Name:	
Class:	

YEAR 12

ASSESSMENT TEST 2
TERM 1, 2013

MATHEMATICS EXTENSION 2

Time Allowed – 90 Minutes
(Plus 5 minutes Reading Time)

General Instructions:

- *All* questions may be attempted
- *All* questions are of equal value
- Standard Integral Tables will be supplied
- Department of Education approved calculators and templates are permitted
- In every Question, show all relevant mathematical reasoning and/or calculations.
- Marks may not be awarded for careless or badly arranged work
- No grid paper is to be used unless provided with the examination paper

The answers to all questions are to be returned in separate bundles clearly labelled Question 1, Question 2, etc.
Each question must show your (in the top right hand corner) Candidate Number.

Question 1.

2013 Year 12 Term 1 Extension II Assessment

Marks

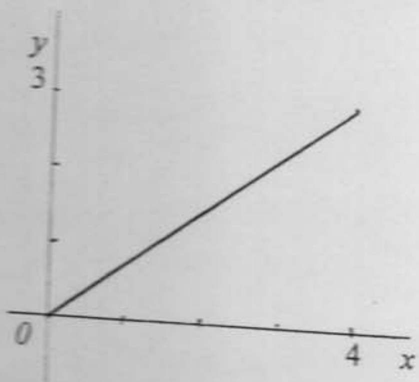
(a) Using integral tables find the primitive function of $\frac{1}{\sqrt{x^2+4}}$. 1

(b) Find $\int \frac{\sin\left(x+\frac{\pi}{4}\right)}{\sin x}\,dx$ 2

(c) Neatly graph the conic with equation : $\frac{x^2}{6}+\frac{y^2}{4}=1$, showing all relevant features. 5
Use a scale of 1 unit = 1 cm.

(d) Find the Cartesian equation for a conic with the Origin as centre,
a focus located at (5,0) and directrix with equation $x = 7$. 2

(e) The graph of $y=f(x)$ for the interval [0, 4] is shown below. 2



On the same axes graph $y = f(x)$, $y = f(8-x)$ and $y = f(x-8)$.
Use a scale of 1 unit =1 cm.

f) A conic has parametric equations $x = \frac{e^{3t}+e^{-3t}}{2}$ and $y = \frac{e^{3t}-e^{-3t}}{4}$.

(i) Find the Cartesian equation 2

(ii) Find the eccentricity of the conic 1

uestion 2.

a) Prove $\int_0^1 x^3(1-x)^{11}dx = \int_0^1 x^{11}(1-x)^3 dx$. 2

b) (i) Find the values a, b and c such that $\frac{ax+b}{x^2+2x+2} - \frac{ax+c}{x^2-2x+2} = \frac{1}{x^4+4}$ 2

ii) Hence find $\int \frac{dx}{x^4+4}$. 4

) Graph $y = 2^{\frac{1}{x}}$ showing all asymptotes and intercepts with the axes. 4
Use a scale of 1 unit = 1 cm.

) Find $\int \sqrt{\frac{4+x}{3-x}}\,dx$ 3

uestion 3.

Marks

) Find $\int e^{\sqrt{x}}\,dx$. 4

) The graph of $y=f(x)$ is shown on the attached sheet.
Graph the relation $y^2=f(x)$ on the attached sheet. 4

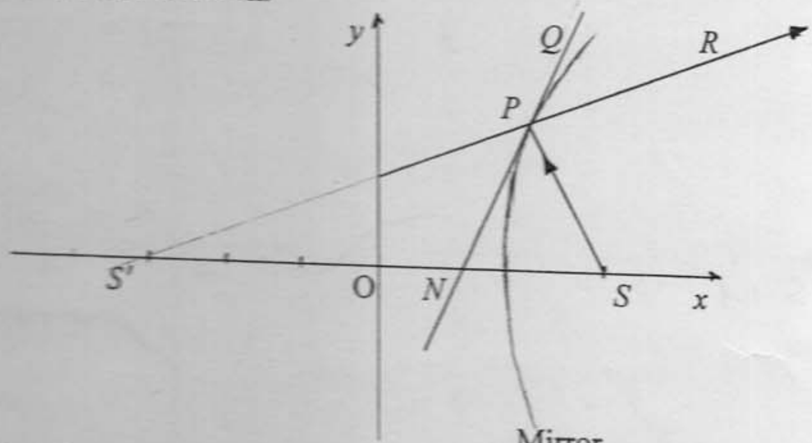
) Let $I_n = \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} cosec^n x\,dx$ where n is a positive integer.

(i) Using integration by parts show $I_n = \frac{2^{n-2}\sqrt{3}}{n-1} + \frac{n-2}{n-1} I_{n-2}$ 4

(ii) Evaluate I_3 3

uestion 4.

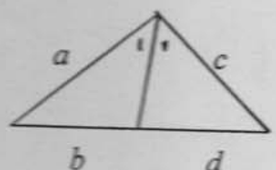
) A hyperbolic mirror with equation $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ and foci $S(ae, 0)$ and $S'(-ae, 0)$ is shown below.
A ray of light emanates from the focus $S(ae, 0)$, and is reflected from the mirror at the point $P(x_1, y_1)$ towards R is shown.



(i) Derive the equation of the tangent at $P\ (x_1, y_1)$. 2

(ii) The tangent at P intersects the x axis at N.
Prove that the tangent at P bisects $\angle SPS'$. 4
Assume the angle bisection result:

$\frac{a}{b} = \frac{c}{d}$



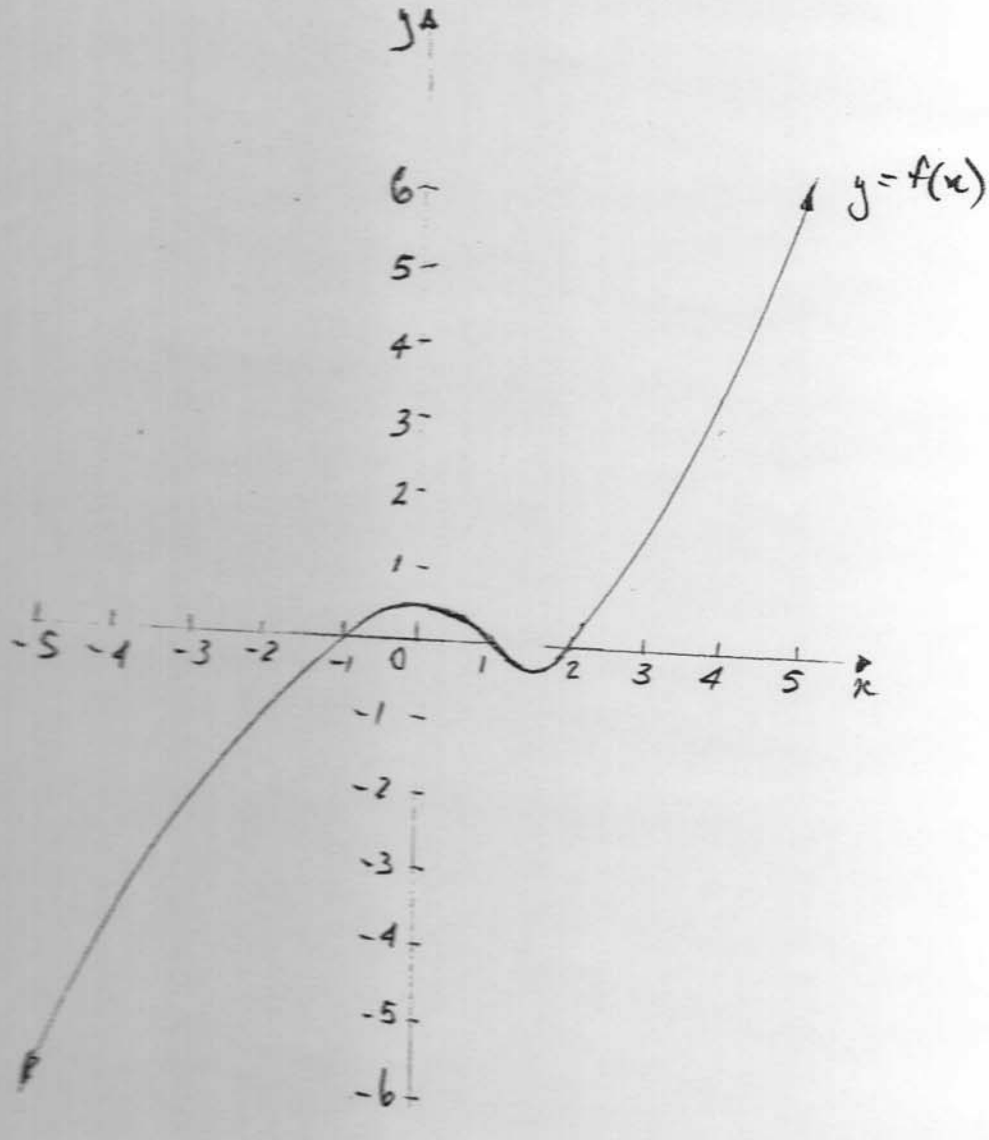
(iii) Given that the angle of incidence equals the angle of reflection ($\angle SPN = \angle QPR$) show that the reflected ray appears to emanate from the other focus $S'(-ae, 0)$. 2

) (i) Show that $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{dx}{4+3\cos x} = \frac{4}{\sqrt{7}} \tan^{-1}\left(\frac{1}{\sqrt{7}}\right)$ 3

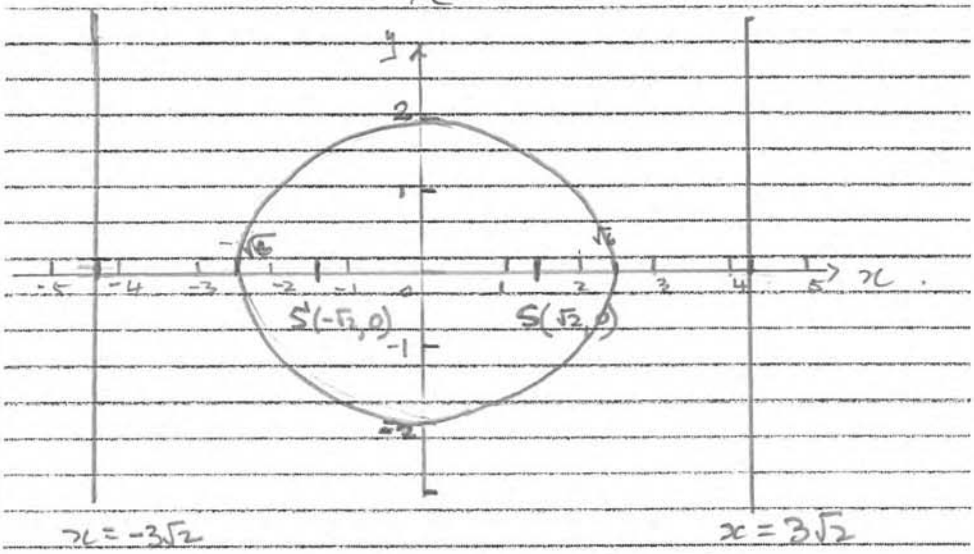
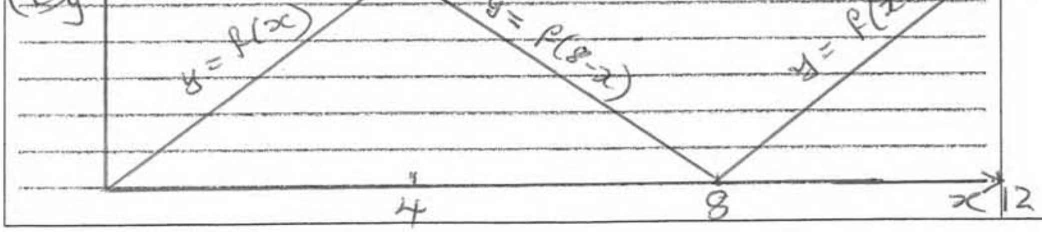
(ii) Using (i) find $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\cos 2x\,dx}{4+3\cos x}$ 4

End Of Exam

3(b)



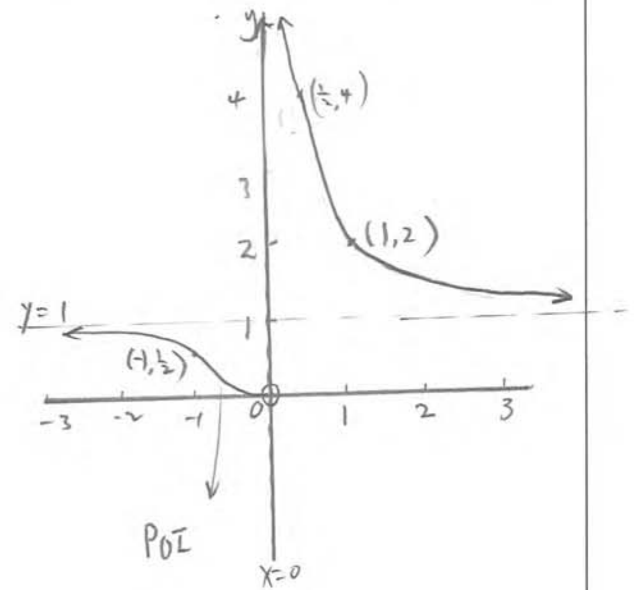
MATHEMATICS Extension 2: Question 1

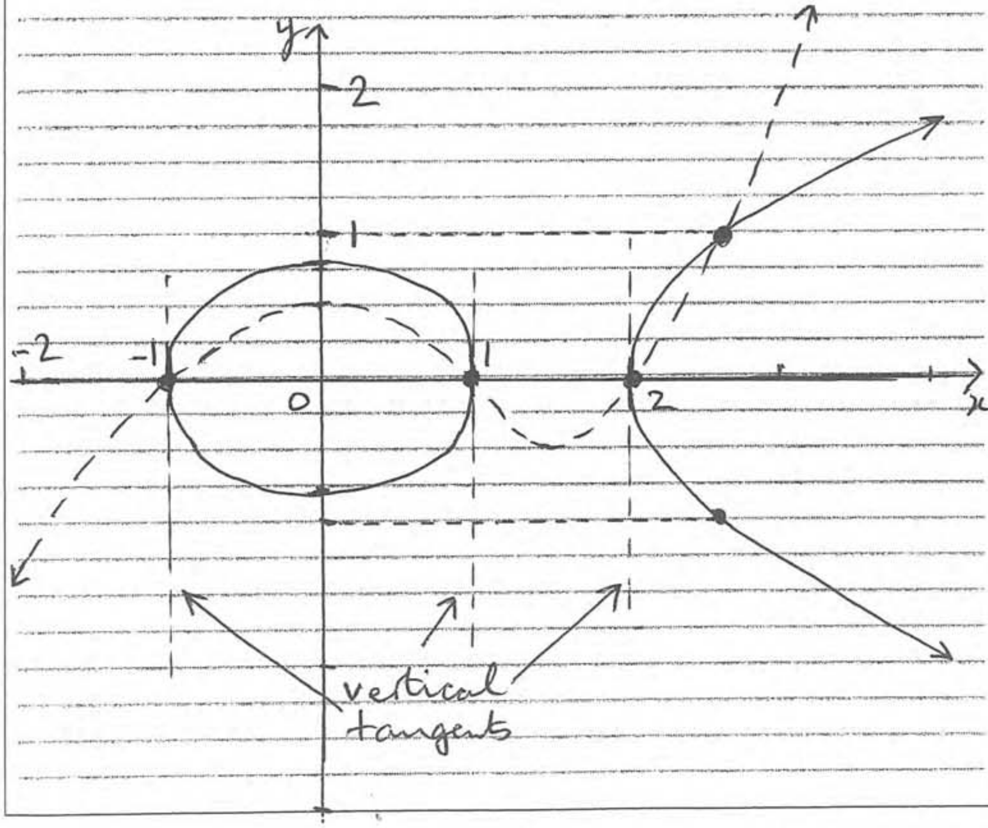
Suggested Solutions	Marks	Marker's Comments
(a) $\int \frac{dx}{\sqrt{x^2+4}} = \ln\left(x+\sqrt{x^2+4}\right)+C$ from integral table	1	1
(b) $\int \frac{\sin(x+\pi/4)}{\sin x}\,dx = \int \frac{\sin x\cos\frac{\pi}{4}+\sin\frac{\pi}{4}\cos x}{\sin x}\,dx$ $=\int\left(\frac{1}{\sqrt2}+\frac{1}{\sqrt2}\frac{\cos x}{\sin x}\right)dx$ $=\frac{1}{\sqrt2}x+\frac{1}{\sqrt2}\ln\lvert\sin x\rvert+C$	2	1 expansion 1 answer.
(c) $\frac{x^2}{6}+\frac{y^2}{4}=1$ $a^2=6 \quad b^2=4 \quad b^2=a^2(1-e^2)$ $e^2=1-\frac{4}{6}$ $\quad a=\sqrt6,\ a>0$ $b=2,\ b>0$ $e^2=\frac{1}{3},\ e=\frac{1}{\sqrt3} \quad 0<e<1$ foci $S(ae,0)=(\sqrt2,0) \quad S'(-ae,0)=(-\sqrt2,0)$ directrices $x=\pm a/e \quad x=\pm3\sqrt2$ 	5	1 directrices 1 foci 1 $\pm a$ 1 $\pm b$ 1 shape/scale/axes.
(d) $ae=5 \quad \frac{a}{e}=7 \quad \therefore a^2=35$ $a=\sqrt{35} \quad a>0$ $\therefore e=\frac{\sqrt5}{\sqrt7} \quad 0<e<1$ $b^2=a^2(1-e^2) \quad b^2=35\left(1-\frac{5}{7}\right)$ $\therefore b^2=10 \quad \frac{x^2}{35}+\frac{y^2}{10}=1$	2	1 e or e^2 must write $a>0$ or $e>0$ if a or e found. 1 answer.
(e) 	2	1 $f(x-8)$ 1 $f(8-x)$

MATHEMATICS Extension 2: Question....1...

Suggested Solutions	Marks	Marker's Comments
(f) (i) $x = \frac{e^{3t} + e^{-3t}}{2}$ $\quad y = \frac{e^{3t} - e^{-3t}}{4}$	(3)	① $\left.\begin{matrix} x^2 \\ y^2 \end{matrix}\right\}$
As it is a conic consider x^2 and y^2.		
$(2x)^2 = e^{6t} + 2 + e^{-6t}$ $\quad (4y)^2 = e^{6t} - 2 + e^{-6t}$		
$4x^2 - 16y^2 = 4$ $\quad \therefore x^2 - 4y^2 = 1$		① formula
$\therefore a^2 = 1$ $\quad b^2 = \frac{1}{4}$		
$\therefore$ hyperbola $\therefore e > 1$		
(ii) $b^2 = a^2(e^2 - 1)$		
$\frac{1}{4} = 1(e^2 - 1)$		
$e^2 = \frac{5}{4}$		① "e" value
$e = \frac{\sqrt{5}}{2}$ $\quad e > 1$		

Suggested Solutions	Marks	Marker's Comments
a) Let $u=1-x$ $du=-dx$ $x=0,\ u=1$; $x=1,\ u=0$		
$\int_0^1 x^3(1-x)^{11}\,dx = -\int_1^0 (1-u)^3 u^{11}\,du$	1	well done some use property $\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$
$= \int_0^1 u^{11}(1-u)^3\,du = \int_0^1 x^{11}(1-x)^3\,dx$ (dummy variable)	1	
b) $(x^2+2-2x)(x^2+2+2x) = (x^2+2)^2-4x^2$ $= x^4+\cancel{4x^2}+4-\cancel{4x^2} = x^4+4$	$\frac{1}{2}$	
$(ax+b)(x^2-2x+2)-(ax+c)(x^2+2x+2)=1$		
Sub $x=0$ $b-c=\frac{1}{2}$ (1) $x=1$ $-4a+b-5c=1$ (2) $x=-1$ $5b-4a-c=1$ (3) Solve (2), (3) simultaneously. $a+c=-\frac{1}{8}$ (4) $c-a=-\frac{3}{8}$ (5)		any revelant equation $\frac{1}{2}$m but if none of the equations are right won't get any mark here
Solve (4), (5) simultaneously $a=\frac{1}{8}$ $c=-\frac{1}{4}$ $\therefore b=\frac{1}{4}$	$\frac{1}{2}+\frac{1}{2}+\frac{1}{2}$	$\frac{1}{2}$m for a, b, c @ if $a=0$, max 1m out of 3 marks
$\int \frac{dx}{x^4+4} = \int \frac{\frac{1}{8}x+\frac{1}{4}\,dx}{x^2+2x+2} - \int \frac{\frac{1}{8}x-\frac{1}{4}\,dx}{x^2-2x+2}$	1	most student got this
$= \frac{1}{16}\int \frac{2x+2}{x^2+2x+2} + \frac{1}{8}\int \frac{dx}{(x+1)^2+1}$ $-\frac{1}{16}\int \frac{2x-2}{x^2-2x+2} + \frac{1}{8}\int \frac{dx}{(x-1)^2+1}$	1	

Suggested Solutions	Marks	Marker's Comments
$= \frac{1}{16} \ln\left(\frac{x^2+2x+2}{x^2-2x+2}\right) + \frac{1}{8}\tan^{-1}(x+1) + \frac{1}{8}\tan^{-1}(x-1) + C$		$1 + \frac{1}{2} + \frac{1}{2}$
c) $y = 2^{\frac{1}{x}}$ 		$\frac{1}{2}$ m for each Asymptotes $x=0$, $y=1$ Pts $(1,2)$ $(-1,\frac{1}{2})$ Right branch / Left branch goes to asymptotes 1 m for POI Very few students get POI. many forgot to number their scales on the axes POI $\doteqdot (-0.3466, 0.1353)$
d) $\int\sqrt{\frac{4+x}{3-x}}\,dx = \int\frac{4+x\,dx}{\sqrt{(4+x)(3-x)}}$ $= \int\frac{4+x\,dx}{\sqrt{12-x-x^2}}$	1 m	
$= -\frac{1}{2}\int\frac{-2x-1\,dx}{\sqrt{12-x-x^2}} + \frac{7}{2}\int\frac{dx}{\sqrt{\frac{49}{4}-(x+\frac{1}{2})^2}}$	$\frac{1}{2}+\frac{1}{2}$	must show $-\frac{1}{2}\int\frac{-2x-1\,dx}{\sqrt{12-x-x^2}}$
$= -\frac{2}{2}\sqrt{12-x-x^2} + \frac{7}{2}\sin^{-1}\left(\frac{2x+1}{7}\right) + c$ $= -\sqrt{12-x-x^2} + \frac{7}{2}\sin^{-1}\left(\frac{2x+1}{7}\right) + c$	$\frac{1}{2}+\frac{1}{2}$	(many fudging answers)

Suggested Solutions	Marks	Marker's Comments
a) $I = \int e^{\sqrt{x}}\,dx$ Let $u^2 = x$ $(u>0)$		Correct substitution
"$2u\,du = dx$"	1	
$= \int 2ue^u\,du$	1	Get to this form
$= 2ue^u - \int 2e^u\,du$ (By Parts)		
$= 2ue^u - 2e^u + k$	1	
$= 2(\sqrt{x}-1)e^{\sqrt{x}} + k$	1	Substitute back (½ off for no 'k')
b) Draw $y=\sqrt{f(x)}$ and reflect in $y=0$ for $y^2=f(x)$	1	Symmetry in $y=0$
Domain $f(x)\geq 0$ $\therefore \{-1\leq x\leq 1\}\cup\{x: x\geq 2\}$	½	Implied domain
Invariant points $f(x)=\sqrt{f(x)} \Rightarrow f(x)=0$ or 1	1	Invariant points
$\frac{dy}{dx} = \frac{f'(x)}{2\sqrt{f(x)}}$ $\Rightarrow 0$ where $f'(x)=0$, ∞ where $f(x)=0$	½	for same max/min but larger absolute values
	1	verticals at $x=2$ and $x=\pm1$. Vertical curve either had to be very obvious or specifically mentioned.

Suggested Solutions	Marks	Marker's Comments
c) i) $I_n = \int_{\pi/6}^{\pi/2} \operatorname{cosec}^2 x \operatorname{cosec}^{n-2} x \, dx$		
$= \left[-\cot x \operatorname{cosec}^{n-2} x\right]_{\pi/6}^{\pi/2} - (n-2)\int_{\pi/6}^{\pi/2} (-\cot x \operatorname{cosec}^{n-3} x)(-\cot x \operatorname{cosec} x)\, dx$	1	(Integration by parts)
$= -0 - \left(-\sqrt{3}.2^{n-2}\right) - (n-2)\int_{\pi/6}^{\pi/2} \cot^2 x \operatorname{cosec}^{n-2} x \, dx$		
$= \sqrt{3}\ 2^{n-2} - (n-2)\int_{\pi/6}^{\pi/2} (\operatorname{cosec}^2 x - 1)\operatorname{cosec}^{n-2} x \, dx$	½	numerical substitution
$= \sqrt{3}\ 2^{n-2} - (n-2)\int_{\pi/6}^{\pi/2} \operatorname{cosec}^n x - \operatorname{cosec}^{n-2} x \, dx$	1	use of $\cot^2 x = \operatorname{cosec}^2 x - 1$
$I_n = \sqrt{3}\ 2^{n-2} - (n-2) I_n + (n-2) I_{n-2}$	½	Revert to the I_k forms
$\therefore (n-1) I_n = \sqrt{3}\ 2^{n-2} + (n-2) I_{n-2}$		
$I_n = \dfrac{\sqrt{3}\ 2^{n-2}}{(n-1)} + \left(\dfrac{n-2}{n-1}\right) I_{n-2}$	1	Rearrangement to final form [Much fudging of signs thro' the integrals]
ii) $I_3 = \dfrac{\sqrt{3} \times 2^1}{2} + \dfrac{1}{2} I_1$		
$I_3 = \sqrt{3} + \dfrac{I_1}{2}$	1	Simple formula
But $I_1 = \int_{\pi/6}^{\pi/2} \operatorname{cosec} x \, dx$		
$= \left[-\ln(\cot x + \operatorname{cosec} x)\right]_{\pi/6}^{\pi/2}$	1	Correct integral
$= -\ln(1) + \ln(2+\sqrt{3})$	½	Correct substitution
$\therefore I_3 = \sqrt{3} + \dfrac{1}{2}\ln(2+\sqrt{3})$	½	

Solution	Marks	Comments
Question 4		
For tangent at P		
$\frac{2x}{a^2} - \frac{2y}{b^2} \cdot \frac{dy}{dx} = 0$		
$\therefore \frac{dy}{dx} = \frac{b^2 x}{a^2 y}$		
$\therefore$ Gradient of Tangent at P = $\frac{b^2 x_1}{a^2 y_1}$	1	
Tangent at P y - y_1 = $\frac{b^2 x_1}{a^2 y_1}$ (x - x_1)		
$a^2 y_1$ y - $a^2 y_1\ y_1$ = $b^2 x_1$ x - $b^2 x_1\ x_1$ $(\div a^2\ b^2)$		
$\frac{x x_1}{a^2} - \frac{y y_1}{b^2} = \frac{{x_1}^2}{a^2} - \frac{{y_1}^2}{b^2}$		
Since P(x_1, y_1) lies on hyperbola	$\frac{1}{2}$	
$\frac{{x_1}^2}{a^2} - \frac{{y_1}^2}{b^2}$ =1		**Some students failed to explain where the = 1 came from. On the whole well done**
$\therefore\ \frac{x x_1}{a^2} - \frac{y y_1}{b^2} = 1$ equation of tangent	$\frac{1}{2}$	
(ii) Coordinates of N $(\frac{a^2}{x_1}, 0)$		
NS = $ae - \frac{a^2}{x_1}$		
= $\frac{a}{x_1}(ex_1 - a)$	1	
NS' = $ae + \frac{a^2}{x_1}$		
= $\frac{a}{x_1}(ex_1 + a)$		
By definition of hyperbola		
$\therefore PS = ePM \quad \therefore PS' = ePM'$		
PS= $e\ (x_1 - \frac{a}{e})$	1	
= $e\,x_1$ - a		

PS'= $e\ (x_1 + \frac{a}{e})$ $= e\ x_1 + a$ $\frac{PS}{NS} = \frac{e\ x_1 - a}{\frac{a}{x_1}(ex_1 - a)}$ $= \frac{x_1}{a}$ $\frac{PS'}{NS'} = \frac{e\ x_1 + a}{\frac{a}{x_1}(ex_1 + a)}$ $= \frac{x_1}{a}$ $\therefore \frac{PS}{NS} = \frac{PS'}{NS'}$ ∠SPN = ∠S′PN (Angle Bisection Theorem) ∴ The Tangent at P bisects the angle S′PS	1 1	**This may be done differently and equivalent marks given** **Some students proved the angle bisection theorem not what was asked.** **Those who did not use PS=ePM ended up on most occasions with a messy algebraic expression**
(iii) ∠SPN = ∠S′PN (from part (i)) ∠S′PN = ∠QPR (given) S'PR is a straight line (vertically opposite angles are equal) ∴ Pr appears to emanate from S'	1 1	**A number students assumed straight line and stated angles were vertically opposite**
(b) t = $\tan\frac{x}{2} \rightarrow \frac{dt}{dx} = \frac{1}{2}\sec^2\frac{x}{2}$ $\frac{dt}{dx} = \frac{1}{2}(1+\tan^2\frac{x}{2})$ $= \frac{1+t^2}{2}$ $dx = \frac{2}{1+t^2}dt$ When x=$\frac{\pi}{2}$ $t = 1$ x=$\frac{-\pi}{2}$ $t = -1$	1	

$\int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} \frac{dx}{4+3\cos x} = \int_{-1}^{1} \frac{\frac{2}{1+t^2}dt}{4+3\frac{1-t^2}{1+t^2}}$ $= \int_{-1}^{1} \frac{2dt}{4+4t^2+3-3t^2}$ $= \int_{-1}^{1} \frac{2dt}{t^2+7}$ $= \left[\frac{2}{\sqrt{7}}(\tan^{-1}\frac{t}{\sqrt{7}})\right]_{-1}^{1}$ $= \frac{2}{\sqrt{7}}(\tan^{-1}\frac{1}{\sqrt{7}}) - \frac{2}{\sqrt{7}}(\tan^{-1}\frac{-1}{\sqrt{7}})$ $= \frac{2}{\sqrt{7}}(\tan^{-1}\frac{1}{\sqrt{7}}) + \frac{2}{\sqrt{7}}(\tan^{-1}\frac{1}{\sqrt{7}})$ $= \frac{4}{\sqrt{7}}(\tan^{-1}\frac{1}{\sqrt{7}})$	1 $\frac{1}{2}$ $\frac{1}{2}$	**Must have this step or earlier stated** $\frac{dx}{4+3\cos x}$ was even function
(ii) $\int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} \frac{\cos 2x dx}{4+3\cos x} = \int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} \frac{2\cos^2 x - 1 dx}{4+3\cos x}$ Using part (i) $= \int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} \frac{2\cos^2 x dx}{4+3\cos x} - \frac{4}{\sqrt{7}}(\tan^{-1}\frac{1}{\sqrt{7}})$	1	$\frac{1}{2}$ if just did first step **The use of integration by parts was unsuccessful for this question**

Divide through $4+3\cos x \overline{\big)\, 2\cos^2 x}$ with quotient $\frac{2}{3}\cos x - \frac{8}{9}$ $-\left(2\cos^2 x + \frac{8}{3}\cos x\right)$ $-\frac{8}{3}\cos x$ $-\frac{8}{3}\cos x - \frac{32}{9}$ $\frac{32}{9}$	1	**Very few got this result many students ran out of time or used unsuccessful methods**
$\int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} \frac{2\cos^2 x - 1dx}{4+3\cos x} = \int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} \frac{2}{3}\cos x - \frac{8}{9} + \frac{\frac{32}{9}-1}{4+3\cos x}dx$	1	
$=\left[\frac{2}{3}\sin x - \frac{8}{9}x\right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} + \frac{23}{9} X \frac{4}{\sqrt{7}}(\tan^{-1}\frac{1}{\sqrt{7}})$ $=\frac{4}{3} - \frac{8\pi}{9} + \frac{92}{9\sqrt{7}}(\tan^{-1}\frac{1}{\sqrt{7}})$	1	