Student Number: ____________________

KAMBALA

Mathematics Extension 2

HSC Assessment Task 2

Half-Yearly Examination

March 2008

General Instructions

- Reading time – 5 minutes.
- Working time – 2 hours.
- Answer all questions in the writing booklets provided. **Start each question in a new booklet.**
- Board-approved calculators may be used.
- A table of standard integrals is provided.
- All necessary working should be shown in every question.

Total marks – 80

- Attempt Questions 1-5.
- All questions are of equal value.

Question 1 16 marks (Begin a new booklet) **Marks**

(a) Evaluate $|5-2i|$. **1**

(b) If $z=-3-4i$ find $\frac{1}{z}$ in the form $a+ib$. **2**

(c) Simplify $\frac{1+i^5}{1-i}$. **2**

(d) (i) Express $z=\frac{-1+i}{\sqrt{3}+i}$ in modulus-argument form. **2**

(ii) Hence evaluate $\cos\frac{7\pi}{12}$ in surd form. **2**

(e) Let ω be a non-real cube root of unity.

(i) Show that $\omega^2+\omega+1=0$. **2**

(ii) Prove that $b^3+c^3=(b+c)(b+c\omega)(b+c\omega^2)$. **2**

(f) Find a polynomial $P(x)$ with real coefficients having $2i$ and $1-3i$ as zeroes. **3**

Question 2 16 marks (Begin a new booklet) Marks

(a) Find $\int \frac{x}{\sqrt{x+1}}\,dx$ **2**

(b) Solve for x: $\frac{x^2-5x}{4-x} \le -3$ **3**

(c) (i) Sketch the function $f(x)=x^2-c^2$, where c is a positive constant, clearly indicating its vertex and intercepts. **1**

(ii) Hence, without using calculus, draw separate sketches, at least $\frac{1}{3}$ of a page, for each of the following curves. Clearly indicate turning points.

(A) $f(x)=\left|x^2-c^2\right|$ **2**

(B) $f(x)=\frac{1}{x^2-c^2}$ **2**

(C) $f(x)=\sqrt{x^2-c^2}$ **2**

(D) $f(x)=\left(x^2-c^2\right)^2$ **2**

(E) $f(x)=\left(x^2-c^2\right)^3$ **2**

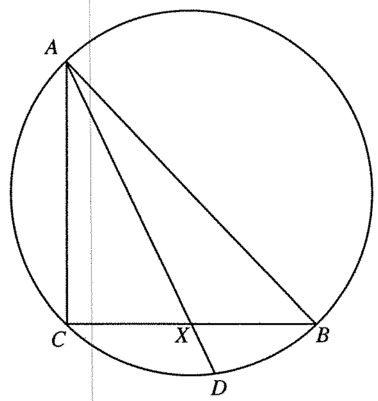
Mathematics Extension 2 HSC Task #2 *Kambala 2008*

Question 3 16 marks (Begin a new booklet) Marks

(a) Consider the function $f(x)=x-\ln\left(1+x^2\right)$. **3**

Show that $f'(x)\ge 0$ for all values of x.

(b)



ABC is a triangle inscribed in a circle as shown above. The bisector of $\angle BAC$ meets BC in X and the circle at D.

(i) Prove that $\Delta ABX \parallel\!| \Delta ADC$ **2**

(ii) Prove that $AB.AC = AD.AX$ **1**

(iii) Prove that $AB.AC = AX^2 + BX.XC$ **2**

Question 3 continues next page

Question 3 continued **Marks**

(c) A cubic equation has roots l, m and n. Given that

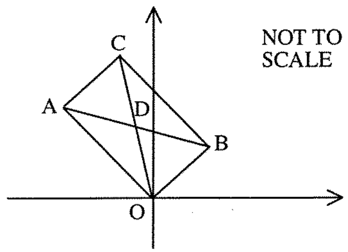
$$l+m+n=-3$$
$$l^2+m^2+n^2=29$$
$$lmn=-6$$

(i) By considering the expansion of $(l+m+n)^2$, show that the cubic equation is given by: **2**

$$x^3+3x^2-10x+6=0$$

(ii) Hence find the values of l, m and n. **3**

(d) OACB is a rectangle where OA = 2OB. D is the point of intersection of the diagonals. The point B represents the complex number z.



Find in terms of z, the complex number represented by:

(i) A **1**

(ii) D **2**

Question 4 16 marks (Begin a new booklet) **Marks**

(a) Assuming the result $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$ and using a suitable substitution, solve the equation $8x^3-6x+1=0$. **3**

(b) (i) If x and y are real, prove that $x^2+y^2 \ge 2xy$. **2**

(ii) Hence show that $a^2+b^2+c^2+d^2 \ge 4abcd$. **2**

(c) $P(2ap, ap^2)$ is a point on the parabola $x^2=4ay$. F is the focus of the parabola. PQ is the perpendicular from P to the directrix d, of the parabola. The tangent at P to the parabola, cuts the axis of the parabola at the point R.

(i) Show that the tangent at the point P to the parabola has equation **2**

$$px-y-ap^2=0$$

(ii) Show that PR and QF bisect each other. **3**

(iii) Show that $PR \perp QF$. **2**

(iv) What type of quadrilateral is $PQRF$? Give reasons for your answer. **2**

Question 5 **16 marks** **(Begin a new booklet)** **Marks**

(a) Let $y = uv$ be the product of u and v, where u and v are functions of x.

(i) Show that $y'' = uv'' + 2u'v' + u''v$. **2**

(ii) Find similar expressions for y''', y'''' and y'''''. **2**

(iii) Hence or otherwise, find and simplify $\frac{d^5}{dx^5}\left((1-x^2)e^{-x}\right)$. **2**

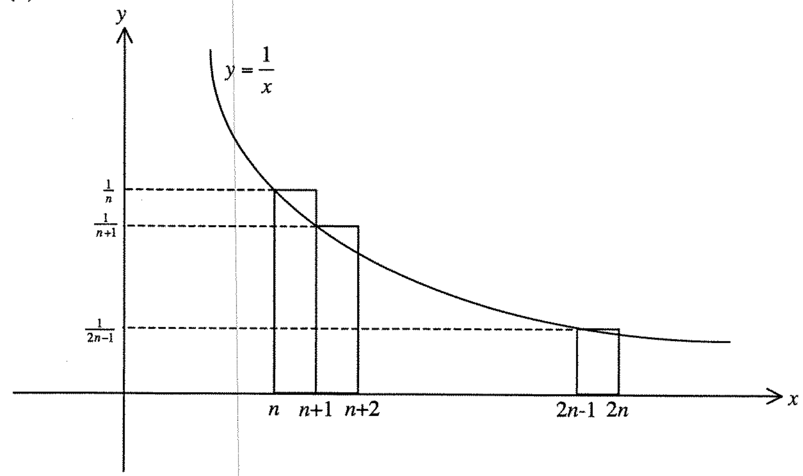
Question 5 continues next page

Question 5 continued **Marks**

(b) For all integers $n \geq 1$, $t_n = \frac{1}{n+1} + \frac{1}{n+2} + \ldots + \frac{1}{2n-1} + \frac{1}{2n}$.

(i) Show that $t_n + \frac{1}{2n} = \frac{1}{n} + \frac{1}{n+1} + \ldots + \frac{1}{2n-1}$ **1**

(ii)



The diagram above shows the graph of the function $y = \frac{1}{x}$ for $n \leq x \leq 2n$.

Using the diagram, show that $t_n + \frac{1}{2n} > \ln 2$. **3**

(iii) For all integers $n \geq 1$, let $s_n = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots + \frac{1}{2n-1} - \frac{1}{2n}$.

Using Mathematical Induction, prove that $s_n = t_n$. **4**

(iv) Hence find, to three decimal places, the value of:

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots + \frac{1}{9999} - \frac{1}{10000}$$ **2**

End of Examination

Qn	Solutions	Marks	Comments
	Kambala Extn 2 Half-Yearly Exam 2008		
	Question 1		
(a)	$\lvert 5-2i \rvert = \sqrt{5^2+2^2} = \sqrt{29}$	1	
(b)	$z = -3-4i$		
	$\frac{1}{z} = \frac{1}{-3-4i} \times \frac{-3+4i}{-3+4i}$	1	
	$= \frac{-3+4i}{9+16}$		
	$= -\frac{3}{25} + \frac{4}{25}i$	1	
(c)	$\frac{1+i^5}{1-i}$ $\quad i^4=1$, $i^5=i$		
	$= \frac{1+i}{1-i} \times \frac{1+i}{1+i}$	1	
	$= \frac{1-1+2i}{1+1}$		
	$= \frac{2i}{2}$		
	$= i$	1	
(d)	(i) $z = \frac{-1+i}{\sqrt{3}+i}$		
	$(-1+i) = \sqrt{2} \operatorname{cis} \frac{3\pi}{4}$ $(\sqrt{3}+i) = 2 \operatorname{cis} \frac{\pi}{6}$	1	
	$z = \frac{\sqrt{2} \operatorname{cis} \frac{3\pi}{4}}{2 \operatorname{cis} \frac{\pi}{6}}$		
	$= \frac{\sqrt{2}}{2} \operatorname{cis}\left(\frac{3\pi}{4} - \frac{\pi}{6}\right)$		
	$= \frac{\sqrt{2}}{2} \operatorname{cis}\left(\frac{7\pi}{12}\right)$	1	
(ii)	$\frac{\sqrt{2}}{2}\left(\cos \frac{7\pi}{12} + i \sin \frac{7\pi}{12}\right) = \frac{-1+i}{\sqrt{3}+i}$		
	$\frac{-1+i}{\sqrt{3}+i} \times \frac{\sqrt{3}-i}{\sqrt{3}-i} = \frac{-\sqrt{3}+i+\sqrt{3}i+1}{3+1} = \frac{1-\sqrt{3}+i(1+\sqrt{3})}{4}$	1	
	Equating real parts		
	$\frac{\sqrt{2}}{2} \cos \frac{7\pi}{12} = \frac{1-\sqrt{3}}{4}$		

$\cos \frac{7\pi}{12} = \frac{1-\sqrt{3}}{4} \times \frac{2}{\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}}$

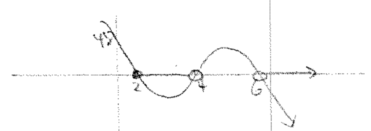
$= \frac{\sqrt{2}(1-\sqrt{3})}{4}$ or $\frac{\sqrt{2}-\sqrt{6}}{4}$ 1

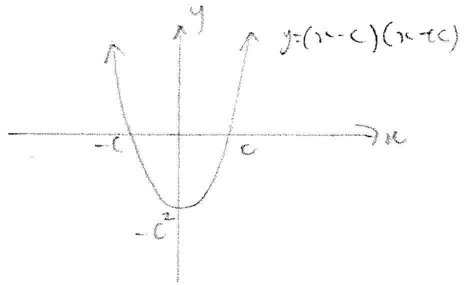
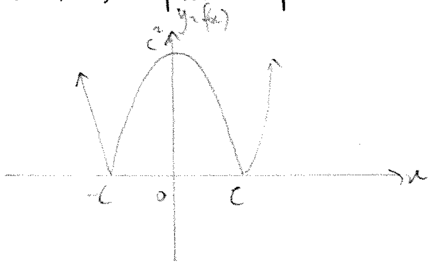
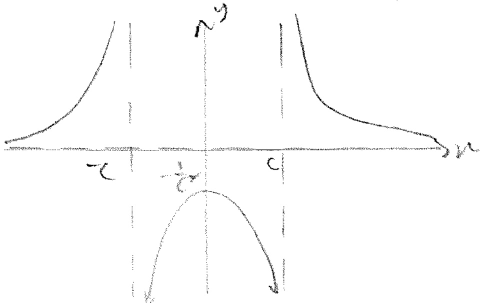
Qn	Solutions	Marks	Comments
	Question 1 ctd.		
(e)	(i) If ω is a non-real cube root of unity,		
	Method I		
	$\omega^3 = 1$		
	$\omega^3 - 1 = 0$		
	$(\omega-1)(\omega^2+\omega+1) = 0$	1	
	$\therefore \omega = 1$ or $\omega^2+\omega+1=0$		
	but ω is a non-real cube root of unity		
	$\omega \neq 1$ so $\omega^2+\omega+1=0$	1	
	Method II		
	$z^3 = 1$		
	let $z = 1 \operatorname{cis} \theta$		
	$\operatorname{cis} 3\theta = \operatorname{cis} 0$		
	$3\theta = 0 + 2k\pi$		
	$\theta = \frac{2k\pi}{3}$		
	when $k=0$ $\quad z_1 = \operatorname{cis} 0 = 1$		
	$k=1$ $\quad z_2 = \operatorname{cis} \frac{2\pi}{3} = \omega$	1	
	$k=2$ $\quad z_3 = \operatorname{cis} \frac{4\pi}{3} = \omega^2$		
	For $z^3-1=0$ Sum roots $=0$		
	$\therefore 1+\omega+\omega^2=0$	1	
	Method III		
	$\omega^3 = 1$		
	$(\omega^2)^3 = (\omega^3)^2 = 1$		
	$\therefore \omega^2$ is also a root		
	$1^3 = 1$ and 1 is obviously a root		
	$\therefore \omega^2$, ω and 1 are cube root of unity		
	For $z^3-1=0$ Sum of roots $=0$		
	$\therefore \omega^2+\omega+1=0$		
(ii)	RHS $= (b+c)(b+c\omega)(b+c\omega^2)$ $\quad \omega^3=1$	1	
	$= (b+c)(b^2+bc\omega^2+bc\omega+c^2\omega^3)$ $\quad \omega^2+\omega+1=0$ $\therefore \omega^2+\omega=-1$		
	$= (b+c)(b^2-bc+c^2)$		
	$= b^3+c^3 = $ LHS $\quad$ Since $x^3+y^3=(x+y)(x^2-xy+y^2)$	1	
	Method II (Similar)		
	LHS $\quad b^3+c^3 = (b+c)(b^2-bc+c^2)$		
	RHS $= (b+c)(b+c\omega)(b+c\omega^2)$		
	$(b+c\omega)(b+c\omega^2) = b^2+bc\omega^2+bc\omega+c^2\omega^3$		
	$= b^2+bc(\omega^2+\omega)+c^2$		

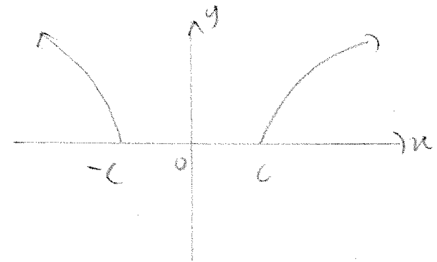
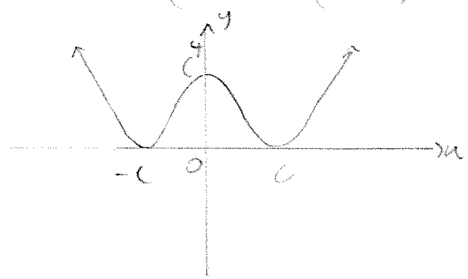
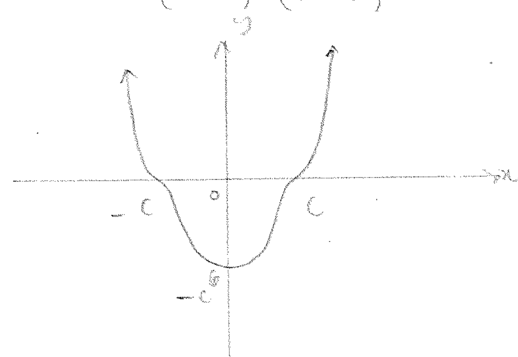
$= b^2-bc+c^2$

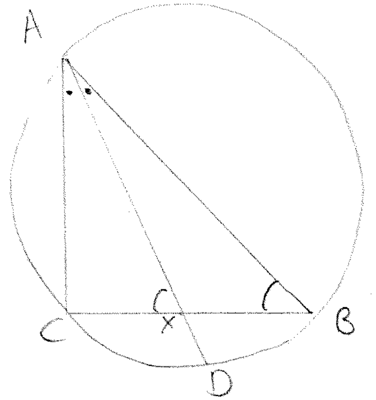
$\therefore$ LHS $=$ RHS

Qn	Solutions	Marks	Comments
	Question 1 ctd.		
(f)	$P(x) = (z-2i)(z-(1-3i))$ $(z+2i)(z-(1+3i))$		
	If $z=2i$ and $1-3i$ are zeroes then $-2i$ and $1+3i$ are also zeroes as $P(x)$ has real coefficients.	1	
	$(z-2i)(z+2i) = z^2+4$		
	$(z-(1-3i))(z-(1+3i))$		
	$= z^2-(1-3i+1+3i)z+1+9$		
	$= z^2-2z+10$	1	
	$P(x) = (z^2+4)(z^2-2z+10)$		
	$= z^4-2z^3+10z^2$ $+4z^2-8z+40$		
	$= z^4-2z^3+14z^2-8z+40$		
	$\therefore P(x)$ could be		
	$P(x) = x^4-2x^3+14x^2-8x+40$	1	

Qn	Solutions	Marks	Comments+Criteria
2	(a) $\int \frac{x\,dx}{\sqrt{x+1}}$		
	Let $u = x+1$		
	$\therefore du = dx$ and $x = u-1$		
	$\int \frac{x}{\sqrt{x+1}}\,dx$		
	$= \int \frac{u-1}{\sqrt{u}}\,.du$		
	$= \int \frac{u}{\sqrt{u}} - \frac{1}{\sqrt{u}}\,.dx$		
	$= \int u^{\frac{1}{2}} - u^{-\frac{1}{2}}\,du$		
	$= \frac{2u^{\frac{3}{2}}}{3} + 2u^{\frac{1}{2}} + C$		
	$= \frac{2}{3}\sqrt{(x+1)^3} + 2\sqrt{x+1} + C$		
	(b) $\frac{x^2-5x}{4-x} \leq -3$		
	$x(x-5)(4-x) \leq -3(4-x)^2$		
	$x(x-5)(4-x)+3(4-x)^2 \leq 0$		
	$(4-x)(x(x-5)+3(4-x)) \leq 0$		
	$(4-x)(x^2-5x+12-3x) \leq 0$		
	$(4-x)(x^2-8x+12) \leq 0$		
	$(4-x)(x-6)(x-2) \leq 0$		
	 $x=2,6,4.$ if $x=0, y=4\cdot-6\cdot-2$		
	$\therefore -2 \leq x < 4$ and $x \geq 6$		

Qn	Solutions	Marks	Comments+Criteria
2 ctd	(c) (i) $f(x) = x^2 - c^2 \quad , c > 0$ 		
	(ii) (A) $f(x) = \lvert x^2 - c^2 \rvert$ 		
	(B) $f(x) = \dfrac{1}{x^2 - c^2} = \dfrac{1}{(x-c)(x+c)}$ $x \neq \pm c$  if $x = 0$, $f(x) = \dfrac{1}{-c^2}$		

Qn	Solutions	Marks	Comments+Criteria
2 ctd	(c) ctd (C) $f(x) = \sqrt{x^2 - c^2}$ 		
	(D) $f(x) = (x^2 - c^2)^2$ $= (x^2 - c^2)(x^2 - c^2)$ $= (x-c)^2 (x+c)^2$ 		
	(E) $f(x) = (x^2 - c^2)^3$ $= (x-c)^3 (x+c)^3$ 		

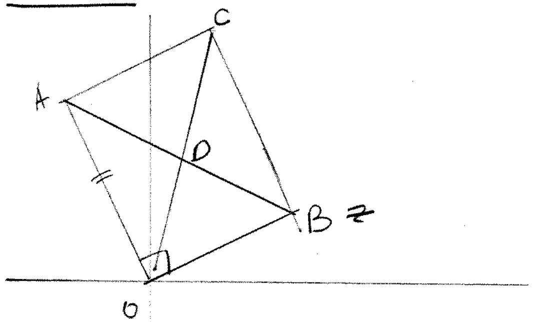
<table>
<tr><th>Qn</th><th>Solutions</th><th>Marks</th><th>Comments+Criteria</th></tr>
<tr><td>3</td><td>(a) $f(x) = x - \ln(1+x^2)$
$f'(x) = 1 - \frac{2x}{1+x^2}$
$= \frac{1+x^2-2x}{1+x^2}$
$= \frac{x^2-2x+1}{1+x^2}$
$= \frac{(x-1)^2}{1+x^2}$
Since $x^2 > 0$, then $1+x^2 > 0$
$(x-1)^2 > 0$ $\therefore f'(x) > 0 \ \forall x$</td><td></td><td></td></tr>
<tr><td></td><td>(b)

(i) Prove $\triangle ABX \parallel\!| \triangle ADC$
In $\triangle ABX$ and $\triangle ADC$:
AD is bisector of $\angle BAC$ (data)
$\therefore \angle CAX = \angle BAX$.
$\angle AXC = \angle ABX$ (angles on same arc =)
$\therefore \angle ACX = \angle AXB$ (angle sum of $\triangle$)
ie $\triangle ABX \parallel\!| \triangle ADC$ (equiangular</td><td></td><td></td></tr>
</table>

<table>
<tr><th>Qn</th><th>Solutions</th><th>Marks</th><th>Comments+Criteria</th></tr>
<tr><td>3 ctd</td><td>(b) ctd
(ii) Prove $AB.AC = AD.AX$
Since $\triangle ABX \parallel\!| \triangle ADC$ from (i), then corresponding sides are in proportion.
ie $\frac{AB}{AX} = \frac{AD}{AC}$
$\therefore AB.AC = AD.AX$</td><td></td><td></td></tr>
<tr><td></td><td>(iii) Prove $AB.AC = AX^2 + BX.XC$
From (ii), $AB.AC = AD.AX$
$= (AX + XD).AX$
$= AX^2 + XD.AX$
$= AX^2 + BX.XC$
since $AX.XD = BX.XC$ by intercept theorem</td><td></td><td></td></tr>
</table>

Qn	Solutions	Marks	Comments+Criteria
3 (c)	$l+m+n=-3 = \Sigma\alpha$ cubic $l^2+m^2+n^2=29$ $\quad \Sigma\alpha\beta=?$ $lmn=-6$ $\quad \Sigma\alpha\beta\gamma$ (i) $(l+m+n)^2$ $= l^2+m^2+n^2 + lm + ln + ml + mn + nl + nm$ $= l^2+m^2+n^2 + 2(lm+mn+nl)$ $(-3)^2 = 29 + 2(lm+mn+nl)$ $9 = 29 + 2(lm+mn+nl)$ $lm+mn+nl = \frac{-20}{2}$ $=-10$ cubic is given by $x^3 - (\text{sum})x^2 + (\text{product } \alpha\beta)x - \text{product} = 0$ $\therefore x^3+3x^2-10x+6=0$ as req'd		
	(ii) $x^3+3x^2-10x+6=0 = P(x)$. $P(1)=0$. $\therefore x-1$ is a factor. $$\begin{array}{r} x^2+4x-6 \\ x-1\ \overline{)\,x^3+3x^2-10x+6} \\ \underline{x^3-x^2} \\ 4x^2-10x \\ \underline{4x^2-4x} \\ -6x+6 \\ \underline{-6x+6} \end{array}$$		

Qn	Solutions	Marks	Comments+Criteria
3 ctd	(c) ctd $\therefore P(x) = (x-1)(x^2+4x-6)$ $x=1$ a root, $\quad x = \frac{-4 \pm \sqrt{16-4.1.-6}}{2.1}$ $= \frac{-4 \pm \sqrt{16+24}}{2}$ $= \frac{-4 \pm \sqrt{40}}{2}$ $= \frac{-4 \pm 2\sqrt{10}}{2}$ $x = -2 \pm \sqrt{10}$ $\therefore l, m, n$ are $1, -2 \pm \sqrt{10}$.		

Q3 (d)



(i) let z be $r(\cos\theta + i\sin\theta)$

$\vec{OB} = z$

$OA = 2OB$

$\vec{OA} = 2zi$ (rotation anticlockwise by $\frac{\pi}{2}$)

$\therefore A$ is $2iz$ 1

(ii) $\vec{OC} = \vec{OB} + \vec{BC}$

$= z + 2iz$ 1

$= (1+2i)z$

$\therefore \vec{OD} = \frac{1}{2}(1+2i)z$

$\therefore D$ is $\frac{1}{2}(1+2i)z$ 1

or $\left(\frac{1}{2} + i\right)z$

Qn	Solutions	Marks	Comments

Question 4

(a) $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$

If $8x^3 - 6x + 1 = 0$

$8x^3 - 6x = -1$

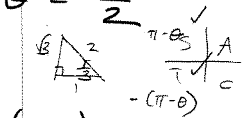
$4x^3 - 3x = -\frac{1}{2}$

Looking for 3 solutions (unique).

Let $x = \cos\theta$

$4\cos^3\theta - 3\cos\theta = -\frac{1}{2}$ 1

$\cos 3\theta = -\frac{1}{2}$



$3\theta = \pi - \frac{\pi}{3},\ -\left(\pi - \frac{\pi}{3}\right)$

$= \frac{2\pi}{3} + 2k\pi,\ -\frac{2\pi}{3} + 2k\pi$

$3\theta = \frac{6k\pi \pm 2\pi}{3} = \frac{2\pi(3k\pm1)}{3}$

$\theta = \frac{2\pi(3k\pm1)}{9}$

when $k=0$ $\theta = \pm\frac{2\pi}{9}$ $\therefore \boxed{x = \cos\frac{2\pi}{9}}$

$x = \cos\left(-\frac{2\pi}{9}\right) = \cos\frac{2\pi}{9}$

When $k=1$ $\theta = \frac{2\pi\times4}{9} = \frac{8\pi}{9}$ $\therefore \boxed{x = \cos\frac{8\pi}{9}}$

$\theta = \frac{2\pi\times2}{9} = \frac{4\pi}{9}$ $\therefore \boxed{x = \cos\frac{4\pi}{9}}$

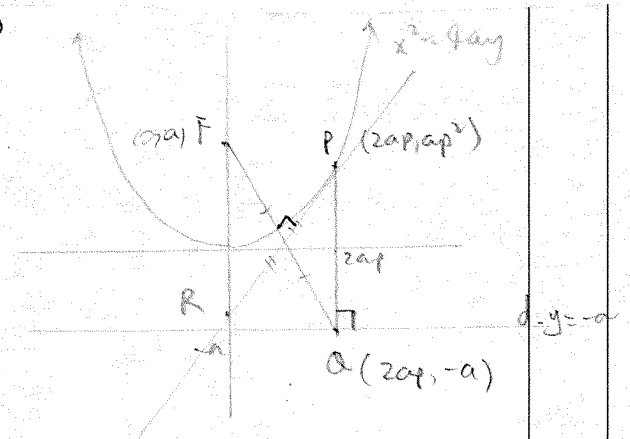
when $k=2$ $\theta = \frac{2\pi\times7}{9} = \frac{14\pi}{9} = -\frac{4\pi}{9}$ $x = \cos\left(-\frac{4\pi}{9}\right) = \cos\frac{4\pi}{9}$

$\theta = \frac{2\pi\times5}{9} = \frac{10\pi}{9} = -\frac{8\pi}{9}$ $x = \cos\left(-\frac{8\pi}{9}\right) = \cos\frac{8\pi}{9}$

$\therefore x = \cos\frac{2\pi}{9},\ \cos\frac{4\pi}{9},\ \cos\frac{8\pi}{9}$

Qn	Solutions	Marks	Comments+Criteria
4	(b) (i) Prove $x^2+y^2 \geq 2xy$ Now $(x-y)^2 \geq 0 \quad \forall\, x, y$ real $\therefore x^2 - 2xy + y^2 \geq 0$ ie $x^2+y^2 \geq 2xy$		

Q4(b) see over

Qn	Solutions	Marks	Comments+Criteria
4	(c) 		

Qn	Solutions	Marks	Comments+Criteria
4	(c) ctd. (i) $P(2ap, ap^2)$ $\quad y = \frac{1}{4a}x^2$ $\frac{dy}{dx} = \frac{1}{2a}x$ $m = \frac{2ap}{2a} = p$. $y - ap^2 = p(x-2ap)$ $y - ap^2 = px - 2ap^2$ $\therefore px - ap^2 - y = 0$.		
	(ii) Show PR and QF bisect. R lies on tangent and axis of parabola. $\therefore$ at $x=0$, $0 - ap^2 - y = 0$ $\therefore y = -ap^2$. R is $(0, -ap^2)$. midpt of PR $= \left(\frac{2ap+0}{2}, \frac{-ap^2+ap^2}{2}\right)$ $= (ap, 0)$ midpt of QF $= \left(\frac{2ap+0}{2}, \frac{-a+a}{2}\right)$ $= (ap, 0)$ since midpt of PR = midpt of QF then PR and QF must bisect		

Qn	Solutions	Marks	Comments+Criteria
4	(c) ctd (iii) Show PR $\perp$ OF $m_{PR} = \dfrac{ap^2+ap^2}{2ap-0}$ $= \dfrac{2ap^2}{2ap}$ $= p$ (already shown in (i)!) $m_{OF} = \dfrac{-a-a}{2ap-0}$ $= \dfrac{-2a}{2ap}$ $= -\dfrac{1}{p}$ $m_{PR} \times m_{OF} = p \times -\dfrac{1}{p}$ $= -1$ $\therefore$ PR $\perp$ OF		
	(iv) PQRF? Diagonals bisect @ right angles, $\therefore$ PQRF is a rhombus. FR $\parallel$ PQ since PQ is perp. to directrix, a horizontal line.		

Qn	Solutions	Marks	Comments+Criteria
5	(a) $y = uv$ (i) $y' = u'v + v'u$ $y'' = u''v + v'u' + v''u + u'v'$ $= uv'' + 2u'v' + u''v$		
	(ii) Find y''', y'''', y''''' $y''' = u'v'' + v'''u + 2u''v' + v''.2u' + u'''v + v'u''$ $= uv''' + 3u'v'' + 3u''v' + u'''v$ $y'''' = u'v''' + v''''u + 3u''v'' + v'''.3u' + 3u'''.v' + v''.3u'' + u''''v + v'u'''$ $= uv'''' + 6u''v'' + 4u'v''' + 4u'''v' + u''''v$ $y''''' = u'v'''' + v'''''u + 6u'''v'' + v'''.6u'' + 4u''v''' + v''''.4u' + 4u''''v' + v''.4u''' + u'''''v + v'u''''$ $= uv''''' + 5u'v'''' + 10u''v''' + 10u'''v'' + 5u''''v' + u'''''v$		1 1 1 1 2 1 1 3 3 1 1 4 6 4 1 1 5 10 10 5 1
	(iii) $\dfrac{d^5}{dx^5}\big(\underbrace{(1-x^2)}_{u}\underbrace{e^{-x}}_{v}\big)$ $u = 1-x^2$, $\dfrac{du}{dx} = u' = -2x$, $u'' = -2$, $u^3 = u^4 = u^5 = 0$ $y^5 = uv^5 + 5u^1v^4 + 10u^2v^3 + 10u^3v^2 + 5u^4v^1 + u^5v$ $\therefore \dfrac{d^5}{dx^5}\left((1-x^2)e^{-x}\right)$ $v = e^{-x}$, $v' = -e^{-x}$, $v'' = e^{-x}$, $v''' = -e^{-x}$, $v^4 = e^{-x}$, $v^5 = -e^{-x}$ $= (1-x^2)(-e^{-x}) + 5(-2x)e^{-x} + 10(-2)(-e^{-x})$ $+ 10.0 \ldots + 5\times 0 \ldots + 0$ $= -e^{-x}(1-x^2+10x-20)$		

$= -e^{-x}(-x^2+10x-19) \quad = e^{-x}(x^2-10x+19)$

Qn	Solutions	Marks	Comments+Criteria
5	(b) $t_n = \frac{1}{n+1} + \frac{1}{n+2} + \dots + \frac{1}{2n-1} + \frac{1}{2n}$		
	(i) $\underbrace{t_n + \frac{1}{2n}} = \frac{1}{n+1} + \frac{1}{n+2} + \dots + \frac{1}{2n-1} + \frac{1}{2n} + \underbrace{\frac{1}{2n}}$ $= \frac{1}{n+1} + \frac{1}{n+2} + \dots + \frac{1}{2n-1} + \frac{2}{2n}$ $= \frac{1}{n} + \frac{1}{n+1} + \dots + \frac{1}{2n-1}$ as required		
	(ii) $\int_n^{2n} \frac{1}{x}\,dx \approx n \times \frac{1}{n} + n \times \frac{1}{n+1} + n \times \frac{1}{n+2} + \dots + n \times \frac{1}{2n-1}$ $= 1 \cdot \left(\frac{1}{n} + \frac{1}{n+1} + \dots + \frac{1}{2n-1}\right) = \left(t_n + \frac{1}{2n}\right)$ also $\int_n^{2n} \frac{1}{x}\,dx = \left[\ln x\right]_n^{2n}$ $= \ln 2n - \ln n$ $= \ln 2 + \ln n - \ln n$ $= \ln 2$ From diagram, area of rectangle is an approximation yielding an area greater than exact area. $\therefore \left(\frac{1}{n} + \frac{1}{n+1} + \dots + \frac{1}{2n-1}\right) > \ln 2$ ie $\left(t_n + \frac{1}{2n}\right) > \ln 2$		

Qn	Solutions	Marks	Comments+Criteria
5	(iii) $S_n = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots + \frac{1}{2n-1} - \frac{1}{2n}$ Prove $S_n = t_n$. $t_n = \frac{1}{n+1} + \frac{1}{n+2} + \dots + \frac{1}{2n-1} + \frac{1}{2n}$ RTP: $1 - \frac{1}{2} + \frac{1}{3} - \dots + \frac{1}{2n-1} - \frac{1}{2n} = \frac{1}{n+1} + \frac{1}{n+2} + \dots + \frac{1}{2n-1} + \frac{1}{2n}$ for $n \geq 1$. test $n=1$: LHS $= 1 - \frac{1}{2} = \frac{1}{2}$ RHS $= \frac{1}{1+1} = \frac{1}{2} =$ LHS $\therefore$ true for $n=1$. Assume true for $n=k$ ie $S_k = t_k$. ie $1 - \frac{1}{2} + \frac{1}{3} - \dots + \frac{1}{2k-1} - \frac{1}{2k} = \frac{1}{k+1} + \frac{1}{k+2} + \dots + \frac{1}{2k-1} + \frac{1}{2k}$ Prove true for $n=k+1$ ie $S_{k+1} = t_{k+1}$. ie RTP: $1 - \frac{1}{2} + \frac{1}{3} - \dots + \frac{1}{2(k+1)-1} - \frac{1}{2(k+1)} = \frac{1}{(k+1)+1} + \frac{1}{(k+2)+1} + \dots + \frac{1}{2(k+1)}$ ie $1 - \frac{1}{2} + \frac{1}{3} - \dots + \frac{1}{2k+1} - \frac{1}{2k+2} = \frac{1}{k+2} + \frac{1}{k+3} + \dots + \frac{1}{2k+2}$ LHS $= 1 - \frac{1}{2} + \frac{1}{3} - \dots + \frac{1}{2k+1} - \frac{1}{2k+2}$ $= \underbrace{1 - \frac{1}{2} + \frac{1}{3} - \dots - \frac{1}{2k}} + \frac{1}{2k+1} - \frac{1}{2k+2}$ $= \frac{1}{k+1} + \frac{1}{k+2} + \dots + \frac{1}{2k-1} + \frac{1}{2k} + \frac{1}{2k+1} - \frac{1}{2k+2}$ $= \frac{1}{k+2} + \frac{1}{k+3} + \dots + \frac{1}{2k+1} + \frac{2k+2-(k+1)}{(2k+2)(k+1)}$ $= \frac{1}{k+2} + \frac{1}{k+3} + \dots + \frac{1}{2k+1} + \frac{\cancel{k+1}}{(2k+2)\cancel{(k+1)}}$ $= \frac{1}{k+2} + \frac{1}{k+3} + \dots + \frac{1}{2k+1} + \frac{1}{2k+2}$ $=$ RHS $\therefore$ since true for $n=1$ ...		

Qn	Solutions	Marks	Comments+Criteria
5	(iv) $1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\dots+\frac{1}{9999}-\frac{1}{10000}$ $S_n = 1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\dots+\frac{1}{2n-1}-\frac{1}{2n}$ So $2n = 10000$ $n = 5000$ $t_n = S_n \therefore t_{5000} = S_{5000}$ $t_n + \frac{1}{2n} > \ln 2$ $t_{5000} + \frac{1}{10000} > \ln 2$ $t_{5000} > \ln 2 - \frac{1}{10000}$ $> 0.693047\dots$ ie $t_{5000} \doteqdot 0.693$		