)$ from midpoint formula $= P$		
$\therefore$ $PM=PL$ (P bisects ML)	1	
c) $b^2=a^2(1-e^2)$ As $e\to0^+$, $b\to a$ $(a>b>0)$ $\therefore$ Shape approaches circle $x^2+y^2=a^2$	1	½ for circle ½ for $a\to b$ or close.

2012 T1

Ext 2

MATHEMATICS: Question.....4..

Suggested Solutions	Marks	Marker's Comments
a) $\frac{dy}{dx} = -\frac{b}{a}\frac{\cos\theta}{\sin\theta}$	$\frac{1}{2}$m	
GRAD of Normal $= \frac{a}{b}\frac{\sin\theta}{\cos\theta}$	$\frac{1}{2}$m	
$y - b\sin\theta = \frac{a\sin\theta}{b\cos\theta}(x - a\cos\theta)$	$\frac{1}{2}$m	
$a\sin\theta\, x - b\cos\theta\, y = (a^2-b^2)\sin\theta\cos\theta$		
$\div \cos\theta\sin\theta$ $\quad \frac{ax}{\cos\theta} - \frac{by}{\sin\theta} = a^2 - b^2$	$\frac{1}{2}$m	
but $b^2 = a^2(1-e^2) = (1-\frac{1}{3})a^2 = \frac{2}{3}a^2$	$\frac{1}{2}$m	
$\therefore \frac{ax}{\cos\theta} - \frac{by}{\sin\theta} = \frac{a^2}{3}$	$\frac{1}{2}$m	
b i) $\cos\sqrt{\lvert x\rvert}$ is even function	1m	
Area $= \int_{-\frac{\pi^2}{4}}^{\frac{\pi^2}{4}} \cos\sqrt{\lvert x\rvert}\, dx = 2\int_0^{\frac{\pi^2}{4}} \cos\sqrt{x}\, dx$		$\cos\sqrt{\lvert -x\rvert} = \cos\sqrt{\lvert x\rvert}$ even function. Many students write $\cos\sqrt{x}$ is even $-\frac{1}{2}$m
ii) $2\int \cos\sqrt{x}\, dx = 2\int \cos u\, 2u\, du$ $\quad u^2 = x$, $2u\,du = dx$	$1+\frac{1}{2}$m	$\frac{1}{2}$m for $2u\,du = dx$
$A = 2\times 2\int_0^{\frac{\pi}{2}} u\cos u\, du = 4(u\sin u)\Big]_0^{\frac{\pi}{2}} - 4\int_0^{\frac{\pi}{2}} \sin u\, du$	$1+\frac{1}{2}$m	$\frac{1}{2}$m for limit,
$= 4\times\frac{\pi}{2}\times 1 - 0 + 4\cos u\Big]_0^{\frac{\pi}{2}}$		Some try to integrate by parts without substitution get a more complicated integrals & go nowhere
$= 2\pi - 4$ unit^2 #	1m	

MATHEMATICS: Question 4

Suggested Solutions	Marks	Marker's Comments
c i) $I_n = \int_1^e (\ln x)^n dx \quad n = 0, 1, 2, \ldots$		many forgot limit $-\frac{1}{2}m$
$I_n = x(\ln x)^n \Big]_1^e - \int_1^e n\frac{(\ln x)^{n-1}}{\cancel{x}} \cdot \cancel{x}\, dx$	1m	must show $(\ln e)^n = 1^n = 1$
$= e(\ln e)^n - 1\cdot(\ln 1)^n - n\int_1^e (\ln x)^{n-1} dx$		$(\ln 1)^n = 0^n = 0$ $-\frac{1}{2}m$
$= e - 0 - n I_{n-1}$		$\frac{n(\ln x)^{n-1}\cancel{x}}{\cancel{x}} = n(\ln x)^{n-1}$
$= e - n I_{n-1}$	1m	cross out $\frac{x}{x}$ $-\frac{1}{2}m$
ii) $\frac{1}{e}(1 + J_{10}) = \frac{1}{e}\left(1 + \frac{I_{10}}{10!}\right)$		must split up
$= \frac{1}{e}\left(1 + \frac{e}{10!} - \frac{10 I_9}{10!}\right)$ since $I_n = e - n I_{n-1}$, $J_n = \frac{I_n}{n!}$	$\frac{1}{2}m$	$\frac{I_{10}}{10!} = \frac{e}{10!} - \frac{10 I_9}{10!}$
$= \frac{1}{e}\left(1 + \frac{e}{10!} - \frac{I_9}{9!}\right)$		$I_0 = \int_1^e (\ln x)^0 dx = e - 1$
$= \frac{1}{e}\left(1 + \frac{e}{10!} - \frac{e - 9I_8}{9!}\right)$		or $I_1 = \int_1^e (\ln x) dx = 1$ 1m
$= \frac{1}{e}\left(1 + \frac{e}{10!} - \frac{e}{9!} + \frac{I_8}{8!}\right)$		must show last few terms
Similarly so on $= \frac{1}{e}\left(1 + \frac{e}{10!} - \frac{e}{9!} + \frac{e}{8!} - \cdots + \frac{e}{2!} - \frac{e}{1!} + \frac{I_0}{0!}\right)$	$\frac{1}{2}m$	many fudging
$= \frac{1}{e}\left(\cancel{1} + \frac{e}{10!} - \frac{e}{9!} + \frac{e}{8!} - \cdots + \frac{e}{2!} - \frac{e}{1!} + \frac{e - \cancel{1}}{1}\right)$	1m	Alternatively can stop at $I_1 = 1$
$= 1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} \cdots + \frac{1}{8!} - \frac{1}{9!} + \frac{1}{10!}$	1m	
iii) $\sum_{r=0}^{\infty} \frac{(-1)^r}{r!} = \lim_{n\to\infty} \frac{1}{e}\left[1 + (-1)^n J_n\right] = \lim_{n\to\infty} \frac{1}{e}\left[1 + \frac{(-1)^n I_n}{n!}\right]$		
$= \frac{1}{e}$	$\frac{I_n}{n!} \to 0$ as $n \to \infty$	1 + 1m