Set By: BRA

Teachers:

*BRA
MAX
HAY

KNOX GRAMMAR SCHOOL
MATHEMATICS DEPARTMENT

2006
HALF YEARLY EXAMINATION

Mathematics Extension 2 (Year 12)

General Instructions

- Reading time – 5 minutes
- Working time – 3 hours
- Write using blue or black pen
- Board-approved calculators may be used
- All necessary working should be shown in every question
- A table of standard integrals is provided at the back of this paper

Total marks (**120**)

- Attempt Questions 1–8
- All questions are of equal value
- Use a **SEPARATE** writing booklet for each question
- Write your Board of Studies (BOS) Number and Teacher's Initials on the front cover of each of your writing booklets

BOS NUMBER: ____________________ **TEACHER'S INITIALS:** ____________

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Total marks (120)
Attempt questions 1 – 8
All questions are of equal value

Answer each question in a SEPARATE writing booklet. Extra writing booklets are available.

Question 1 (15 marks) Use a SEPARATE writing booklet **Marks**

(a) Find z in the form $a + bi$ if $z = \dfrac{2-i}{2+i}$. **2**

(b) If $z = -1 + i\sqrt{3}$ determine:

(i) $\bar{z}$ **1**

(ii) $|z|$ **1**

(iii) z^4 *(In Cartesian Form)* **3**

(iv) $\operatorname{Arg}\left(z^4\right)$ *(Exact Value In Radians)* **1**

(c) Draw a neat, accurate sketch of the region representing the complex number z where z satisfies the inequalities $1 \le |z-1| \le 2$ and $\dfrac{\pi}{4} \le \arg(z) \le \dfrac{\pi}{2}$ **3**

(d) The complex number z satisfies the equation $\arg(z+2) = \dfrac{\pi}{3}$.

(i) Sketch the points z on an Argand diagram for which $\arg(z+2) = \dfrac{\pi}{3}$. **2**

(ii) What is the minimum value of $|z|$? **2**

Question 2 (15 marks) Use a SEPARATE writing booklet **Marks**

(a) Prove by induction that $\sin x + \sin 2x + \sin 3x + \cdots + \sin nx = \dfrac{\sin\left(\dfrac{nx}{2}\right)\sin\left(\dfrac{n+1}{2}\right)x}{\sin\left(\dfrac{x}{2}\right)}$ **4**

for all counting numbers n.

(Hint: You may assume that $2\cos A \sin B = \sin(A+B) - \sin(A-B)$*.)*

(b) (i) Given $z = \cos\theta + i\sin\theta$, simplify $z^n - z^{-n}$ where n is a counting number. **2**

(ii) By expanding $\left(z - z^{-1}\right)^5$ find the values of the constants a, b and c so that: **4**

$$\sin^5\theta = a\sin 5\theta + b\sin 3\theta + c\sin\theta.$$

(Hint: Use the coefficients 1, 5, 10, 10, 5, 1 in the expansion of $\left(z - z^{-1}\right)^5$*.)*

(c) ω is a complex cube root of unity.

(i) Prove that the other complex root is ω^2. **2**

(ii) Prove that $1 + \omega + \omega^2 = 0$. **1**

(iii) Simplify $\left(1 - 3\omega + \omega^2\right)\left(1 + \omega - 8\omega^2\right)$. **2**

Question 3 (15 marks) Use a SEPARATE writing booklet **Marks**

(a) Factorise $x^3 - 8$ over the field of

(i) real numbers **1**

(ii) complex numbers **2**

(b) $(x-2)$ is a factor of $P(x) = x^3 + ax + b$. When $P(x)$ is divided by $(x+3)$ the remainder is -10.

(i) Find the values of a and b. **3**

(ii) Using the values for a and b obtained in (i), what is the remainder when $P(x)$ is divided by $(x-2)(x+3)$? **2**

(c) Given that $2 + 3i$ is a root of the equation $x^4 - 4x^3 + 4x^2 + 36x - 117 = 0$, find the other roots. **4**

(d) If α, β and γ are the rots of $2x^3 - 3x^2 - 3x + 2 = 0$, form the cubic equation whose roots are α^2, β^2 and γ^2. **3**

Question 4 (15marks) Use a SEPARATE writing booklet **Marks**

(a) Sketch the graph of $y = \ln x$. Use your graph to sketch on separate number planes the graphs of: **5**

(i) $y = |\ln x|$

(ii) $y = \ln|x|$

(iii) $|y| = \ln x$

(iv) $y = \ln(-x)$

(b) (i) If $f(x) = \tan^{-1} x + \tan^{-1}\left(\frac{1}{x}\right)$ for $x > 0$, show that $f'(x) = 0$. **2**

(ii) Hence, or otherwise, sketch the graph of $f(x)$. **2**

(c) On a number plane, sketch the region given by $y \le |x^2 - 1|$. **2**

(d) Without using calculus, draw a neat sketch of $y = \frac{1}{(x-1)(x+2)}$ showing all important features. **4**

Question 5 (15marks) Use a SEPARATE writing booklet **Marks**

(a)

(i) Show that if the polynomial equation $P(x) = 0$ has a root α of multiplicity m then the equation $P'(x) = 0$ has a root of multiplicity $m - 1$. **3**

(ii) Find the roots of the equation $x^4 + 5x^3 - 21x^2 + 23x - 8 = 0$ given that it has a root of multiplicity 3. **3**

(b) Find the values of p and q such that $x^4 + 6x^3 + 13x^2 + px + q$ will be a perfect square. **3**

(c) If α, β and γ are the roots of the cubic equation $x^3 - 5x^2 + 2x - 1 = 0$, write down the values of :

(i) $\dfrac{1}{\alpha} + \dfrac{1}{\beta} + \dfrac{1}{\gamma}$ **2**

(ii) $\alpha^2 + \beta^2 + \gamma^2$ **2**

(iii) $\dfrac{\alpha\beta}{\gamma} + \dfrac{\alpha\gamma}{\beta} + \dfrac{\beta\gamma}{\alpha}$ **2**

Question 6 (15 marks) Use a SEPARATE writing booklet **Marks**

(a) Sketch the following graphs on separate number planes, clearly showing all features.

(i) $y = \tan^{-1}|x|$ **2**

(ii) $y = \dfrac{\pi}{2} + \sin^{-1}(x-1)$ **2**

(iii) $\sin(x+y) = 0$ **2**

$\sin^{-1}(\theta) = x + y$

$\sin^{-1}(\theta) - x = y$

(b) You are given that the graphs for the following equations each have just <u>one</u> line of symmetry. State in each case the equation of the line of symmetry, giving reasons for your answers.

(i) $x^4 = y^3$ **2**

(ii) $x^3 + y^3 = 1$ **2**

(c) (i) Find the equation of the locus of w if $w = \dfrac{z-2}{z-1}$ and z describes the unit circle centred at the origin. **3**

(ii) Hence, sketch the locus of w on an Argand diagram. **2**

Question 7 (15 marks) Use a SEPARATE writing booklet **Marks**

(a) State the domain and range of $y = \sin^{-1}\left(\frac{x}{2}\right)$. **2**

(b) (i) Write $\sin\theta + \sqrt{3}\cos\theta$ in the form $A\cos(\theta - \alpha)$ where $A > 0$ and α is acute. **2**

(ii) Hence, solve $\sin\theta + \sqrt{3}\cos\theta = 1$ for $0 \le \theta \le 2\pi$. **2**

(c) Find, in terms of π, the value of:

(i) $\int_0^2 \frac{1}{4 + x^2}\,dx$ **2**

(ii) $\int_0^{\frac{1}{6}} \left(\frac{1}{\sqrt{1 - 9x^2}}\right) dx$ **2**

(d) If n is a counting number, n factorial, which is written as $n!$ is defined by:

$$n! = n(n-1)(n-2)\cdots.3\cdot 2\cdot 1$$

For example, $4! = 4\cdot 3\cdot 2\cdot 1$ or 24.

(i) Find the smallest counting number A for which $n! \ge 3^n$. **1**

(ii) Hence, prove by mathematical induction that $n! \ge 3^n$ for $n \ge A$. **4**

Question 8 (15 marks) Use a SEPARATE writing booklet **Marks**

(a) Find the general solution to $2\sin^2\theta + \cos\theta = 1$. **4**

(b) Differentiate $\cos^4\theta$ and use the result to evaluate $\int_0^{\pi/4} (\cos^3\theta \sin\theta)d\theta$ **3**

(c) Find $\int \frac{1-\cos 2\theta}{1+\cos 2\theta} d\theta$ **2**

(d) (i) Write down $\sin\theta, \cos\theta$ and $\tan\theta$ in terms of t where $t = \tan\frac{\theta}{2}$ **1**

(ii) Use these results to prove that $\sec\theta - \tan\theta = \frac{1-t}{1+t}$. **2**

(iii) Hence, find $\int (\sec\theta - \tan\theta) d\theta$ **3**

End of Paper

(a) $\frac{2-i}{2+i}\cdot\frac{2-i}{2-i} = \frac{(2-i)^2}{2^2+1^2}$ or $\frac{3}{5}-\frac{4}{5}i$ ✓

(b) (i) $\bar{z} = -1-\sqrt{3}i$ ✓

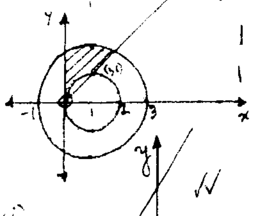
(ii) $|z| = \sqrt{(-1)^2+(\sqrt{3})^2}$ or 2 ✓

(iii) $z = 2\operatorname{cis}\frac{2\pi}{3}$ ✓ $\Rightarrow z^4 = 16\operatorname{cis}\frac{8\pi}{3}$ ✓

$= 16\operatorname{cis}\frac{2\pi}{3} = -8+8\sqrt{3}i$ ✓

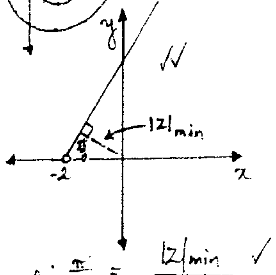
(iv) $\operatorname{Arg}(z^4) = \frac{2\pi}{3}$ ✓

(c)



1 for annulus
1 for ray
1 for shading

(d) (i)



✓✓

(ii) $\sin\frac{\pi}{3} = \frac{|z|_{min}}{2}$ ✓

$\Rightarrow |z|_{min} = \sqrt{3}$. ✓

When $n=1$, LHS $= \sin x$, RHS $= \frac{\sin\frac{x}{2}\cdot\sin x}{\sin\frac{x}{2}}$ or $\sin x$ $\Rightarrow$ true for $n=1$. ✓

If the statement is true for $n=K$; $k\in\mathbb{N}$, then:- ✓

$\ldots + \sin Kx + \sin(K+1)x = \frac{\sin\frac{Kx}{2}\cdot\sin\left(\frac{K+1}{2}\right)x}{\sin\frac{x}{2}} + \sin(K+1)x$

$= \frac{\sin\frac{Kx}{2}\cdot\sin\left(\frac{K+1}{2}\right)x + 2\sin\left(\frac{K+1}{2}\right)x\cdot\cos\left(\frac{K+1}{2}\right)x}{\sin x/2}$ ✓

$= \frac{\sin\left(\frac{K+1}{2}\right)x}{\sin\frac{x}{2}}\left[\sin\frac{Kx}{2} + 2\cos\left(\frac{K+1}{2}\right)x\cdot\sin\frac{x}{2}\right]$

$= \frac{\sin\left(\frac{K+1}{2}\right)x}{\sin\frac{x}{2}}\left[\sin\frac{Kx}{2} + \underbrace{\sin\left(\frac{K+2}{2}\right)x - \sin\frac{Kx}{2}}_{\text{using hint}}\right]$

$= \frac{\sin\frac{(K+1)x}{2}\cdot\sin\frac{(K+2)x}{2}}{\sin\frac{x}{2}}$. ✓

Hence, if true for $n=K \Rightarrow$ true for $n=K+1$. As it is true for $n=1$, then by induction it is true $\forall\, n\in\mathbb{N}$. ✓

$z^n - z^{-n} = 2i\sin n\theta$ ✓ (consequence of De Moivre)

$(z-z^{-1})^5 = z^5 - 5z^4z^{-1} + 10z^3z^{-2} - 10z^2\cdot z^{-3} + 5zz^{-4} - z^{-5}$ ✓

$= (z^5-z^{-5}) - 5(z^3-z^{-3}) + 10(z-z^{-1})$ ✓

$\therefore (2i\sin\theta)^5 = 2i\sin 5\theta - 5(2i\sin 3\theta) + 10(2i\sin\theta)$ ✓

$\Rightarrow \sin^5\theta = \frac{1}{16}\sin 5\theta - \frac{5}{16}\sin 3\theta + \frac{5}{8}\sin\theta$.

$\therefore a = \frac{1}{16},\ b = -\frac{5}{16}$ & $c = \frac{5}{8}$. ✓

(c) (i) $\omega^3 = 1 \Rightarrow (\omega^3)^2 = 1^2$ ✓

$\Rightarrow (\omega^2)^3 = 1$ ✓

$\therefore \omega^2$ is also a complex cube root of unity

(ii) From $z^3-1=0 \Rightarrow \sum\alpha = 0$

$\Rightarrow 1+\omega+\omega^2 = 0$. ✓

(iii) $(1-3\omega+\omega^2)(1+\omega-8\omega^2)$

$= (1+\omega+\omega^2-4\omega)(1+\omega+\omega^2-9\omega^2)$

$= (-4\omega)(-9\omega^2)$ using (ii) ✓

$= 36\omega^3$ or $\underline{\underline{36}}$ ✓

3.(b) $P(2) = 0$ (Factor th'm) & $P(-3) = -10$ ✓ (Rem. th'm)

(i) $\Rightarrow \begin{cases} 2a+b = -8 & (i) \\ -3a+b = 17 & (ii) \end{cases}$ ✓

$\Rightarrow a = -5$ & $b = 2$ ✓

(ii) By div. transformation:

$x^3-5x+2 \equiv (x-2)(x+3)Q(x) + px + q$. $p, q\in\mathbb{R}$.

$\Rightarrow \begin{cases} 2p+q = 0 & (\text{sub. } x=2) \\ -3p+q = -10 & (\text{sub. } x=-3) \end{cases}$

$\Rightarrow p = 2;\ q = -4$ ✓✓

Remainder $= 2x-4$. ✓

(a) (i) $(x-2)(x^2+2x+4)$ ✓

(ii) $(x-2)\left[(x+1)^2-(\sqrt{3}i)^2\right]$ ✓ completing square

$= (x-2)(x+1+i\sqrt{3})(x+1-i\sqrt{3})$ ✓

(c) Co-eff'ts real $\Rightarrow 2-3i$ is also a root. ✓

$\therefore (x-(2+3i))(x-(2-3i))$ are linear factors.

$\Rightarrow x^2-4x+13$ is a quad. factor. ✓

$\therefore x^4-4x^3+4x^2+36x-117 = (x^2-4x+13)(x^2-9)$ ✓
by inspection or equivalent

$\Rightarrow$ other roots are $2-3i$, ± 3. ✓

Q3 Cont.

(d) Let $x = \sqrt{y} \Rightarrow 2y\sqrt{y} - 3y - 3\sqrt{y} + 2 = 0$ ✓

and $\alpha^2, \beta^2, \gamma^2$ satisfy this eq'n iff α, β, γ satisfy $2x^3-3x^2-3x+2 = 0$. ✓

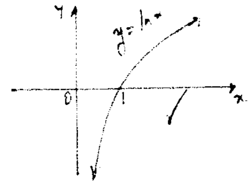
$\Rightarrow \sqrt{y}(2y-3) = 3y-2$

$\Rightarrow y(2y-3)^2 = (3y-2)^2$

$\Rightarrow 2y^3-12y^2+9y = 9y^2-12y+4$

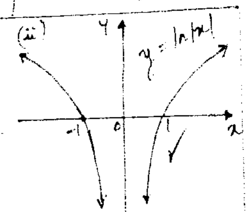
$\Rightarrow 2y^3-21y^2+21y-4 = 0$. ✓

4. (a)



✓

(ii)



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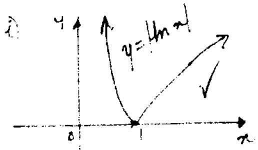
(i)



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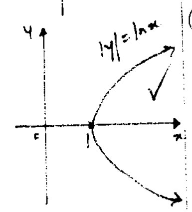
(iii)



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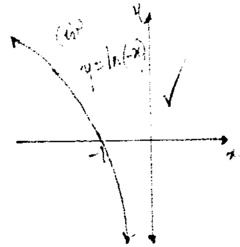
(iv)



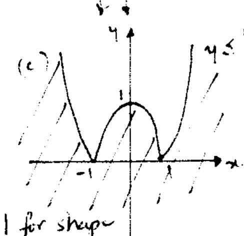
✓

(b) (i) $f'(x) = \frac{1}{1+x^2} + \frac{1}{1+\left(\frac{1}{x}\right)^2}\cdot\frac{-1}{x^2}$ ✓

$= \frac{1}{1+x^2} - \frac{1}{1+x^2}$ ✓

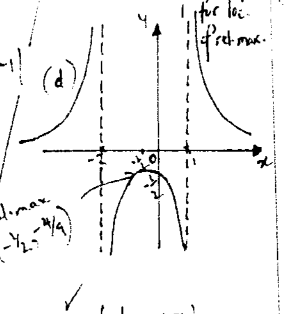
$= 0$.

(c)



1 for shape
1 for shading.

(d)



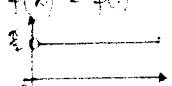
rel. max $(-\frac{1}{2}, -\frac{4}{9})$

1 for shape
1 for asym.
1 for loc. of rel. max.

(b) (ii) $f(x) = $ CONSTANT. ✓ Sub. $x=1$

$\Rightarrow f(x) = f(1) = \frac{\pi}{4}+\frac{\pi}{4}$ or $\frac{\pi}{2}$.

$\therefore f(x) = \frac{\pi}{2};\ x>0$



✓

3. (i) $P(x) = (x-\alpha)^m Q(x)$ ✓; $Q(\alpha)\neq 0$

$\Rightarrow P'(x) = (x-\alpha)^m Q'(x) + m(x-\alpha)^{m-1}\cdot Q(x)$ ✓

$= (x-\alpha)^{m-1}\left[(x-\alpha)\cdot Q'(x) + mQ(x)\right]$

$= (x-\alpha)^{m-1}\cdot Q_1(x)$ ✓ & $Q_1(\alpha)\neq 0$.

(ii) Let $P(x) = x^4+5x^3-21x^2+23x-8$

$\Rightarrow P'(x) = 4x^3+15x^2-42x+23$

$\Rightarrow P''(x) = 12x^2+30x-42$

$\therefore P''(x) = 0$ when $6(2x^2+5x-7) = 0$

$\Rightarrow 6(2x+7)(x-1) = 0$

$\Rightarrow x = -\frac{7}{2}$ or 1. ✓

As $P(1) = 0 = P'(1) \Rightarrow$ 3 fold root is $x=1$.

$\therefore P(x) = (x-1)^3(x+8)$ by inspection. ✓

$\Rightarrow$ roots are -8, 1 (3 fold).

(b) $(x^2+ax+b)^2 = x^4+6x^3+13x^2+px+q$ ✓
for some $a, b\in\mathbb{R}$.

Equating co-eff'ts of $x^3 \Rightarrow 2a = 6$
$\Rightarrow a = 3$.

Equating co-eff'ts of $x^2 \Rightarrow a^2+2b = 13$
$\Rightarrow b = 2$.

$\therefore (x^2+3x+2)^2 = x^4+6x^3+13x^2+px+q$

Equating co-eff'ts of $x \Rightarrow \underline{p = 12}$ ✓

Equating constant terms $\Rightarrow \underline{q = 4}$ ✓

(c) (i) $\sum\frac{1}{\alpha} = \frac{1}{\alpha}+\frac{1}{\beta}+\frac{1}{\gamma}$

$= \frac{\alpha\beta+\alpha\gamma+\beta\gamma}{\alpha\beta\gamma}$ ✓

$= \frac{2}{-(-1)}$ or $\underline{\underline{2}}$. ✓

(ii) $\sum\alpha^2 = \alpha^2+\beta^2+\gamma^2$ ✓

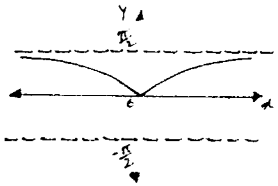
$= (\alpha+\beta+\gamma)^2 - 2(\alpha\beta+\alpha\gamma+\beta\gamma)$

$= (-(-5))^2 - 2\times 2$ or $\underline{21}$ ✓

(iii) $\sum\frac{\gamma}{\alpha\beta} = \frac{\alpha}{\beta\gamma}+\frac{\beta}{\alpha\gamma}+\frac{\gamma}{\alpha\beta}$

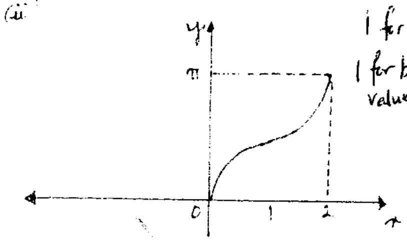
$= \frac{\alpha^2+\beta^2+\gamma^2}{\alpha\beta\gamma}$ ✓ $= \frac{21}{-(-1)}$ or $\underline{\underline{21}}$ ✓

(a) (i)



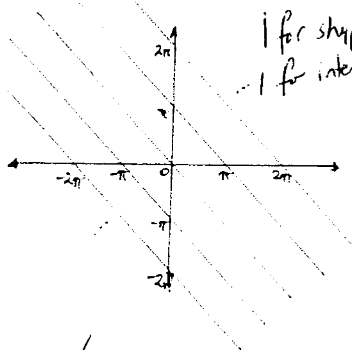
1 for shape
1 for asymp.

(ii)



1 for shape
1 for boundary values.

(iii)



1 for shape
1 for intercepts

(b)(i) $x = 0$ (y axis) as $(-x)^n = x^n$. ✓ ✓

(ii) $y = x$ ✓ as interchanging x & y introduces the same eq'n. ✓

(c) $|z| = 1$ ✓

Making z the subject: $wz - w = z - 2$

$\Rightarrow z(w-1) = w-2$

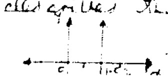
$\Rightarrow z = \frac{w-2}{w-1}$

$\Rightarrow 1 = \left|\frac{w-2}{w-1}\right|$ ✓ as $|z| = 1$

$\Rightarrow |w-1| = |w-2|$ ✓

∴ w describes the ⊥ bisector of the interval from $(1,0)$ to $(2,0)$.

i.e. w describes the vertical line $x = 1\frac{1}{2}$. ✓✓



7.(a) Domain $-2 \le x \le 2$; Range $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$. ✓ ✓

(b)(i) $A = \sqrt{1^2 + (\sqrt{3})^2}$
$= 2$ ✓

& $2\left(\cos\theta \cdot \frac{\sqrt{3}}{2} + \sin\theta \cdot \frac{1}{2}\right) = 2\cos(\theta - \alpha)$

$\Rightarrow \begin{cases} \cos\alpha = \sqrt{3}/2 \\ \sin\alpha = 1/2 \end{cases} \Rightarrow \alpha = \frac{\pi}{6}$ ✓

∴ $\sin\theta + \sqrt{3}\cos\theta = 2\cos\left(\theta - \frac{\pi}{6}\right)$.

(ii) $\cos\left(\theta - \frac{\pi}{6}\right) = \frac{1}{2}$

$\Rightarrow \theta - \frac{\pi}{6} = \ldots -\frac{\pi}{3}, \frac{\pi}{3}, 2\pi - \frac{\pi}{3}, 2\pi + \frac{\pi}{3} \ldots$ ✓ ✓

$\Rightarrow \theta = \frac{\pi}{2}, \frac{11\pi}{6}$ only for $0 \le \theta \le 2\pi$.

(c)(i) $\left[\frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right)\right]_0^2 = \frac{1}{2}\left(\tan^{-1}(1) - \tan^{-1}(0)\right)$ ✓
$= \frac{\pi}{8}$ ✓

(ii) $\frac{1}{3}\int_0^{1/6} \frac{1}{\sqrt{(\frac{1}{3})^2 - x^2}}\,dx = \left[\frac{1}{3}\sin^{-1}(3x)\right]_0^{1/6}$ ✓
$= \frac{1}{3}\cdot\sin^{-1}\left(\frac{1}{2}\right)$ or $\frac{\pi}{18}$ ✓

(d)(i)

n	n!	3^n
1	1	3
2	2	9
3	6	27
4	24	81
5	120	243
6	720	729
7	5040	2187 ← 1st occurrence
8	40320	6561
9	362880	19683

Hence, $A = 7$. ✓

(ii) From (i) statement is true for $n = 7$ ✓

Assume $k! \ge 3^k$; k some integer ≥ 7.

Then $(k+1)! = (k+1)\cdot(k)!$ ✓

$\ge (k+1)\cdot 3^k$ ✓ by induction hypothesis

$\ge 3\cdot 3^k$ ✓ as k is at least 7

$= 3^{k+1}$

Hence if it is true for $n = k$, then also true for $n = k+1$. As it is true for $n = 7$, conclude by induction that it is true for all higher counting no's.

8.(a) $2\sin^2\theta + \cos\theta = 1$

$\Rightarrow 2(1 - \cos^2\theta) + \cos\theta = 1$

$\Rightarrow 2\cos^2\theta - \cos\theta - 1 = 0$

$\Rightarrow (2\cos\theta + 1)(\cos\theta - 1) = 0$ ✓

$\Rightarrow \cos\theta = -\frac{1}{2}$ or 1. ✓

$\Rightarrow \theta = 2k\pi \pm \cos^{-1}(-\frac{1}{2})$ or $2k\pi \pm \cos^{-1}(1)$ ✓

$\Rightarrow \theta = 2k\pi \pm \left(\pi - \frac{\pi}{3}\right)$, $2k\pi \pm 0$ ✓

$\Rightarrow \theta = 2k\pi \pm \frac{2\pi}{3}, 2k\pi$; $k \in \mathbb{Z}$.

(b) $\frac{d}{d\theta}(\cos^4\theta) = 4\cos^3\theta \cdot -\sin\theta$ by the chain rule.
$= -4\cos^3\theta\cdot\sin\theta$ ✓

∴ $\int_0^{\frac{\pi}{4}} \cos^3\theta\cdot\sin\theta\,d\theta = -\frac{1}{4}\int_0^{\frac{\pi}{4}} -4\cos^3\theta\sin\theta\,d\theta$

$= \left[-\frac{1}{4}\cos^4\theta\right]_0^{\frac{\pi}{4}}$ ✓

$= -\frac{1}{4}\left[\cos^4\frac{\pi}{4} - \cos^4 0\right]$

$= -\frac{1}{4}\left[\frac{1}{4} - 1\right]$

$= \frac{3}{16}$ ✓

(c) $\int \frac{2\sin^2\theta}{2\cos^2\theta}\,d\theta$ ✓ $= \int \tan^2\theta\,d\theta$

$= \int(\sec^2\theta - 1)\,d\theta$

$= \tan\theta - \theta + C$ ✓

(d)(i) $\sin\theta = \frac{2t}{1+t^2}$; $\cos\theta = \frac{1-t^2}{1+t^2}$; $\tan\theta = \frac{2t}{1-t^2}$. 1 mark only if all correct.

(ii) $\sec\theta - \tan\theta = \frac{1}{\cos\theta} - \frac{\sin\theta}{\cos\theta}$

$= \frac{1 - \sin\theta}{\cos\theta}$ ✓

$= \frac{1 - \frac{2t}{1+t^2}}{\frac{1-t^2}{1+t^2}}$ × by $\frac{1+t^2}{1+t^2}$

$= \frac{1+t^2-2t}{1-t^2}$

$= \frac{t^2-2t+1}{1-t^2}$

$= \frac{(1-t)^2}{(1-t)(1+t)}$ ✓

$= \frac{1-t}{1+t}$.

(iii) $\int \frac{1 - \tan\frac{\theta}{2}}{1 + \tan\frac{\theta}{2}}\,d\theta$ ✓

$= \int \frac{\tan\frac{\pi}{4} - \tan\frac{\theta}{2}}{1 + \tan\frac{\pi}{4}\cdot\tan\frac{\theta}{2}}\,d\theta$ ✓

$= \int \tan\left(\frac{\pi}{4} - \frac{\theta}{2}\right)d\theta$ ✓

$= 2\ln\left|\cos\left(\frac{\pi}{4} - \frac{\theta}{2}\right)\right| + C$.