Class:

JAMES RUSE AHS

GESTA NON VERBA

Term 1 Task 2 2014

MATHEMATICS EXTENSION 2

General Instructions:

- Reading Time: 5 minutes.
- Working Time: 2 hours.
- Write in black pen.
- Board approved calculators & templates may be used
- A Standard Integral Sheet is provided.
- In Question 6 - 9, show all relevant mathematical reasoning and/or calculations.
- Marks may not be awarded for careless or badly arranged working.

Total Marks 85

Section I: 5 marks

- Attempt Question 1 – 5.
- Answer on the Multiple Choice answer sheet provided.
- Allow about 8 minutes for this section.

Section II: 80 Marks

- Attempt Question 6 - 9
- Answer on paper provided unless otherwise instructed. Start a new page for each new question.
- Allow about 1 hours & 42 minutes for this section.

The answers to all questions are to be returned in separate *stapled* bundles clearly labelled Question 6, Question 7, etc. Each question must show your Candidate Number.

For the following questions colour the most correct answer on your multiple choice answer sheet

1 Which of the following are the foci for the ellipse

$$\frac{x^2}{9}+\frac{y^2}{16}=1$$

(A) $(0,\pm\frac{3\sqrt{7}}{4})$

(B) $(\pm\sqrt{7},0)$

(C) $(0,\pm\sqrt{7})$

(D) $(\pm\frac{3\sqrt{7}}{4},0)$

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2 Using implicit differentiation on $y^3 = x^2 + xy$ then $\frac{dy}{dx} =$

(A) $\frac{3y^2-2x}{x}$

(B) $\frac{2x+y}{3y^2-x}$

(C) $\frac{2x}{3y^2+y}$

(D) $\frac{2x-y}{3y^2+y}$

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3 Using a suitable substitution what is the correct expression for $\int_0^{\frac{\pi}{3}} \sin^3 x\cos^4 x\,dx$

(A) $\int_0^{\frac{\sqrt{3}}{2}} (u^4-u^6)du$

(B) $\int_1^{\frac{1}{2}} (u^6-u^4)du$

(C) $\int_{\frac{1}{2}}^{1} (u^6-u^4)du$

(D) $\int_0^{\frac{\sqrt{3}}{2}} (u^6-u^4)du$

4 The points P$(a\cos\theta, b\sin\theta)$ and Q $(a\cos\phi, b\sin\phi)$ lie on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ and the chord PQ subtends a right angle at (0,0). Which of the following is the correct expression?

(A) $\tan\theta\tan\phi=-\frac{b^2}{a^2}$

(B) $\tan\theta\tan\phi=\frac{a^2}{b^2}$

(C) $\tan\theta\tan\phi=\frac{b^2}{a^2}$

(D) $\tan\theta\tan\phi=-\frac{a^2}{b^2}$

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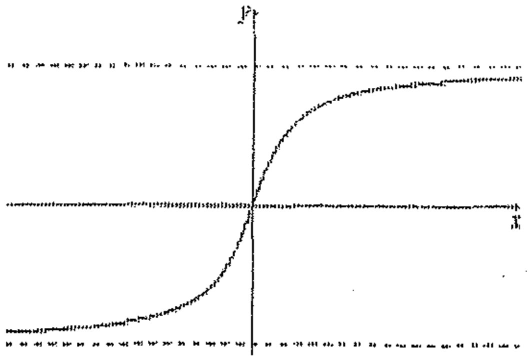
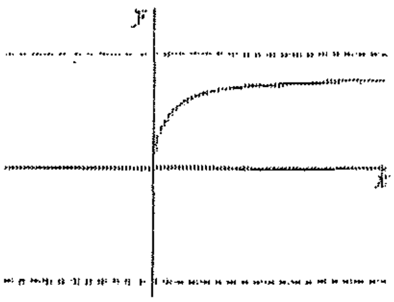
5 The graph of $y=f(x)$ is shown below

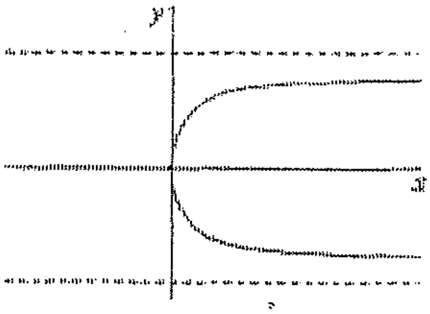


Which of the following best represents $y^2=f(x)$

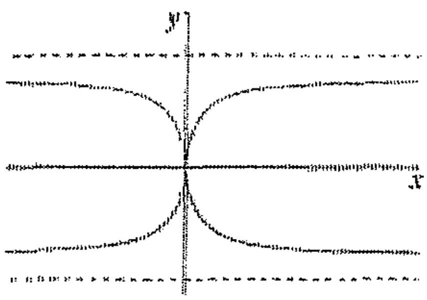
(A)



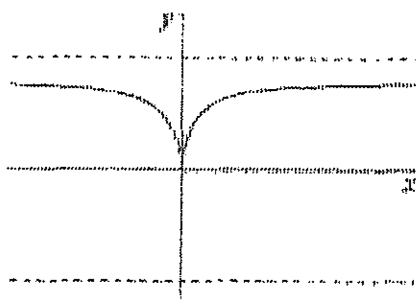
(B)



(C)



(D)



QUESTION 6 (20 Marks) **MARKS**

(a) Find $\int \frac{dx}{\sqrt{6x - x^2}}$ **2**

(b) Find $\int \frac{1-x}{1-\sqrt{x}}dx$ **2**

(c) (i) Show that $\int_0^1 \frac{5-5x^2}{(1+2x)(1+x^2)}dx = \frac{1}{2}(\pi + \ln\frac{27}{16})$ **4**

(ii) Hence find $\int_0^{\frac{\pi}{2}} \frac{\cos x}{(1+2\sin x + \cos x)}dx$ using the substitution $t = \tan\frac{x}{2}$ **3**

(d) Find $\int_1^3 x^2 \ln x \, dx$ **4**

(e) (i) On the same set of axes sketch and label $y = x^{\frac{1}{3}}$ and $y = e^{-x}$ **2**

(ii) Hence, on a different set of axes, without using calculus, sketch and label clearly the graph of the function $y = x^{\frac{1}{3}}e^{-x}$ **2**

(iii) Use your sketch to determine the values of m for which the equation $x^{\frac{1}{3}}e^{-x} = mx - 1$ has exactly one solution **1**

QUESTION 7 (20 Marks) Start a new page **MARKS**

(a) Let P be the point (x_1, y_1) on the ellipse E: $\frac{x^2}{4}+\frac{y^2}{3}=1$ with $x_1>0$.

(i) Find the eccentricity of the ellipse. **1**

(ii) Prove that the equation of the tangent at P is $\frac{xx_1}{4}+\frac{yy_1}{3}=1$. **4**

(iii) Find the coordinates of the point T where the tangent meets the nearest directrix **2**

(iv) Prove that the segment of the tangent to the ellipse E between the point of contact and the directrix subtends a right angle to the corresponding focus. **3**

(v) Write down the equation of the chord of contact PQ from $R(x_0, y_0)$ to E **1**

(vi) Show that if PQ passes through the focus then R lies on the directrix . **2**

(b) Show that if $y = px+q$ is a tangent to the hyperbola

$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ then $p^2a^2-b^2=q^2$ **3**

(c) Consider the complex number $z = x+iy$ where $z^2 = a+ib$

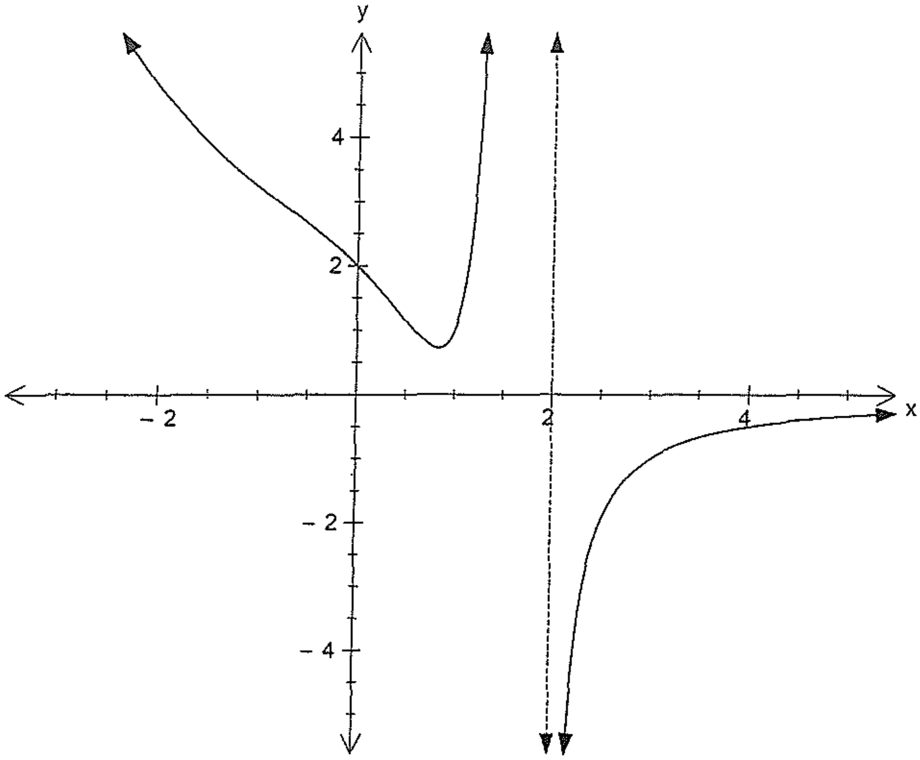
(i) Sketch on the same set of axes the graphs of x^2-y^2=a and 2xy=b where both a and b are positive
The foci and directrices of the curves need not be found **2**

(ii) Use the graphs to explain why there are two distinct square roots of the complex number a+ib if a>0,b>0 **1**

(iii) Consider how the sketch changes when b is negative . What is the relationship between the new square roots and those found when b is positive? **1**

QUESTION 8 (20 Marks) Start a new page **MARKS**

(a) The diagram below shows the graph of $y = f(x)$



Draw separate sketches of the following (indicate important features)

(i) $y = \dfrac{1}{0.5 - f(x)}$ **3**

(ii) $y = [f(x)]^2$ **2**

(iii) $y = f(|x|)$ **2**

(iv) $y = \int_{-2}^{x} f(u)du$ for $-2 \le x \le 1$ **2**

(b) (i) Show that $(1-\sqrt{x})^{n-1}\sqrt{x} = (1-\sqrt{x})^{n-1} - (1-\sqrt{x})^n$ **2**

(ii) If $I_n = \int_0^1 (1-\sqrt{x})^n \, dx$ for $n \ge 0$ show that $I_n = \dfrac{n}{n+2} I_{n-1}$ for $n \ge 1$ **3**

(iii) Hence find I_3 **2**

QUESTION 8 (cont)

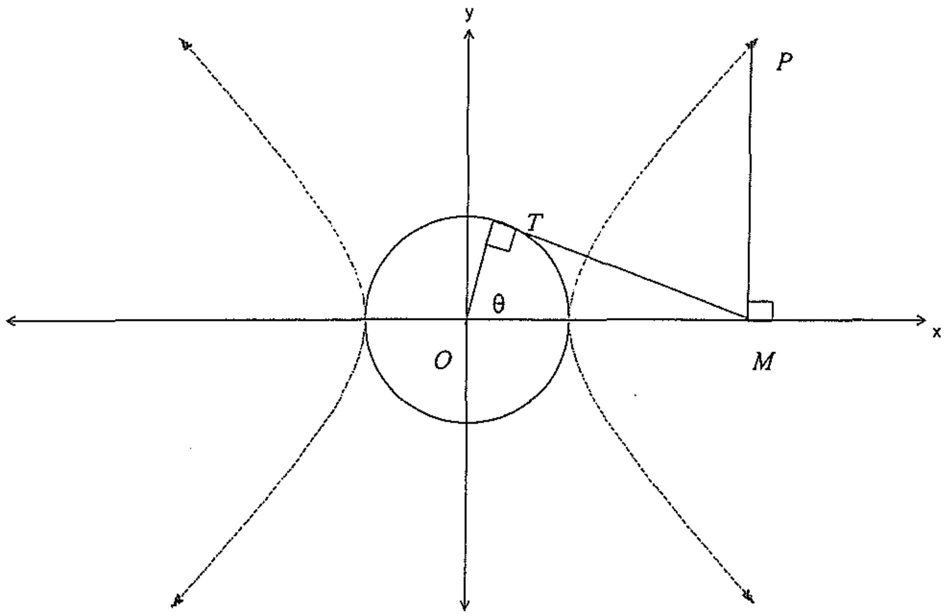
(c) Find $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^5\theta \, d\theta$ **1**

(d) Prove that for all complex numbers z and w

(i) $|z+w|^2 + |z-w|^2 = 2(|z|^2 + |w|^2)$ **2**

(ii) Give a geometrical interpretation of this equation in the complex plane **1**

QUESTION 9 (20 Marks) Start a new page MARKS



(a) The sketch shows the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ and the circle $x^2 + y^2 = a^2$ with $a, b \geq 0$. T lies on the circle where $\angle TOX = \theta$ and $0 \leq \theta \leq \frac{\pi}{2}$. The tangent at T meets OX at M and MP is perpendicular to OX with P on the hyperbola.

i) Find the equation of the tangent TM and hence the coordinates of M. **3**

ii) Hence show that the coordinates of P are $(a\sec\theta, b\tan\theta)$ **1**

QUESTION 9 (cont)

iii) Assume $\theta \neq \frac{\pi}{4}$, if $Q(a\sec\beta, b\tan\beta)$ is another point on the hyperbola, where $\theta + \beta = \frac{\pi}{2}$ show that the equation of PQ is $ay = b(\cos\theta + \sin\theta)x - ab$. **3**

iv) Every such chord PQ passes through a fixed point. Find the coordinates of this fixed point.. **1**

v) Show that as θ approaches $\frac{\pi}{2}$ the chord PQ approaches a line parallel to an asymptote of the hyperbola **2**

(b) (i) Consider $f(x)=\frac{1}{1+\tan x}$ where $0 \le x \le \frac{\pi}{2}$ and $f(\frac{\pi}{2})=0$ **2**

Show that $f(x) + f(\frac{\pi}{2}-x) = 1$

~~(iii)~~ (ii) Hence or otherwise evaluate $\int_0^{\frac{\pi}{2}} \frac{1}{1+\tan x}dx$ **3**

(c) Using the fact that $\tan^{-1}x + \tan^{-1}y = \tan^{-1}\left(\frac{x+y}{1-xy}\right)$ for $0 \le x \le 1$ and $0 \le y \le 1$. Prove by mathematical induction that for all positive integers n **5**

$$\tan^{-1}\left(\frac{1}{2\times 1^2}\right) + \tan^{-1}\left(\frac{1}{2\times 2^2}\right) + \ldots\ldots + \tan^{-1}\left(\frac{1}{2\times n^2}\right) = \frac{\pi}{4} - \tan^{-1}\left(\frac{1}{2n+1}\right)$$

END OF EXAM

MATHEMATICS Extension 2: Question 6

Suggested Solutions	Marks	Marker's Comments
(a) $\int \frac{dx}{\sqrt{9-(x-3)^2}}$	1	Multiple Choice Answers Q1 C Q2 B Q3 B Q4 D Q5 B
$= \sin^{-1}\left(\frac{x-3}{3}\right) + C$	1	
(b) $\int \frac{1-x}{1-\sqrt{x}}\,dx$		
$= \int \frac{(1-\sqrt{x})(1+\sqrt{x})}{1-\sqrt{x}}\,dx$		
$= \int (1+\sqrt{x})\,dx$	1	
$= x + \frac{2}{3}x^{3/2} + C$	1	
(c)(i) $\int_0^1 \frac{5-5x^2}{(1+2x)(1+x^2)}\,dx$		
$= \int_0^1 \left(\frac{a}{1+2x} + \frac{bx+c}{1+x^2}\right)dx$		
$\therefore 5-5x^2 = a(1+x^2) + (bx+c)(1+2x)$	1	
put in $x=0$: $5 = a + c$		
put in $x=-\frac{1}{2}$; $5-\frac{5}{4} = a\left(\frac{5}{4}\right)$		
$a = 3$	1	
$\therefore c = 2$		
put in $x=1$; $0 = 6+6+3b$		
$b = -4$		
$\therefore I = \int_0^1 \left(\frac{3}{1+2x} - \frac{4x+2}{1+x^2}\right)dx$		
$= \int_0^1 \left(\frac{3}{1+2x} - \frac{4x}{1+x^2} + \frac{2}{1+x^2}\right)dx$	1	
$= \left[\frac{3}{2}\ln(1+2x) - 2\ln(1+x^2) + 2\tan^{-1}x\right]_0^1$		
$= \left(\frac{3}{2}\ln 3 + 2\tan^{-1}1\right) - \left(\frac{3}{2}\ln 1 + 2\tan^{-1}0\right) - (2\ln 2)$		
$= \frac{3}{2}\ln 3 + \frac{\pi}{2} - 2\ln 2$	1	
$= \frac{1}{2}(\ln 27 + \pi - \ln 4^2) = \frac{1}{2}\left(\ln\frac{27}{16} + \pi\right)$		

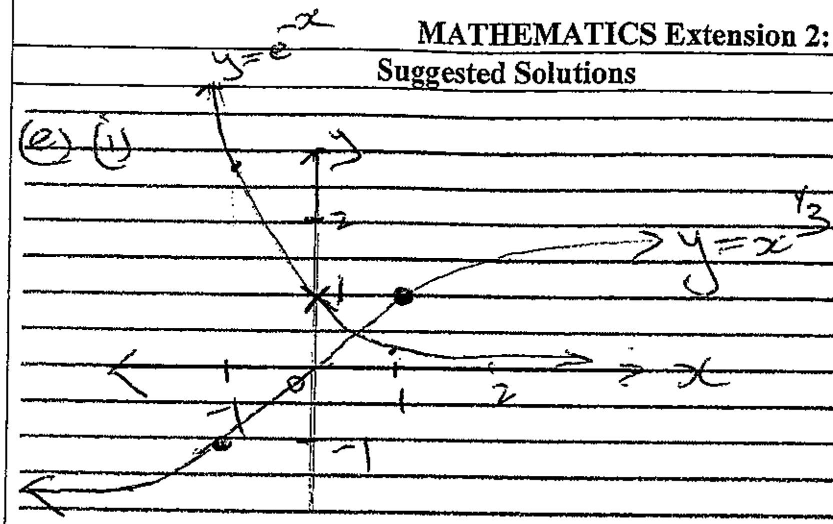
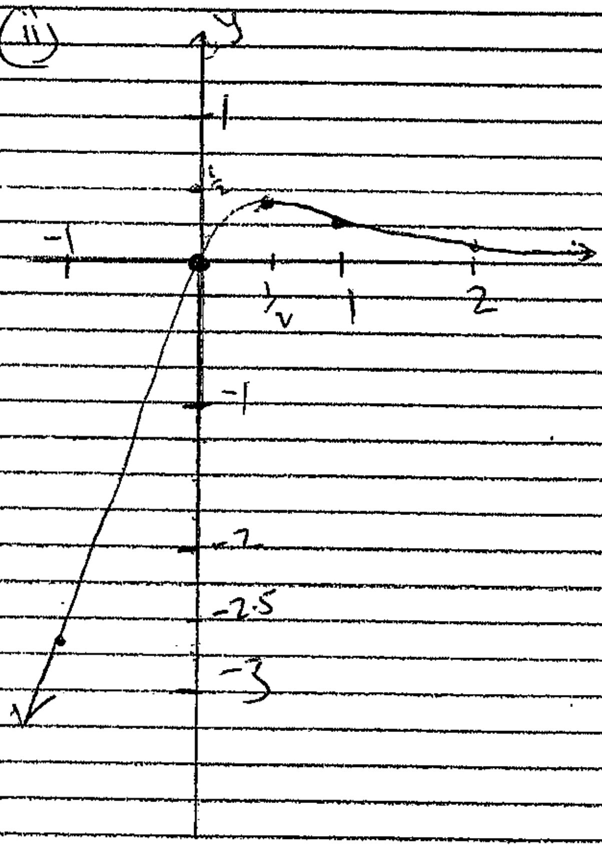
2/3

MATHEMATICS Extension 2: Question..6...

Suggested Solutions	Marks	Marker's Comment
(c)(ii) let $t = \tan \frac{x}{2}$ when $x=0$ $t=0$ when $x=\frac{\pi}{2}$ $t=1$		
$\therefore I = \int_0^1 \frac{\frac{1-t^2}{1+t^2}}{1+\frac{4t}{1+t^2}+\frac{1-t^2}{1+t^2}} \times \frac{2dt}{1+t^2}$	1	
$= 2\int_0^1 \frac{1-t^2}{1+t^2+4t+1-t^2} \frac{dt}{(1+t^2)}$		
$= 2\int_0^1 \frac{1-t^2}{(2+4t)(1+t^2)} dt$	1	
$= \int_0^1 \frac{1-t^2}{(1+2t)(1+t^2)} dt$		
$= \frac{1}{5} \times \frac{1}{2}\left(\pi + \ln \frac{27}{16}\right)$		
$= \frac{1}{10}\left(\pi + \ln\frac{27}{16}\right)$	1	
(d) $u = \ln x$ $\frac{du}{dx} = \frac{1}{x}$ $\frac{dv}{dx} = x^2$ $v = \frac{1}{3}x^3$		
$\int_1^3 x^2 \ln x\, dx = \left[\frac{1}{3}x^3 \ln x\right]_1^3 - \frac{1}{3}\int_1^3 x^2\, dx$	1	
$= (9\ln 3 - 0) - \frac{1}{3}\left[\frac{1}{3}x^3\right]_1^3$	1	
$= 9\ln 3 - \frac{1}{9}(27-1)$	1	
$= 9\ln 3 - 26/9$	1	

3/3

MATHEMATICS Extension 2: Question……..

Suggested Solutions	Marks	Marker's Comments
(e) (i)		*common error was students drew $y=x^3$ instead of $y=x^{\frac{1}{3}}$
		*1 for each graph
(ii) 		*no scale, lost a mark *poorly done. ½ for scale ½ for horizontal asymptote for $x>0$ ½ for shape ½ for when $x<0$
(iii) $m \leq 0$	1	$m<0 \Rightarrow \frac{1}{2}$ mk $m=0 \Rightarrow 0$ mk

a i) $e = \frac{1}{2}$ — 1m well done

ii) $\frac{dy}{dx} = \frac{-3x}{4y}$ — 1m

Eq of tangent $y - y_1 = \frac{-3x_1}{4y_1}(x - x_1)$ — 1m

$4yy_1 - 4y_1^2 = -3xx_1 + 3x_1^2$

$\frac{xx_1}{4} + \frac{yy_1}{3} = \frac{x_1^2}{4} + \frac{y_1^2}{3}$ — 1m

Since $P(x_1, y_1)$ lies on ellipse $\frac{x_1^2}{4} + \frac{y_1^2}{3} = 1$ — 1m mention $P(x_1, y_1)$ lies on ellipse

$\therefore$ Eq of tangent at P: $\frac{xx_1}{4} + \frac{yy_1}{3} = 1$ #

iii) Directrix $x = \frac{a}{e} = \frac{2}{\frac{1}{2}} = 4$

Sub $x=4$ into $\frac{xx_1}{4} + \frac{yy_1}{3} = 1$

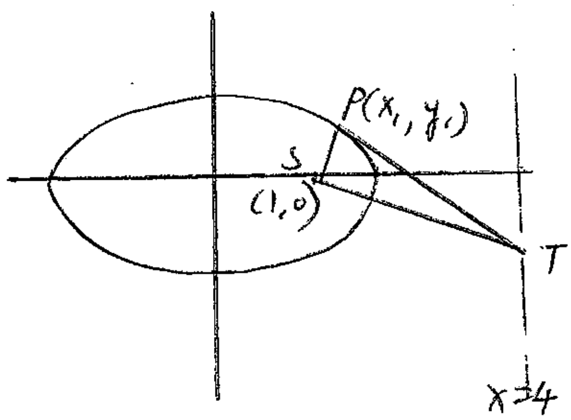
$x_1 + \frac{yy_1}{3} = 1$

$y = \frac{3(1-x_1)}{y_1}$ — 1m + 1m well done

$\therefore T = \left(4, \frac{3(1-x_1)}{y_1}\right)$

or $T = \left(-4, \frac{3(1+x_1)}{y_1}\right)$ optional

iv)



$m(ST) = \frac{3(1-x_1)}{y_1} - 0 \Big/ 4-1$

$m(ST) = \frac{(1-x_1)}{y_1}$

$S = (1, 0)$

$m(ST) = \frac{1-x_1}{y_1}$

$m(PS) = \frac{y_1}{x_1 - 1}$

any 2 correct get 1m

All 3 correct get 2m

iv) $m(PS) = \frac{y_1 - 0}{x_1 - 1} = \frac{y_1}{x_1 - 1}$

$m(ST) \times m(PS) = \frac{1 - x_1}{y_1} \times \frac{y_1}{x_1 - 1}$ — 1 m

$= -1$

$\therefore PS \perp ST$ #

v) Eq of chord of contact from $R(x_0, y_0)$ is $\frac{x x_0}{4} + \frac{y y_0}{3} = 1$ # — 1 m — well done

vi) $S(ae, 0) = (1, 0)$

$\frac{x_0 \cdot 1}{4} + \frac{y_0 \cdot 0}{3} = 1$

$\therefore x_0 = 4$ — easy 2 m.

$\therefore R$ lies on directrix #

b) $\frac{x^2}{a^2} - \frac{(px+q)^2}{b^2} = 1$

$x^2(b^2 - a^2p^2) - 2(a^2pq)x - (a^2q^2 + a^2b^2) = 0$ — 1 m

For tangent, $\Delta = 0$

$[-2(a^2pq)]^2 - 4(b^2 - a^2p^2)(a^2q^2 + a^2b^2) = 0$ — 1 m

$4a^4p^2q^2 + 4a^2(b^2 - a^2p^2)(q^2 + b^2) = 0$

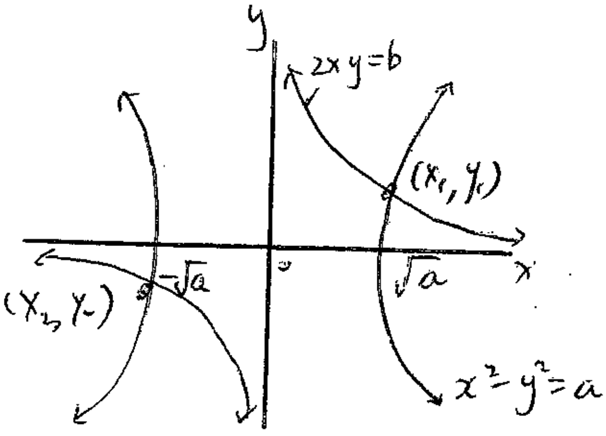
$\div 4a^2 \quad \cancel{a^2p^2q^2} + b^2q^2 - \cancel{a^2p^2q^2} + b^4 - a^2b^2p^2 = 0$ — MANY fudge.

$b^2q^2 + b^4 - a^2b^2p^2 = 0$

$\div b^2 \quad q^2 + b^2 - a^2p^2 = 0$ — 1 m.

$\therefore q^2 = p^2a^2 - b^2$ #

c) i)



1m for each graph

must show x intercepts $\sqrt{a}$, $-\sqrt{a}$

many wrote a, $-a$ instead (−1m)

ii) $z = x+iy$

$z^2 = x^2-y^2+2ixy = a+ib$

$\therefore x^2-y^2=a$

$2xy = b$

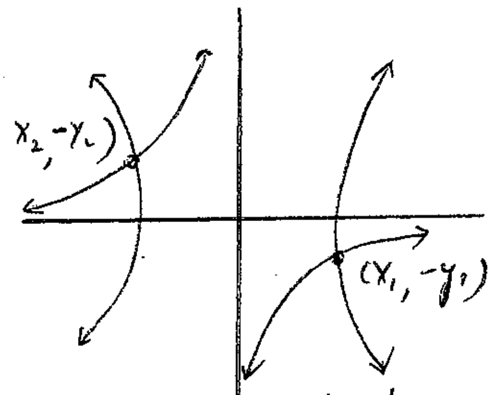
must show the algebraic solutions 1m

From graph in (i), there are 2 distinct points of intersection

$\therefore$ 2 distinct square roots of $z = a+ib$

i.e. (x_1,y_1) & (x_2,y_2)

iii)



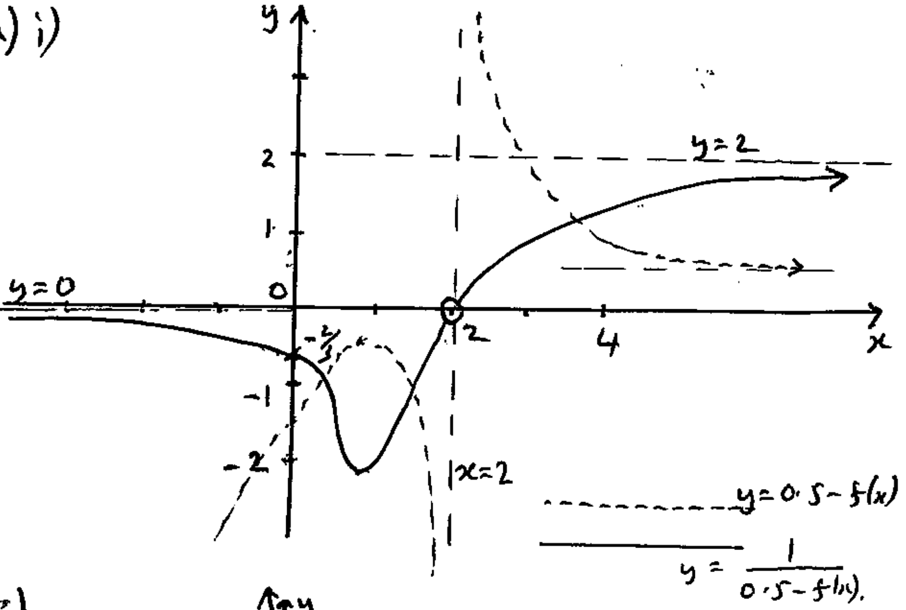
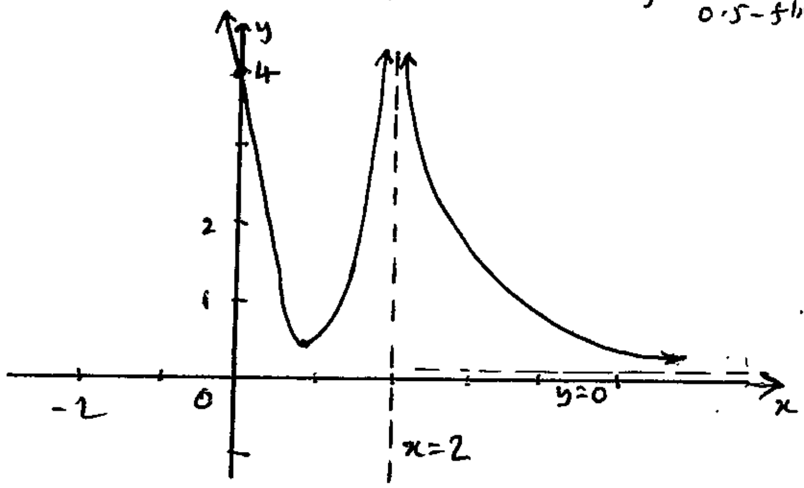
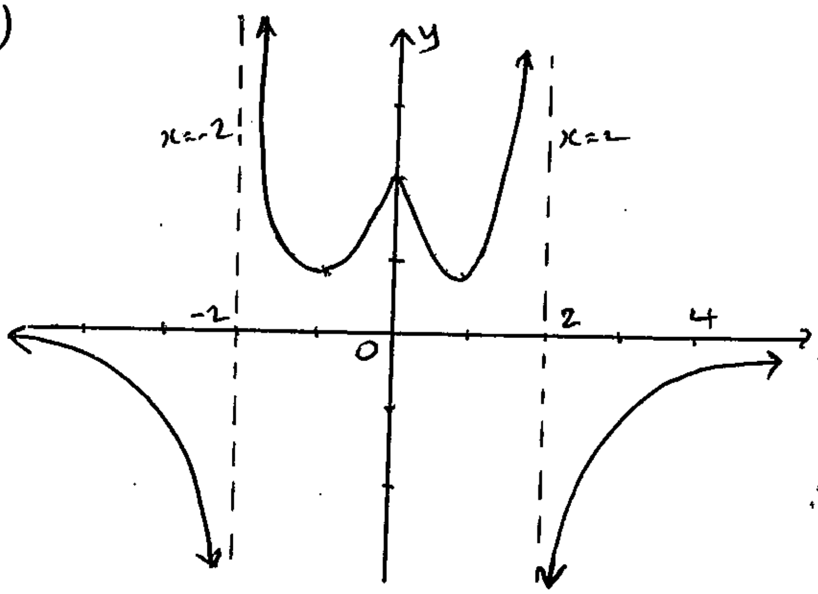
must mention conjugates 1m

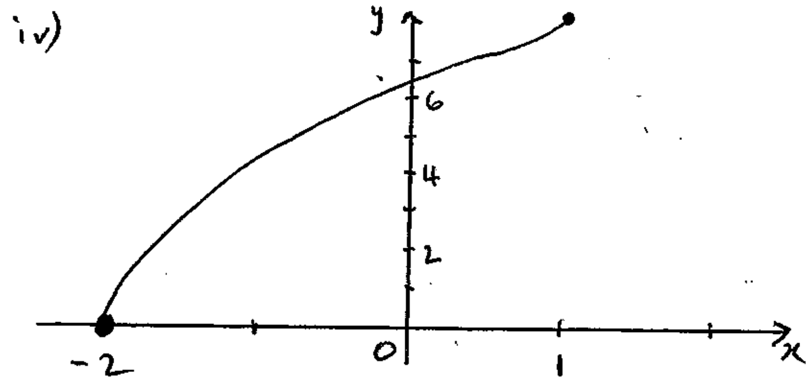
For $b<0$, 2 distinct square roots of $z=a+ib$ will be

$(x_1,-y_1)$, & $(x_2,-y_2)$

$\therefore$ conjugates of the solns in part ii.

X2 MATHEMATICS: Question 8

Suggested Solutions	Marks	Marker's Comments
a) i) 	1 ½ ½ ½ ½	Asymptote y=2 Asymptote y=0 Intercept at $(0,-\frac{2}{3})$ Hole at (2,0) Minimum near (1,−2)
ii) 	1 1	Intercept at (0,4) Minimum near (1, ½) (Some ½ given here) Mark deducted if right hand section wrong.
iii) 	1 1	for symmetry in x=0 for making the join "pointy". (Some ½ marks for indeterminate slush at x=0)

Suggested Solutions	Marks	Marker's Comments
iv) 	1 1	for starting at $(-2,0)$ for monotonic increasing. The intercept was indeterminate but would be somewhere near 7.
b) i) $(1-\sqrt{x})^{n-1}\sqrt{x} = (1-\sqrt{x})^{n-1}(1+\sqrt{x}-1)$ $= (1-\sqrt{x})^{n-1} + (1-\sqrt{x})^{n-1}(\sqrt{x}-1)$ $= \underline{\underline{(1-\sqrt{x})^{n-1} - (1-\sqrt{x})^{n}}}$	2	Most people got two marks here. Need to be careful with given result
ii) $I_n = \int_0^1 1.(1-\sqrt{x})^n\,dx$ $= \left[x(1-\sqrt{x})^n\right]_0^1 - \int_0^1 x\,n(1-\sqrt{x})^{n-1}\left(\frac{-1}{2\sqrt{x}}\right)dx$ (Integration by Parts)	1	for integration by parts successfully completed
$= 0 + \frac{n}{2}\int_0^1 (1-\sqrt{x})^{n-1}\sqrt{x}\,dx$ $= \frac{n}{2}\int_0^1 (1-\sqrt{x})^{n-1} - (1-\sqrt{x})^n\,dx$ (Part (i)) $= \frac{n}{2}\left(I_{n-1} - I_n\right)$	1	for use of (i) after integration. If used before, only ½ given for that by itself.
$\therefore I_n\left(1+\frac{n}{2}\right) = \frac{n}{2}I_{n-1}$ $\therefore \quad \underline{\underline{I_n = \frac{n\,I_{n-1}}{n+2}}}$	1	manipulation and answer.

Suggested Solutions	Marks	Marker's Comments
iii) $I_1 = \int_0^1 1-\sqrt{x}\,dx$ $= \left[x - \frac{2x^{3/2}}{3}\right]_0^1 = \left(1-\frac{2}{3}\right) - 0 = \frac{1}{3}.$	1	for $I_1 = \frac{1}{3}$ or $I_0 = 1$
$\therefore I_2 = \frac{2}{4} I_1 = \frac{1}{6}$	1	for manipulation.
$I_3 = \frac{3}{5} I_2 = \frac{3}{5} \times \frac{1}{6} = \frac{1}{10}$		
c) $\int_{-\pi/2}^{\pi/2} \sin^5\theta\, d\theta = 0$ (Integral of an odd function between symmetric limits)	1	No mark if no reason at all. Very few people mentioned symmetric limits
d) i) Using result $\lvert z\rvert^2 = z\bar{z}$ $\lvert z+w\rvert^2 + \lvert z-w\rvert^2 = (z+w)(\overline{z+w}) + (z-w)(\overline{z-w})$ $= (z+w)(\bar{z}+\bar{w}) + (z-w)(\bar{z}-\bar{w})$ $= z\bar{z} + w\bar{z} + z\bar{w} + w\bar{w} + z\bar{z} - z\bar{w} - w\bar{z} + w\bar{w}$ $\leftarrow$ $= 2(z\bar{z} + w\bar{w})$ $= 2(\lvert z\rvert^2 + \lvert w\rvert^2)$	2	In a given result it is important to show all steps. This line (or equivalent) is the critical line. Several people lost a mark for not including it. Many people used $x+iy$ forms.
ii) (These numbers can be represented by the diagonals and sides of a parallelogram.) The sum of the squares of the diagonals in a parallelogram equals the sum of the squares of the four sides.	1	Some ½ marks given for "typos".

MATHEMATICS: Question 9 Ext 2 (1)

Suggested Solutions	Marks	Marker's Comments
Question 9 a) (i) T is $(a\cos\theta, a\sin\theta)$ $x^2+y^2=a^2$ $2x+2y\frac{dy}{dx}=0 \Rightarrow \frac{dy}{dx}=-\frac{x}{y}$ at $(a\cos\theta, a\sin\theta)$, $\frac{dy}{dx}=\frac{-a\cos\theta}{a\sin\theta}=\frac{-\cos\theta}{\sin\theta}$ (1)	(1)	Gradient
$\therefore$ Equation of TM is $y-a\sin\theta=\frac{-\cos\theta}{\sin\theta}(x-a\cos\theta)$ $y\sin\theta-a\sin^2\theta=-x\cos\theta+a\cos^2\theta$ $x\cos\theta+y\sin\theta=a$ since $\sin^2\theta+\cos^2\theta=1$ (1)	(1)	Equation (any form)
When $y=0$, $x=\frac{a}{\cos\theta} \Rightarrow M\ (a\sec\theta, 0)$ (1)	(1)	$x=\frac{a}{\cos\theta}$ $y=0$.
(ii) substitute $x=a\sec\theta$ into $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ $\sec^2\theta-\frac{y^2}{b^2}=1$ $y^2=b^2(\sec^2\theta-1)$ $\quad$ $\sec^2\theta+\tan^2\theta=1$ $y=\pm b\tan\theta$ (1) $\therefore$ P is $(a\sec\theta, b\tan\theta)$	(1)	proof
(iii) $Q\ (a\sec\beta, b\tan\beta)$ but $\beta=\frac{\pi}{2}-\theta$ $\therefore$ Q is $(a\sec(\frac{\pi}{2}-\theta), b\tan(\frac{\pi}{2}-\theta))$ i.e. $(a\csc\theta, b\cot\theta)$ $m_{PQ}=\frac{b\cot\theta-b\tan\theta}{a\csc\theta-a\sec\theta}$ $=\frac{\frac{b}{a}\left\{\frac{\cos\theta}{\sin\theta}-\frac{\sin\theta}{\cos\theta}\right\}}{\left\{\frac{1}{\sin\theta}-\frac{1}{\cos\theta}\right\}}$ $\times$ above & below by $\sin\theta\cos\theta$	(1)	Gradient in terms of θ
$m_{PQ}=\frac{b}{a}\left\{\frac{\cos^2\theta-\sin^2\theta}{\cos\theta-\sin\theta}\right\}$ $=\frac{b}{a}\{\cos\theta+\sin\theta\}$ (1)		
Equ. of PQ is $y-b\tan\theta=\frac{b}{a}\{\cos\theta+\sin\theta\}\{x-a\sec\theta\}$ (1) $ay-\frac{ab\sin\theta}{\cos\theta}=bx\cos\theta-ab+bx\sin\theta-\frac{ab\sin\theta}{\cos\theta}$	(1)	Straight line formula with some simplification
$ay=(\cos\theta+\sin\theta)bx-ab$ (1)	(1)	answer

MATHEMATICS: Question 9 EXT 2 (3)

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c) Let $P(n)$ be the proposition that

$\tan^{-1}\left(\frac{1}{2\times1^2}\right) + \tan^{-1}\left(\frac{1}{2\times2^2}\right) + \cdots \tan^{-1}\left(\frac{1}{2\times n^2}\right) = \frac{\pi}{4} - \tan^{-1}\left(\frac{1}{2n+1}\right)$

test $n=1$

LHS $= \tan^{-1}\left(\frac{1}{2}\right)$ RHS $= \frac{\pi}{4} - \tan^{-1}\left(\frac{1}{3}\right)$ (1)

as $\tan^{-1}x + \tan^{-1}y = \tan^{-1}\left(\frac{x+y}{1-xy}\right)$ $0 \le x \le 1$ $0 \le y \le 1$

consider $\tan^{-1}\left(\frac{1}{2}\right) + \tan\left(\frac{1}{3}\right) = \tan^{-1}\left[\frac{\frac{1}{2}+\frac{1}{3}}{1-\frac{1}{2}\times\frac{1}{3}}\right] = \frac{5/6}{5/6} = 1$

$= \frac{\pi}{4}$ (1)

$\therefore \tan^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{4} - \tan^{-1}\left(\frac{1}{3}\right)$ $\therefore P(1)$ is true.

Assume $P(k)$ is true for $k \in \mathbb{Z}^+$

ie $\tan^{-1}\left(\frac{1}{2\times1^2}\right) + \tan^{-1}\left(\frac{1}{2\times2^2}\right) \cdots \tan^{-1}\left(\frac{1}{2\times k^2}\right) = \frac{\pi}{4} - \tan^{-1}\left(\frac{1}{2k+1}\right)$

To prove $P(k+1)$ is true.

ie $\tan^{-1}\left(\frac{1}{2\times1^2}\right) + \tan^{-1}\left(\frac{1}{2\times2^2}\right) \cdots \tan^{-1}\left(\frac{1}{2k^2}\right) + \tan^{-1}\left(\frac{1}{2\times(k+1)^2}\right)$

$= \frac{\pi}{4} - \tan^{-1}\left(\frac{1}{2(k+1)+1}\right)$

$= \frac{\pi}{4} - \tan^{-1}\left(\frac{1}{2k+3}\right)$

$\therefore LHS = \tan^{-1}\left(\frac{1}{2\times1^2}\right) + \tan^{-1}\left(\frac{1}{2\times2^2}\right) \cdots \tan^{-1}\left(\frac{1}{2\times k^2}\right) + \tan^{-1}\left(\frac{1}{2(k+1)^2}\right)$

$= \frac{\pi}{4} - \tan^{-1}\left(\frac{1}{2k+1}\right) + \tan^{-1}\left(\frac{1}{2(k+1)^2}\right)$ by assumption

consider $\tan^{-1}\left(\frac{1}{2(k+1)^2}\right) + \tan^{-1}\left(\frac{1}{2k+3}\right)$ (1)

$= \tan^{-1}\left[\frac{\frac{1}{2(k+1)^2} + \frac{1}{2k+3}}{1 - \frac{1}{2(k+1)^2}\times\frac{1}{2k+3}}\right]$ from above (1)

$= \tan^{-1}\left[\frac{(2k+3) + 2(k+1)^2}{2(k+1)^2(2k+3) - 1}\right]$

$= \tan^{-1}\left[\frac{2k+3+2k^2+4k+2}{(2k^2+4k+2)(2k+3)-1}\right]$

$= \tan^{-1}\left[\frac{2k^2+6k+5}{4k^2+14k^2+16k+5}\right]$

$= \tan^{-1}\left[\frac{2k^2+6k+5}{(2k^2+6k+5)(2x+1)}\right]$

$\tan^{-1}\left(\frac{1}{2k+1}\right)$

$\therefore \tan^{-1}\left(\frac{1}{2(k+1)^2}\right) + \tan^{-1}\left(\frac{1}{2k+3}\right) = \tan^{-1}\left(\frac{1}{2k+1}\right)$

$-\tan^{-1}\left(\frac{1}{2k+1}\right) + \tan^{-1}\left(\frac{1}{2(k+1)^2}\right) = -\tan^{-1}\left(\frac{1}{2k+3}\right)$

$\frac{\pi}{4} - \tan^{-1}\left(\frac{1}{2k+1}\right) + \tan^{-1}\left(\frac{1}{2(k+1)^2}\right) = \pi - \tan\frac{1}{(2k+3)}$

$\therefore P(k+1)$ is true. (1)

$\therefore P(n)$ is true by principle of mathematical induction

Marks: (1) (1)

Marker's Comments:

Must read question and not change given information

Note: $0 \le y \le 1$
$0 \le x \le 1$

formula was given for $\tan^{-1}x + \tan^{-1}y$

Only (1) if $y = -\frac{1}{3}$ used.

(1) Assumption line

(1) using formula

No loss of mark if "–" used.

correct derivation of

(1) final answer (complete proof).

MATHEMATICS: Question 9 ExT 2 (2)

Suggested Solutions	Marks	Marker's Comments
(iv) y coordinate for PQ is always $y=-b$ $\therefore$ The required point is $(0,-b)$ (1)	(1)	$(0,-b)$.
(v) as $\theta \to \frac{\pi}{2}$ $ay \to (\cos\theta + \sin\theta)\,bx - ab$ becomes $ay \to (0+1)\,bx - ab$ $ay \to bx - ab$ $y \to \frac{b}{a}x - b$ (1)	(1)	$y \to \frac{b}{a}x - b$
$\therefore$ PQ is parallel to the asymptote $y = \frac{bx}{a}$ (1)	(1)	Equation of asymptote.
b) (i) $f(x) = \frac{1}{1+\tan x}$ $f(x) + f(\frac{\pi}{2}-x) = \frac{1}{1+\tan x} + \frac{1}{1+\tan(\frac{\pi}{2}-x)}$ $= \frac{1}{1+\tan x} + \frac{1}{1+\cot x}$ (1) $= \frac{1}{1+\tan x} + \frac{\tan x}{\tan x+1}$ $= \frac{1+\tan x}{1+\tan x}$ (1) $= 1$	(1) (1)	changing $\tan(\frac{\pi}{2}-x)$. answer
b) (ii) $\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$ $\therefore \int_0^{\pi/2} \frac{1}{1+\tan x}\,dx = \int_0^{\frac{\pi}{2}} \frac{1}{1+\tan(\frac{\pi}{2}-x)}\,dx$ (1)	(1)	Quoting or proving shift formula.
But $\frac{1}{1+\tan x} + \frac{1}{1+\tan(\frac{\pi}{2}-x)} = 1$ $\therefore \int_0^{\pi/2} \frac{1}{1+\tan x}\,dx = \int_0^{\pi/2} \left(1 - \frac{1}{1+\tan x}\right)dx$ $2\int_0^{\pi/2} \frac{1}{1+\tan x}\,dx = \int_0^{\pi/2} 1\,dx$ (1) $2\int_0^{\pi/2} \frac{1}{1+\tan x}\,dx = \left[x\right]_0^{\pi/2}$	(1)	Showing 2I
$\therefore \int_0^{\pi/2} \frac{1}{1+\tan x}\,dx = \frac{1}{2}\left(\frac{\pi}{2}-0\right)$ $= \frac{\pi}{4}$ (1)	(1)	Answer