בס"ד

MASADA

MY ROCK AND צלעי ומצודתי MY FORTRESS

# MATHEMATICS

## YEAR 12
## 1999 HALF YEARLY EXAM

## 4 UNIT

*Time allowed: Three hours*
*(Plus 5minutes reading time)*

DIRECTIONS TO CANDIDATES

- Attempt ALL questions.
- ALL questions are of equal value.
- All necessary working should be shown in every question. Marks may be deducted for careless or badly arranged work.
- Standard integrals are printed on page 10.
- Board-approved calculators may be used.
- Answer each question in a *separate* Writing Booklet.
- You may ask for extra Writing Booklets if you need them.

**Marks**

## QUESTION 1

(a) Find the modulus and argument of $Z = 3 + 4i$ (Express the argument in radians) **2**

(b) For any complex number $Z$ where $Z = -\overline{Z}$ prove that $Z$ must be purely imaginary **3**

(c) Find the square root of $Z = 5 - 12i$ **4**

(d) Draw a neat sketch to illustrate the following region of the Argand diagram **2**

$$-\frac{\pi}{6} \le \arg(Z - 1) \le \frac{\pi}{6} \; and \; |Z - 1| \le 1$$

(e) If $Z$ is a complex number such that **4**

$|Z - 6| + |Z + 6| = 60$ describe geometrically the locus of $Z$ and find its Cartesian equation.

**Marks**

## QUESTION 2

(a) If $1 + i$ is a solution of $x^4 - 6x^3 + 5x^2 + 2x - 10 = 0$ solve the equation over the field of real numbers. **3**

(b) If $\alpha, \beta, \delta$ are the roots of $x^3 - px + q = 0$ find in terms of $p$ and $q$ a cubic equation with roots **4**

i. $\frac{1}{\alpha}, \frac{1}{\beta}, \frac{1}{\delta}$

ii. $\alpha^3, \beta^3, \delta^3$

(c) If the cubic equation $2x^3 - 9x^2 + 12x + k = 0$ has two equal roots, find the value of $k$ **4**

(d) Find the condition (i.e. the relationship between $a$ and $b$) that $x^4 - 3ax + b = 0$ has a repeated root **4**

## QUESTION 3

Marks

(a) Express $z$ in the form $a + ib$ if 3

$$\arg (z+1) = \frac{\pi}{6} \text{ and } \arg (z-1) = \frac{2\pi}{3}$$

(b) If $z$ is a complex number show that $z^2 + (\bar{z})^2 = 2$ is a hyperbola and state its eccentricity 4

(c) By writing each factor in the modulus-argument form, simplify 3

$$(\sqrt{3}+i)^6 \div (1-i)^4$$

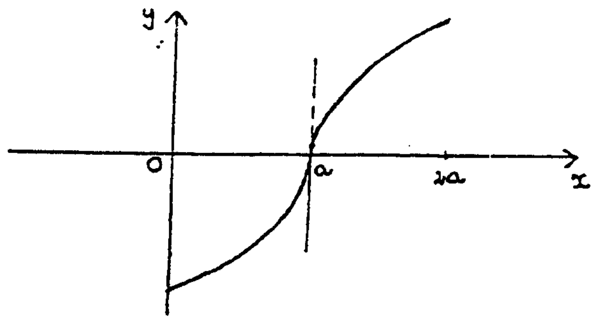
(d) i. Find the four complex roots of $z^4 + 4 = 0$ 5

ii. Plot these roots on an Argand diagram

**Marks**

## QUESTION 4

(a) Consider the graph of $y = f(x)$ for $0 \le x \le 2a$ **3**




The graph has point symmetry and a vertical tangent exists at $x = a$.
Sketch:

i. $y = f'(x)$

ii. $y = f''(x)$

iii. $y = \int_0^x f(t)\,dt$

(b) i. Given $F(x) = \dfrac{x^2 - 1}{x^2 + 1}$, sketch the following on separate axes **12**

1. $y = F(x)$

2. $[F(x)]^2 = \dfrac{x^2 - 1}{x^2 + 1}$

3. $y = [F(x)]^2$

4. $y = \log_e F(x)$

5. $y = \dfrac{|x + 1|(x - 1)}{x^2 + 1}$

ii. Use your graph in (5) to solve the inequality $x^2 + 1 > 2|x + 1|(x - 1)$

## QUESTION 5

**Marks**

(a) i. Obtain the equation of the tangent to the curve $\sqrt{x}+\sqrt{y}=\sqrt{c}$ at the point $P(a,b)$ on the curve **5**

ii. This tangent meets the $x$ and $y$ axes at Q and R respectively. Show that $OQ+OR=c$ for all positions of P, where O is the origin

(b) i. Find the eccentricity, the equations of the directrices and the co-ordinates of the foci of the ellipse with equation $7x^2+16y^2=112$ **10**

ii. Sketch the ellipse showing the above information on your diagram. Also sketch the auxiliary circle on your diagram

iii. Set up the integrals that give:

1. The area of a quadrant of the circle with equation $x^2+y^2=16$

2. The area of the quadrant of the ellipse $7x^2+16y^2=112$

iv. Show that the integral in (iii) ~~(i)~~2. above is $\frac{b}{a}$ times the integral in (iii) ~~(i)~~1. and deduce the area of the ellipse from the known area of the circle. Hence write down a general formula for the area of an ellipse

**QUESTION 6** **Marks**

(a) The point P $(cp, \frac{c}{p})$ lies on the rectangular hyperbola $xy = c^2$ in the first quadrant. The tangent to the hyperbola at the point P, crosses the $x$ axis at the point A and the $y$ axis at the point B. **10**

i. Find the equation of the tangent to the hyperbola at the point P

ii. Show that the equation of the normal to the hyperbola at the point P is $p^3x - py = cp^4 - c$

iii. If the normal at P meets the other branch of the hyperbola at the point Q, determine the coordinates of Q

iv Show that the area of the triangle ABQ is $c^2\left(p^2 + \frac{1}{p^2}\right)^2$

v. Prove that the area of the triangle is a minimum when $p = 1$

(b) If $a$, $b$ and $c$ are positive real numbers such that $a \neq b \neq c$, prove, **5**

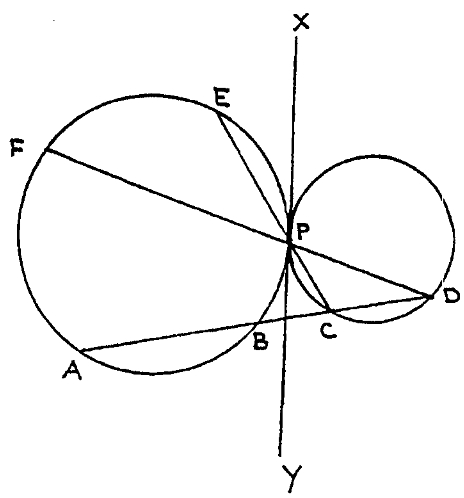
i. $\frac{a}{b} + \frac{b}{a} > 2$

ii. $(a + b + c)\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right) > 9$

## QUESTION 7

**Marks**

(a) Two circles touch externally at point P. The line ABCD cuts the first circle at A and B and the second circle at C and D. The lines CPE and DPF meet the first circle at E and F respectively. XPY is the common tangent. **7**




Copy the diagram onto your answer paper.
Prove that:

i. FE||AD

ii. $\angle FPA = \angle BPC$

iii. $\Delta FPA ||| \Delta BPC$

(b) $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is the equation of a hyperbola with eccentricity $e$. **8**

i. Prove the perpendicular from the focus $S(ae,0)$ to the asymptote $y = \frac{b}{a}x$ meets it on the directrix.

ii. Prove that the angle between the two asymptotes is $2\tan^{-1}\sqrt{e^2 - 1}$

**QUESTION 8** **Marks**

(a) Find, as a relation between $k$, $l$, and $m$, the condition for the quadratic equation in $x$, **3**

$$(k^2+l^2)x^2+2l(k+m)x+(l^2+m^2)=0$$

to have real roots. Simplify your answer as far as possible.

(b) If $|a|>2|b|$, prove $2|a-b|>|a|$ **3**

(c) i. Show that $\int_0^{\frac{\pi}{2}} \cos^4 x\,dx=\frac{3\pi}{16}$ **9**

ii. Prove $3(\cos^4 x+\sin^4 x)-2(\cos^6 x+\sin^6 x)=1$

iii. Without attempting to evaluate any integrals, explain why:

$\int_0^{\frac{\pi}{2}} \sin^n x\,dx=\int_0^{\frac{\pi}{2}} \cos^n x\,dx$, for all positive integers $n$

iv By integrating the identity in part (ii), and using parts (i) and (iii),

find $\int_0^{\frac{\pi}{2}} \cos^6 x\,dx$

v. Without attempting to evaluate any integrals, explain why:

$\int_0^{\frac{\pi}{2}} \sin^{n+1} x\,dx<\int_0^{\frac{\pi}{2}} \sin^n x\,dx$. for all positive integers $n$

4 unit Solutions

1 a $3+4i = 5\left(\frac{3}{5} + \frac{4i}{5}\right)$

$\therefore \cos\theta = \frac{3}{5}$

$\theta = \cos^{-1}\frac{3}{5}$

$= \cdot 927295218^c$

$= \cdot 93^c$

$\therefore 3+4i = 5\,\text{cis}\,(\cdot 93)$

$\therefore$ modulus $=5$, argument $= 0.93^c$

b) let $z = x+iy$

if $z = -\bar{z}$

$x+iy = -(x-iy)$

i.e $2x = 0$

$x = 0$

Hence $z = 0+iy$

$= iy$

which is purely imaginary

c) let $x+iy = \sqrt{5-12i}$

ie $(x+iy)^2 = 5-12i$

$x^2-y^2+2xyi = 5-12i$

Comparing real and imaginary parts

$x^2-y^2 = 5$ ....1)

$2xy = -12$ .....2)

from 2) $y = -\frac{6}{x}$ .... 3)

Substitute 3) into 1)

$x^2 - \frac{36}{x^2} = 5$

$x^4 - 5x^2 - 36 = 0$

$(x^2+4)(x^2-9) = 0$

Hence $x = \pm 3$ ($x$ is real)

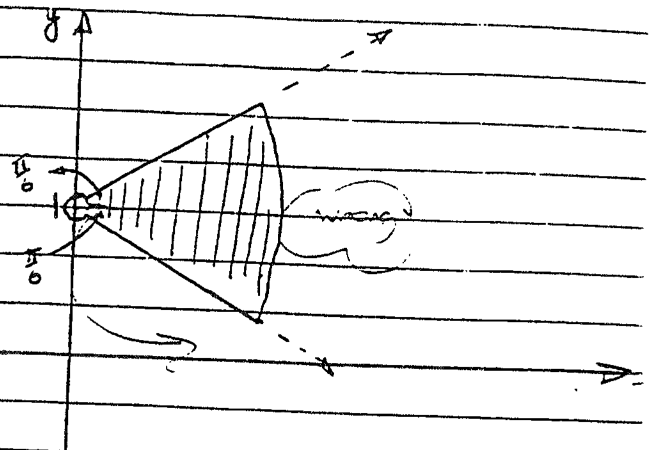
When $x = 3$ $y = -2$

$x = -3$ $y = 2$

$\therefore \sqrt{5-12i} = \pm(3-2i)$

d) $-\frac{\pi}{6} \le \arg(z-1) \le \frac{\pi}{6}$

and $|z-1| \le 1$




e) $|z-6| + |z+6| = 60$

The locus of $z$ is an ellipse, with foci at $(6,0)$ and $(-6,0)$. The length of the major axis is 60, ie. ($a=30$)

$\therefore a = 30$

$ae = 6$

$e = \frac{1}{5}$

Now

$b^2 = a^2(1-e^2)$

$= 900\left(1-\frac{1}{25}\right)$

$= 864$

Hence the equation is

$$\frac{x^2}{900} + \frac{y^2}{864} = 1$$

Question 2.

a) Let $f(x) = x^4 - 6x^3 + 5x^2 + 2x - 10$

$f(x)$ has real coefficients

$\therefore$ if $1+i$ is a root $1-i$ is also

~~and~~

and $(x-1-i)(x-1+i)$ is a factor

ie. $x^2 - 2x + 2$ is a factor

$$
\begin{array}{r|l}
 & x^2 - 4x - 5 \\
\hline
x^2 - 2x + 2 & x^4 - 6x^3 + 5x^2 + 2x - 10 \\
 & \underline{x^4 - 2x^3 + 2x^2} \\
 & -4x^3 + 3x^2 + 2x \\
 & \underline{-4x^3 + 8x^2 - 8x} \\
 & -5x^2 + 10x - 10 \\
 & \underline{-5x^2 + 10x - 10} \\
 & 0
\end{array}
$$

$\therefore f(x) = (x^2 - 2x + 2)(x^2 - 4x - 5)$

$= (x^2 - 2x + 2)(x+1)(x-5)$

Hence the real roots are $5, -1$

b) Let $f(x) = x^3 - px + q$

i) Put $u = \frac{1}{x}$ $\therefore x = \frac{1}{u}$

Hence $\left(\frac{1}{u}\right)^3 - \frac{p}{u} + q = 0$ has roots $\frac{1}{\alpha}, \frac{1}{\beta}, \frac{1}{\gamma}$

$\frac{1}{u^3} - \frac{p}{u} + q = 0$

$1 - pu^2 + qu^3 = 0$

ie. $qu^3 - pu^2 + 1 = 0$

So the required function is

$qx^3 - px^2 + 1 = 0$

ii) Put $u = x^3 \Rightarrow x = u^{1/3}$

$(u^{1/3})^3 - pu^{1/3} + q = 0$ has roots $\alpha^3, \beta^3, \gamma^3$

$u - pu^{1/3} + q = 0$

$u + q = pu^{1/3}$

$(u+q)^3 = (pu^{1/3})^3$

$u^3 + 3u^2q + 3uq^2 + q^3 = p^3u$

$u^3 + 3u^2q + (3q^2 - p^3)u + q^3 = 0$

ie

$x^3 + 3x^2q + (3q^2 - p^3)x + q^3 = 0$

c) $f(x) = 2x^3 - 9x^2 + 12x + k = 0$

$f'(x) = 6x^2 - 18x + 12$

If $X$ is a double root then $f'(X) = 0$ and $f(X) = 0$

$f'(x) = 0$ when $6(x^2 - 3x + 2) = 0$

ie $(x-1)(x-2) = 0$

$x = 1, 2$

$f(1) = 2 - 9 + 12 + k = 0 \Rightarrow k = -5$

$f(2) = 16 - 36 + 24 + k = 0 \Rightarrow k = -4$

$\therefore$ the equation has equal roots when $k = -4, -5$.

Question 2.

a) $f(x) = x^4 - 3ax + b$ ---- 1)

$f'(x) = 4x^3 - 3a$ ---- 2)

If there is a repeated root $f'(x) = f(x) = 0$

2) × x $\quad 4x^4 - 3ax = 0$ --- 3)

$x^4 - 3ax + b = 0$ ---- 1)

3) − 1) $\quad 3x^4 - b = 0$

$x^4 = \frac{b}{3}$ ---- 4)

Since $4x^3 - 3a = 0$

$x^3 = \frac{3a}{4}$ ----- 5)

from 4) $\quad x^{12} = \left(\frac{b}{3}\right)^3 = \frac{b^3}{27}$

from 5) $\quad x^{12} = \left(\frac{3a}{4}\right)^4 = \frac{81a^4}{256}$

$\therefore \frac{b^3}{27} = \frac{81a^4}{256}$

ie. $256b^3 = 2187a^4$

b) Let $z = x + iy$, $\bar{z} = x - iy$

Hence $z^2 + \bar{z}^2 = 2$

$\Rightarrow$

$(x+iy)^2 + (x-iy)^2 = 2$

$x^2 + 2ixy + i^2y^2 + x^2 - 2ixy + i^2y^2 = 2$

$2x^2 - 2y^2 = 2$

$x^2 - y^2 = 1$

which is a hyperbola

Now $b^2 = a^2(e^2 - 1)$ $\quad a = 1, b = 1$

$\therefore e^2 = 2$

$e = \sqrt{2}$

So we have a hyperbola with eccentricity $\sqrt{2}$

c) Let $z_1 = \sqrt{3} + i$ $\quad |z_1| = \sqrt{3+1} = 2$

$\arg z_1 = \tan^{-1}\frac{1}{\sqrt{3}} = \frac{\pi}{6}$

$\therefore z_1 = 2 \operatorname{cis} \frac{\pi}{6}$

$z_2 = 1 - i$ $\quad |z_2| = \sqrt{1+1} = \sqrt{2}$

$\arg z_2 = -\tan^{-1} 1 = -\frac{\pi}{4}$

$\therefore z_2 = \sqrt{2} \operatorname{cis}\left(\frac{-\pi}{4}\right)$

$(\sqrt{3}+i)^6 \div (1-i)^4 = \left(2 \operatorname{cis} \frac{\pi}{6}\right)^6 \div \left(\sqrt{2} \operatorname{cis}\right.$

$= 2^6 \operatorname{cis} \pi \div 2^2 \operatorname{cis}(-\pi)$

$= 2^4 \operatorname{cis}(\pi - (-\pi))$

$= 2^4 \operatorname{cis} 2\pi$

$= 2^4 \cos 2\pi + i \sin 2\pi$

$= 2^4$

$= 16$

Question 3

a) $\arg(z-1) - \arg(z+1) = \frac{2\pi}{3} - \frac{\pi}{6}$

$\therefore \arg\left(\frac{z-1}{z+1}\right) = \frac{\pi}{2}$

ie. the locus is the top semi-circle of $x^2 + y^2 = 1$ $\quad x \neq \pm 1$

ie $y = \sqrt{1 - x^2}$ $\quad x \neq \pm 1$

ie $|z| = 1$

Hence $z = x + \left(\sqrt{1-x^2}\right)i$

d) $z^4 + 4 = 0$

i) $z^4 = -4$

$\therefore r^4(\cos\theta + i\sin\theta)^4 = -4$

$r^4(\cos 4\theta + i\sin 4\theta) = -4$

$\therefore r^4 = 4$ $\qquad \cos 4\theta + i\sin 4\theta = -1$

$r = \sqrt{2}$ $\qquad \cos 4\theta = -1$ and $\sin 4\theta = 0$

$4\theta = \pi, 3\pi, 5\pi, 7\pi$

$\theta = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$

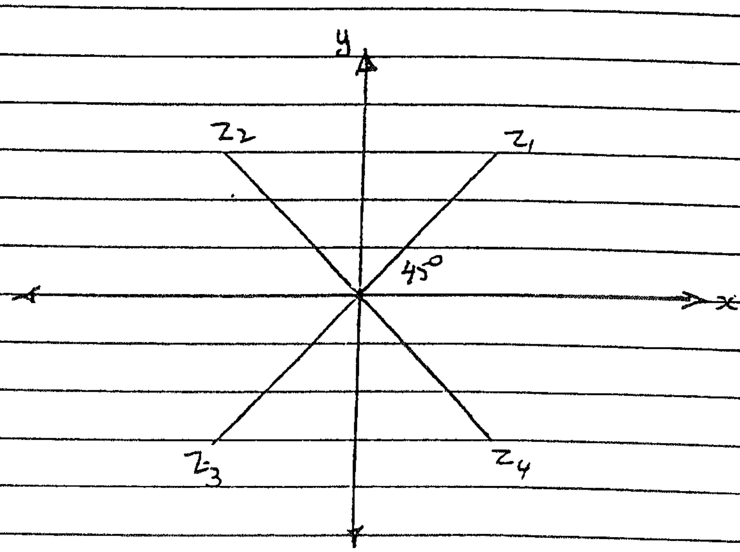
Roots $z_1 = \sqrt{2}\ \text{cis}\ \frac{\pi}{4},$

$z_2 = \sqrt{2}\ \text{cis}\ \frac{3\pi}{4}$

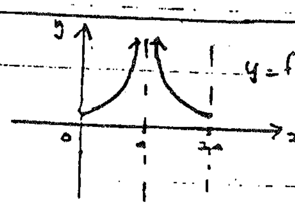
$z_3 = \sqrt{2}\ \text{cis}\ \frac{5\pi}{4}$

$z_4 = \sqrt{2}\ \text{cis}\ \frac{7\pi}{4}$

ii)



4 (a). i.



$y = f'(x)$

$-\frac{1}{2}$ each mistake

Check for:

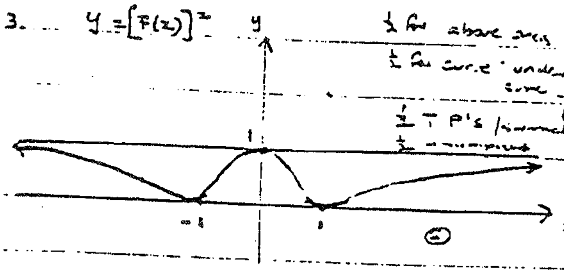
Symmetry

Asymptote

Always +ve slope

Start above $y=0$

3. $y = [F(x)]^2$



$\frac{1}{2}$ for above axis

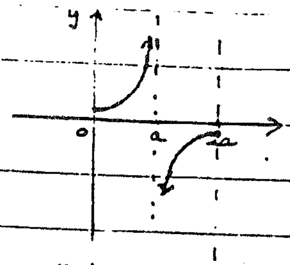
$\frac{1}{2}$ for curve "under" $F(x)$ curve $-1 \le F(x) \le 1$

$\frac{1}{2}$ T P's / symmetry / shape

$\frac{1}{2}$ asymptotes

(2)

ii



$y = f''(x)$ $-\frac{1}{2}$ each mistake

Check for

point symmetry

asymptote

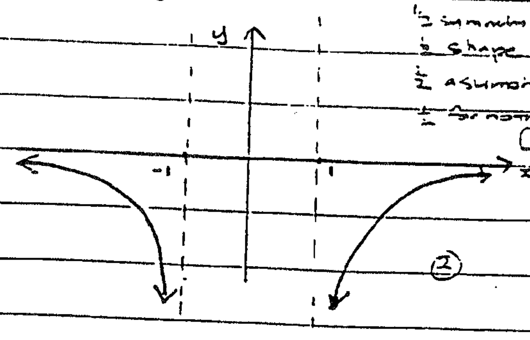
Shape

$x<a,\ f''>0$

$x>a,\ f''<0$

(3)

4. $y = \log_e F(x)$



$\frac{1}{2}$ symmetry

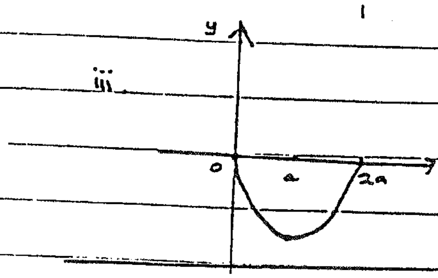
$\frac{1}{2}$ shape

$\frac{1}{2}$ asymptotes

$\frac{1}{2}$ for nothing between $[-1, 1]$

(2)

iii



$y = \int_0^x f(t)\,dt$

$-\frac{1}{2}$ each mistake

Check for:

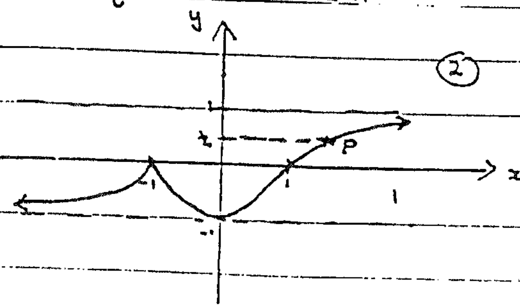
symmetry

T P at $x = a$

0 at $x=0,\ x=2a$

5. $y = \dfrac{|x+1|(x-1)}{x^2+1}$

$= \begin{cases} F(x) & \text{if } x \ge -1 \\ -F(x) & \text{if } x < -1 \end{cases}$



(2)

(b) i. $F(x) = \dfrac{x^2-1}{x^2+1}$

1. even function, $F(x) = 1 - \dfrac{2}{x^2+1}$

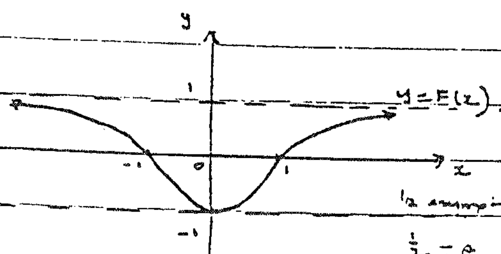
$F(x) = 0,\ x = \pm 1$; $x = 0,\ F(x) = -1$ (min value)

$-1 \le F(x) < 1$

As $x \to \infty$, $F(x) \to 1$

As $x \to -\infty$, $F(x) \to 1$

(2)



$\frac{1}{2}$ asymptotes

$\frac{1}{2}$ T P

$\frac{1}{2}$ symmetry/shape

$\frac{1}{2}$ x, y intercepts

(ii) $x^2+1 > 2|x+1|(x-1)$

$\dfrac{|x+1|(x-1)}{x^2+1} < \frac{1}{2}$ $\frac{1}{2}$

To find P,

(2)

Solve $F(x) = \frac{1}{2}$ (since $x \ge -1$)

$2(x^2-1) = x^2+1$

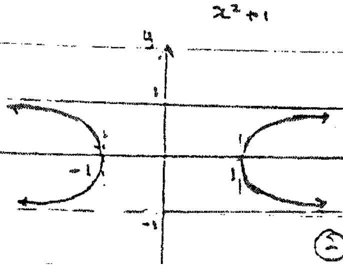
$x^2 - 3 = 0$

$x = \sqrt{3}$ $(x > 0)$

$\therefore x^2+1 > 2|x+1|(x-1)$ for all $x < \sqrt{3}$

(From graph) $\frac{1}{2}$

2. $[F(x)]^2 = \dfrac{x^2-1}{x^2+1}$



$\frac{1}{2}$ Symmetry about the x axis

$\frac{1}{2}$ Steeper than 1 / shape

$\frac{1}{2}$ Symmetric about y axis

$\frac{1}{2}$ asymptotes

(2)

5 (a) (i) $\sqrt{x}+\sqrt{y}=\sqrt{c}$

Diff. w.r.t. $x$: $\frac{1}{2}x^{-\frac{1}{2}}+\frac{1}{2}y^{-\frac{1}{2}}\frac{dy}{dx}=0$

$\frac{dy}{dx}=-\frac{\sqrt{y}}{\sqrt{x}}$

$\therefore$ At $P(a,b)$, $\frac{dy}{dx}=-\sqrt{\frac{b}{a}}$ 1

$\therefore$ Eqn. of tangent is

$y-b=-\sqrt{\frac{b}{a}}(x-a)$

$\sqrt{a}y-b\sqrt{a}=-\sqrt{b}x+a\sqrt{b}$

ie. $\sqrt{b}x+\sqrt{a}y=a\sqrt{b}+b\sqrt{a}$ 1 (5)

(ii) when $y=0$, $x=\frac{a\sqrt{b}+b\sqrt{a}}{\sqrt{b}}$

$=a+\sqrt{ab}$ $\therefore Q(a+\sqrt{ab},0)$

$x=0$, $y=\frac{a\sqrt{b}+b\sqrt{a}}{\sqrt{a}}$ 1

$=\sqrt{ab}+b$ $\therefore R(0,b+\sqrt{ab})$

$\therefore OQ=a+\sqrt{ab}$ units

$OR=b+\sqrt{ab}$ units 1

$OQ+OR=a+b+2\sqrt{ab}$

$=(\sqrt{a}+\sqrt{b})^2$ 1

$=(\sqrt{c})^2=c$ (since $a,b$ satisfy eqn of curve).

(b) (i) $7x^2+16y^2=112$

ie $\frac{x^2}{16}+\frac{y^2}{7}=1$ $\therefore a=4, b=\sqrt{7}$ 1

$b^2=a^2(1-e^2)$

$7=16(1-e^2)$

$e^2=1-\frac{7}{16}=\frac{9}{16}$

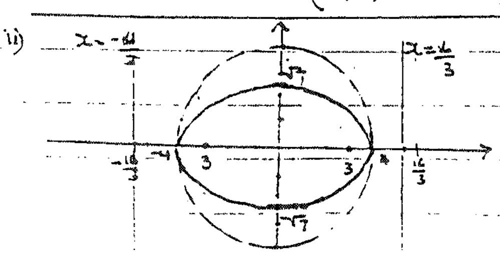
$\therefore e=\frac{3}{4}$ $(e>0)$ 1 (4)

$\therefore$ eccentricity $=\frac{3}{4}$

Directrices are $x=\pm\frac{16}{3}$ 1

Foci are $(3,0)$ and $(-3,0)$ 1

(ii)




(1)

(iii) 1. $\int_0^4\sqrt{16-x^2}\,dx$ 1

2. $\int_0^4\sqrt{\frac{112-7x^2}{16}}\,dx$ (2)

$=\frac{1}{4}\int_0^4\sqrt{112-7x^2}\,dx$

(iv) $\frac{1}{4}\int_0^4\sqrt{112-7x^2}\,dx$

$=\frac{1}{4}\int_0^4\sqrt{7}.\sqrt{16-x^2}\,dx$

$=\frac{\sqrt{7}}{4}\int_0^4\sqrt{16-x^2}\,dx$ (3)

$=\frac{b}{a}\int_0^4\sqrt{16-x^2}\,dx$ 1

$\therefore$ Area of ellipse $=\frac{\sqrt{7}}{4}\times\frac{\pi r^2}{4}\times 4$ where $r=4$.

$=\frac{\sqrt{7}}{16}\times 16\pi$ 1

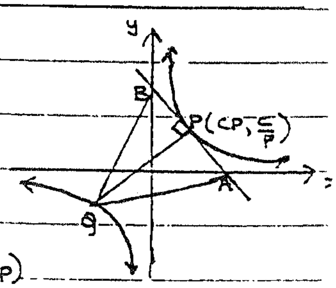
$=4\sqrt{7}\pi$ units$^2$

$\therefore$ General form for area of ellipse is:

$A=\frac{b}{a}\times\pi a^2$

$=\pi ab$ units$^2$ 1

6 (i) $xy=c^2$




$\frac{dy}{dx}=-\frac{c^2}{x^2}$

$\therefore$ At $P$, $m=-\frac{1}{p^2}$ 1

$\therefore$ Eqn of tangent at P is

$y-\frac{c}{p}=-\frac{1}{p^2}(x-cp)$

$p^2y-cp=-x+cp$ (2)

ie $x+p^2y=2cp$ 1

(ii) Eqn of normal is

$y-\frac{c}{p}=p^2(x-cp)$

$py-c=p^3x-cp^4$ (1)

ie $p^3x-py=cp^4-c$

Alternatively, Q must be of form $(cq, \frac{c}{q})$
sub. into ②

$cp^3q - c\frac{p}{q} = cp^4 - c$
which leads to
$q = -\frac{1}{p^3}$ etc.

(iii) Solve $xy = c^2$ — ①

$p^3x - py = cp^4 - c$ — ② simultaneously.

From ① $y = \frac{c^2}{x}$, sub into ②

$p^3x - \frac{pc^2}{x} = cp^4 - c$

$p^3x^2 - pc^2 = (cp^4 - c)x$

$p^3x^2 - (cp^4 - c)x - pc^2 = 0$ 1

We know $x = cp$ is one solution ∴ factorise:

$(x - cp)(p^3x + c) = 0$

∴ $x = cp$ or $x = -\frac{c}{p^3}$

$x = cp$ corr. to 1st quad. ②

$x = -\frac{c}{p^3}$ is the x value at Q.

Sub. into ②

$y = \frac{c^2 x - p^3}{c}$ 1

$= -cp^3$

∴ $Q\left(-\frac{c}{p^3}, -cp^3\right)$

(iv) Now $A(2cp, 0)$ and $B\left(0, \frac{2c}{p}\right)$

∴ $d_{AB} = \sqrt{4c^2p^2 + \frac{4c^2}{p^2}}$

$= \frac{2c}{p}\sqrt{p^4 + 1}$ 1

$d_{PQ} = \sqrt{\left(cp + \frac{c}{p^3}\right)^2 + \left(\frac{c}{p} + cp^3\right)^2}$ ③

$= \frac{c}{p^3}\sqrt{(p^4+1)^2 + p^2(1+p^4)^2}$

$= \frac{c(p^4+1)}{p^3}\sqrt{1+p^2}$ 1

∴ Area of $\Delta ABQ = \frac{1}{2} AB . PQ$

$= \frac{1}{2} \times \frac{2c}{p}\sqrt{p^4+1} \times \frac{c(p^4+1)}{p^3}\sqrt{p^4+1}$

$= \frac{c^2}{p^4}(p^4+1)^2$

$= c^2\left(p^2 + \frac{1}{p^2}\right)^2$ 1

(v) $A = c^2\left(p^2 + \frac{1}{p^2}\right)^2$

$= c^2\left[p^4 + 2 + \frac{1}{p^4}\right]$

$\frac{dA}{dp} = c^2\left(4p^3 - \frac{4}{p^5}\right]$ 1

For min., $\frac{dA}{dp} = 0$

$4p^3 = \frac{4}{p^5}$

$p^8 = 1$ ②

∴ $p = 1$. 1

$\frac{d^2A}{dp^2} = 4c^2\left[3p^2 + \frac{5}{p^6}\right]$

$= 4c^2[8]$

$> 0$ when $p = 1$,

∴ $p = 1$ corresponds to a minimum.

[i.e. when P is at the point $(c, c)$]

⑥

(b) (i) To prove: $\frac{a}{b} + \frac{b}{a} > 2$  $a \neq b$

a, b real pos. ...

Proof: $\left(\sqrt{\frac{a}{b}} - \sqrt{\frac{b}{a}}\right)^2 > 0$ 1  $a, b > 0$

∴ $\frac{a}{b} + \frac{b}{a} - 2\sqrt{\frac{a}{b}}\sqrt{\frac{b}{a}} > 0$

②

∴ $\frac{a}{b} + \frac{b}{a} > 2$ 1

(ii) To prove: $(a+b+c)\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right) > 9$

Proof: $(a+b+c)\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right)$

$= 1 + 1 + 1 + \frac{a}{b} + \frac{a}{c} + \frac{b}{a} + \frac{b}{c} + \frac{c}{a} + \frac{c}{b}$

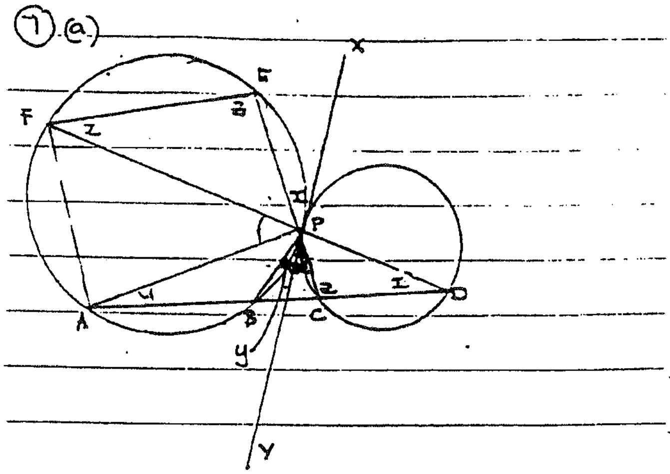
$= 3 + \frac{a}{b} + \frac{b}{a} + \frac{a}{c} + \frac{c}{a} + \frac{b}{c} + \frac{c}{b}$

$> 3 + 2 + 2 + 2 = 9$ 1

(from (i), $\frac{a}{c} + \frac{c}{a} > 2$ etc).

1

(7) (a)




(i) To Prove: $FE \parallel AD$

Proof: Let $\angle EPX = x$

then $\angle PFE = x$ (angle bet. chord & tangent equals $\angle$ in alt. segment). ½

Also $\angle YPC = x$ (vertically opp. $\angle$'s) ½

$\therefore \angle CDP = x$ ($\angle$ bet. chord & tangent equals $\angle$ in alt. segment). ½

(2)

ie $\angle EFP = \angle CDP$ ½

$\therefore FE \parallel AD$ (alternate $\angle$'s equal)

(ii) To prove: $\angle FPA = \angle BPC$

Proof. Let $\angle BPY = y$.

then $\angle BPC = x+y$.

Let $\angle FEP = z$ (3)

Then $\angle DCP = z$ (alternate $\angle$'s, $FE \parallel AD$) ½

$\therefore \angle PBC = z-(x+y)$ (exterior $\angle$ of $\Delta PBC$) ½

Also, $\angle PAB = y$ ($\angle$ bet chord & tangent equals $\angle$ in alt segment) ½

$\therefore \angle BPA = z-(x+y)-y$ (ext $\angle$ of $\Delta PBA$)

$= z-x-2y$ ½

And $\angle EPF = 180^\circ-(x+z)$ ($\angle$ sum $\Delta FEP$) ½

$\therefore \angle FPA = 180^\circ - [180^\circ-(x+z)] - [z-x-2y] - [x+y]$ (EPC is straight $\angle$) ½

$= 180 - 180 + x + z - z + x + 2y - x - y$

$= x+y = \angle BPC$

(iii) To prove: $\Delta FPA \parallel\!\mid \Delta BPC$

Proof: In $\Delta FPA$ and $\Delta BPC$,

$\angle FPA = \angle BPC$ (from (ii) above) ½

$\angle FAP = 180^\circ - z$ (opp $\angle$'s of a cyclic quad. supp.) ½ (2)

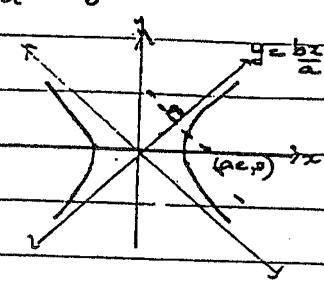
And $\angle BCP = 180^\circ - z$ (ABCD straight $\angle$)

$\therefore \angle FAP = \angle BCP$ ½

$\therefore \Delta FPA \parallel\!\mid \Delta BPC$ (equiangular) ½

(b) $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$

(i)




Line $\perp$ to $y = \frac{bx}{a}$ has gradient $-\frac{a}{b}$

$\therefore$ Eqn of perp. through focus is

$y - 0 = -\frac{a}{b}(x - ae)$

$by = -ax + a^2e$

ie $ax + by = a^2e$ 1

This meets the line $y = \frac{b}{a}x$ when

$ax + b\left(\frac{b}{a}x\right) = a^2e$

$a^2x + b^2x = a^3e$

$x = \dfrac{a^3e}{a^2+b^2}$ 1 (3)

But $b^2 = a^2(e^2-1)$

so $a^2+b^2 = a^2e^2$ 1

$\therefore x = \dfrac{a^3e}{a^2e^2} = \dfrac{a}{e}$ which is a pt on the directrix.

$\therefore$ the perp meets the asymptote on the directrix.

(ii) Angle between the asymptotes is

$\tan\theta = \left|\frac{m_1-m_2}{1+m_1m_2}\right|$ where $m_1 = \frac{b}{a}$

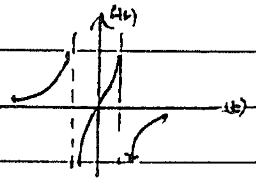
$m_2 = -\frac{b}{a}$

∴ if $t = \tan\frac{\theta}{2}$,

$\frac{2t}{1-t^2} = \frac{\frac{b}{a}+\frac{b}{a}}{1-\frac{b^2}{a^2}}$

∴ $\frac{2t}{1-t^2} = \frac{2\left(\frac{b}{a}\right)}{1-\left(\frac{b}{a}\right)^2}$ 1

∴ $t = \frac{b}{a}$ 1




(since $f(t) = \frac{2t}{1-t^2}$ is a function, only one value of f(t) for each t.)

But $b^2 = a^2(e^2-1)$

∴ $\frac{b}{a} = \sqrt{e^2-1}$ ($\frac{b}{a}>0$)

5

∴ $\tan\frac{\theta}{2} = \sqrt{e^2-1}$ 1

∴ $\frac{\theta}{2} = \tan^{-1}\sqrt{e^2-1}$

∴ $\theta = 2\tan^{-1}\sqrt{e^2-1}$. 1

8 (a) $(k^2+l^2)x^2 + 2l(k+m)x + (l^2+m^2) = 0$

is to have real roots. ∴ $\Delta \ge 0$

$\Delta = 4l^2(k+m)^2 - 4(l^2+m^2)(k^2+l^2)$

$= 4l^2(k^2+2km+m^2)$ ½

$- 4(l^2k^2 + l^4 + m^2k^2 + m^2l^2)$

$= 4l^2k^2 + 8l^2km + 4\cancel{l^2m^2}$

$- 4l^2k^2 - 4l^4 - 4m^2k^2 - 4m^2l^2$ ½

$= -4(l^4 - 2l^2km + m^2k^2)$

$= -4(l^2-mk)^2$ ½

If $\Delta \ge 0$, $-4(l^2-mk)^2 \ge 0$ ½

∴ $(l^2-mk)^2 \le 0$

3

But $(l^2-mk)^2 \ge 0$ ½ as this is a square. ∴ Only possible value is

$l^2 - mk = 0$ i.e. $l^2 = mk$

(b) To prove: $2|a-b| > |a|$ if $|a| > 2|b|$

Proof: $2|a-b| = 2\left|a\left(1-\frac{b}{a}\right)\right|$ ½

$= 2|a|\left|1-\frac{b}{a}\right|$ ½

$\ge 2|a|\left[1-\left|\frac{b}{a}\right|\right]$ ½

(Since $|a-b| \ge |a|-|b|$) ½ $a \ne b$.

∴ $2|a-b| > 2|a|\left[1-\frac{1}{2}\right]$ ½

(since $\left|\frac{b}{a}\right| < \frac{1}{2}$)

$= |a|$ ½

∴ $2|a-b| > |a|$.

(c) i) $\cos^2x = \frac{\cos2x+1}{2}$

$\cos^4x = \frac{1}{4}(1+\cos2x)^2$ 1

$= \frac{1}{4}(1+2\cos2x+\cos^22x)$

$= \frac{1}{4}\left[1+2\cos2x+\frac{\cos4x+1}{2}\right]$ 1

∴ $\int_0^{\frac{\pi}{2}}\cos^4x\,dx = \frac{1}{4}\int_0^{\frac{\pi}{2}}\left(\frac{3}{2}+2\cos2x+\frac{\cos4x}{2}\right)dx$ 3

$= \left[\frac{3x}{8}\right]_0^{\frac{\pi}{2}} + 0 + 0$ (these graphs are symmetrical about the x axis for $0<x<\frac{\pi}{2}$)

$= \frac{3\pi}{16}$ 1

ii) To prove:

$3(\cos^4x+\sin^4x) - 2(\cos^6x+\sin^6x) = 1$

L.S $= \cos^4x+\sin^4x+2\cos^4x+2\sin^4x$

$-2\cos^6x-2\sin^6x$

$= \cos^4x+\sin^4x+2\cos^4x(1-\cos^2x)$

$+2\sin^4x(1-\sin^2x)$ 1

$= \cos^4x+\sin^4x+2\cos^4x\sin^2x$

$+2\sin^4x\cos^2x$ 2

$= \cos^4x+\sin^4x+2\sin^2x\cos^2x(\cos^2x+\sin^2x)$

$= \cos^4x+\sin^4x+2\sin^2x\cos^2x$

$= (\cos^2x+\sin^2x)^2 = 1$ = R.S.

(cos π – – – 1

(iii) $\cos(\frac{\pi}{2} - x) = \sin x$ ∴ $y = \cos x$ and $y = \sin x$ are reflections of each other in the line $x = \frac{\pi}{4}$


$\therefore \int_0^{\frac{\pi}{2}} \cos x \, dx = \int_0^{\frac{\pi}{2}} \sin x \, dx$ (1)

Similarly $y = \cos^n x$ and $y = \sin^n x$ must be reflections of each other in the line $y = \frac{\pi}{4}$

$\therefore \int_0^{\frac{\pi}{2}} \cos^n x \, dx = \int_0^{\frac{\pi}{2}} \sin^n x \, dx$

(iv)

$$3\int_0^{\frac{\pi}{2}} (\cos^4 x + \sin^4 x)\,dx - 2\int_0^{\frac{\pi}{2}} (\cos^6 x + \sin^6 x)\,dx = \frac{\pi}{2}$$

But $\int_0^{\frac{\pi}{2}} \sin^4 x \, dx = \int_0^{\frac{\pi}{2}} \cos^4 x \, dx$ and similarly for $\int_0^{\frac{\pi}{2}} \sin^6 x \, dx$

$$\therefore 3\int_0^{\frac{\pi}{2}} 2\cos^4 x \, dx - 2\int_0^{\frac{\pi}{2}} 2\cos^6 x \, dx = \frac{\pi}{2} \quad 1$$

∴ From (i)

$$6\left(\frac{3\pi}{16}\right) - 4\int_0^{\frac{\pi}{2}} \cos^6 x \, dx = \frac{\pi}{2}$$

$$\therefore 4\int_0^{\frac{\pi}{2}} \cos^6 x \, dx = \frac{9\pi}{8} - \frac{\pi}{2} \quad (2)$$

$$= \frac{5\pi}{8} \quad 1$$

$$\therefore \int_0^{\frac{\pi}{2}} \cos^6 x \, dx = \frac{5\pi}{32}.$$

(v) Since $0 \le \sin x < 1$ for $0 < x < \frac{\pi}{2}$

then $\sin^n x \times \sin x < \sin^n x$

ie $\sin^{n+1} x < \sin^n x$ for all $0 < x < \frac{\pi}{2}$ (1)

(for all +ve integers $n$).

$$\therefore \int_0^{\frac{\pi}{2}} \sin^{n+1} x \, dx < \int_0^{\frac{\pi}{2}} \sin^n x \, dx$$

for all +ve integers $n$.