MORIAH COLLEGE

Year 12

2008 EXTENSION 2 MATHEMATICS

ASSESSSMENT TASK 2 (PRE-TRIAL)

Time Allowed: 3 hours plus 5 minutes reading time.

Examiners: Greg Wagner, Evie Apfelbaum

Instructions:

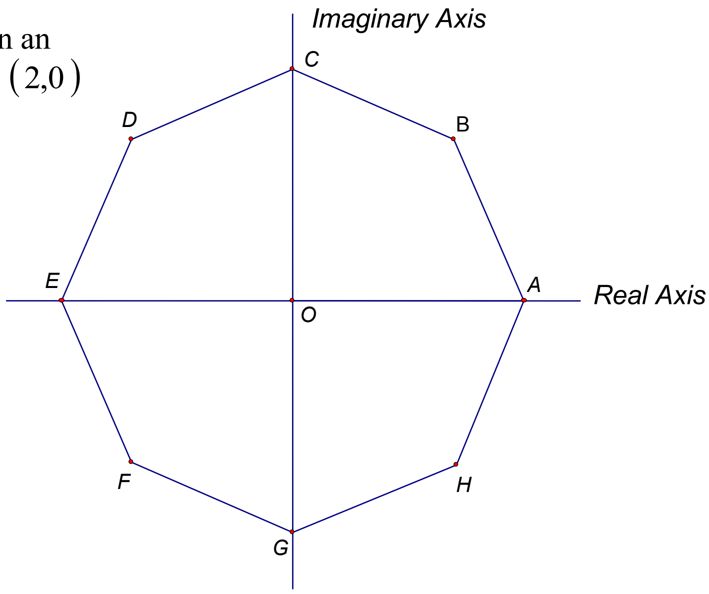
- All necessary working should be shown in every question.
- Marks may be deducted for careless or badly arranged work.
- Standard integrals are at the end of the paper.

Question 1 (15 marks)

a) Show that $\int x\sin x\,dx = \sin x - x\cos x$ **(2)**

b) Hence, find the exact value of $\int_0^{\frac{\pi}{2}} x^2 \cos x\,dx$ **(3)**

c) *ABCDEFGH* is a regular octagon drawn on an Argand diagram with vertex A at the point $(2,0)$
The origin O is the centre of the octagon.



i) Find, in the form $r\text{cis}\theta$, the complex number ω that represents the point B. **(1)**

ii) Hence, express ω in the form $x+iy$ **(1)**

iii) If B is represented by the complex number ω, then which points are represented by the following complex numbers:

α) $i\omega$ **(1)**

β) $-\omega$ **(1)**

iv) Use ΔEBO to explain why $\arg(w+2)=\frac{\pi}{8}$ **(1)**

v) Find the exact value of $|w+2|^2$ **(2)**

vi) Hence, find the exact value of $\cos^2\frac{\pi}{8}$ **(2)**

vii) Write down a polynomial equation whose real and complex solutions are represented by the 8 vertices of the octagon. **(1)**

Question 2 (15 marks)

a)

i) By expanding $(cis\theta)^3$ in two different ways, show that $\sin 3\theta = 3\sin\theta - 4\sin^3\theta$ **(2)**

ii) Hence, find $\int \sin 3\theta \cos\theta \, d\theta$ **(2)**

b)

i) Find B and C such that $\dfrac{4x-3}{(x-1)(x-3)} = \dfrac{B}{(x-1)} + \dfrac{C}{(x-3)}$ **(2)**

ii) Hence, find $\int \dfrac{4x-3}{x^2-4x+3}\,dx$ **(2)**

c) Use the substitution $t = \tan\dfrac{x}{2}$ to evaluate $\displaystyle\int_0^{\frac{\pi}{2}} \dfrac{1}{2+\cos x}\,dx$ **(3)**

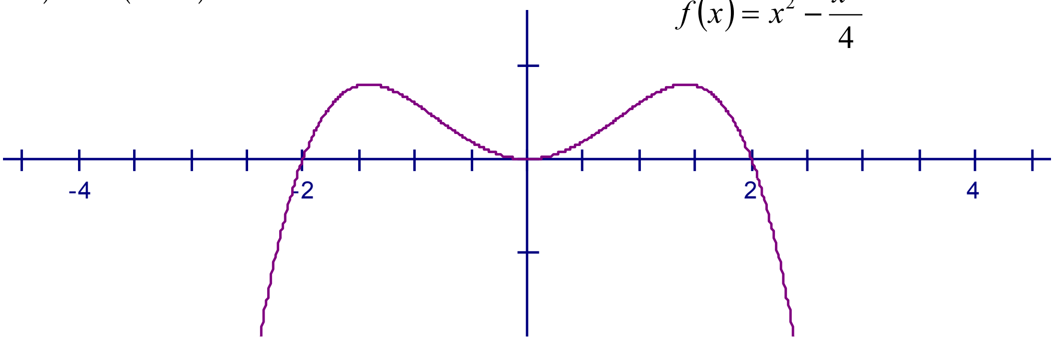
d)

i) Show that $\displaystyle\int \dfrac{1}{x\sqrt{x^2-1}}\,dx = \cos^{-1}\left(\dfrac{1}{x}\right) + c$ using the substitution $x = \sec\theta$ **(2)**

ii) Hence, show that $\displaystyle\int_2^3 \dfrac{1}{x\sqrt{x^2-1}}\,dx = \cos^{-1}\left(\dfrac{1+2\sqrt{6}}{6}\right)$ **(2)**

Question 3 (15 marks)

a) The graph below shows the curve $f(x) = x^2 - \frac{x^4}{4}$. The maximum turning points are at $\left(-\sqrt{2}, 1\right)$ and $\left(\sqrt{2}, 1\right)$



Sketch the graphs of the following on separate number planes.
Show the coordinates of all stationary points on the graphs of i), ii) and iii).
There is no need to show the stationary points on iv).

i) $y = \frac{1}{f(x)}$ **(2)**

ii) $y = [f(x)]^2$ **(2)**

iii) $y = \tan^{-1}(f(x))$ **(2)**

iv) $y = xf(x)$ **(2)**

b) Consider the curve given implicitly by the equation $4y^3 - 3y = x^3 + x - 1$

i) Use implicit differentiation to show that $\frac{dy}{dx} = \frac{3x^2 + 1}{12y^2 - 3}$ **(2)**

ii) Find the equation of the tangent to $4y^3 - 3y = x^3 + x - 1$ at the point $(0, -1)$ **(2)**

iii) The curve intersects or meets the y-axis at one other point. **(3)**
Find the equation of the tangent at this point.

Question 4 (15 marks)

a) The complex number z satisfies the following equations:

$$|z| = 1 \text{ and } \arg z = \frac{\pi}{6}$$

i) Find the quadratic equation for which z and $\bar{z}$ are both roots. **(2)**

ii) If the points represented by the complex numbers z, 0 and ω form an equilateral triangle, then find all complex numbers ω. **(2)**

b) Consider the function $f(x) = \dfrac{1}{\sqrt{(x-1)(x-3)}}$

i) Find the domain of $f(x)$ **(1)**

ii) Find $\displaystyle\int \frac{1}{\sqrt{(x-1)(x-3)}}\,dx$ **(2)**

c) The equation $x^3 - 2x^2 + bx - 16 = 0$ has roots α, β and γ such that $\alpha\beta = 8$.

i) Show that $b = 8$ **(2)**

ii) Hence, find α and β. **(2)**

iii) Find the cubic equation with roots given by α^2, β^2 and γ^2 **(2)**

iv) Hence find the value of $\alpha^3 + \beta^3 + \gamma^3$. **(2)**

Question 5 (15 marks)

a) Consider the function $f(x) = \dfrac{x}{\log_e x}$

i) Find the equation of the vertical asymptote of $y = f(x)$ **(1)**

ii) Show that the coordinates of the minimum turning point are (e, e) **(2)**

iii) Find $\lim\limits_{x \to 0^+} f'(x)$ **(1)**

iv) Sketch the curve $y = \dfrac{x}{\log_e x}$ **(2)**

v) Explain using your graph and part ii), why $\dfrac{e}{\log e} < \dfrac{\pi}{\log \pi}$. **(2)**

vi) Hence, show that $\pi^e < e^\pi$. **(2)**

b) The curve $y = x^3 - 2x^2 + x$ intersects the line $y = k$ at three distinct points *A*, *B* and *C*. The *x*-coordinates of these points are α, β and γ respectively.

i) By considering the stationary points of $y = x^3 - 2x^2 + x$, find the possible values of k. **(2)**

ii) Find the value of k such that *B* is the midpoint of *AC*. **(2)**

iii) Use symmetry to explain why *B* is the inflexion point of $y = x^3 - 2x^2 + x$. **(1)**

Question 6 (15 marks)

a) Factorise the cubic polynomial $z^3 + 8$

i) over the real field of numbers. **(1)**

ii) over the complex field of numbers. **(2)**

b) Let w be one of the complex roots of the equation $z^3 + 8 = 0$.

i) Using part a), show that $w^2 + 4 = 2w$ **(1)**

ii) Hence simplify $(w^2 + 4)^3$. **(1)**

c) Copy and complete the following in order to write $\frac{x}{\sqrt{x^2}}$ in piecewise form. **(1)**

$$\frac{x}{\sqrt{x^2}} = \begin{cases} 1 & \text{if } \ 0 < x \le 1 \\ & \text{if } \ -1 \le x < 0 \\ & \text{if } \ x = 0 \end{cases}$$

d) Consider the even function $f(x) = \sin^{-1}\sqrt{1 - x^2}$ where $-1 \le x \le 1$

i) Use part c) to find $f'(x)$ in piecewise form, carefully defining its domain **(3)**

ii) The curve $y = f(x)$ has a cusp at the point where its two branches meet the y-axis. Show that these two branches meet at $90°$. **(2)**

iii) Carefully sketch the curve $y = f(x)$ **(2)**

iv) The region bounded by the curve $y = f(x)$ and the x-axis is rotated through $180°$ about the y-axis to form a solid of revolution. Find the volume of this solid. **(2)**

Question 7 (15 marks)

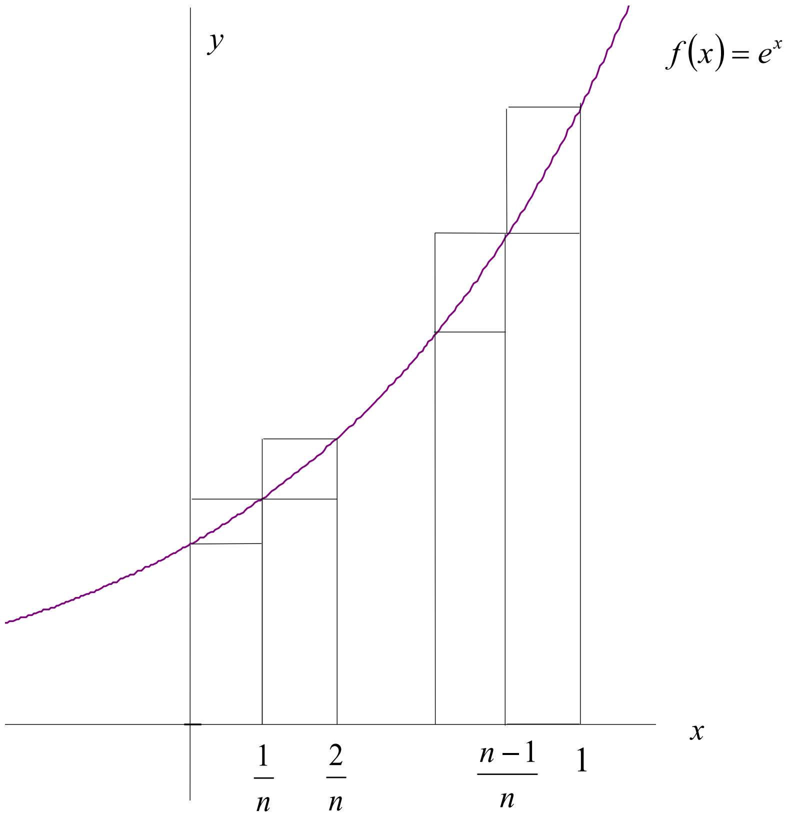
a) Consider the polynomial $P(x) = (x+2)^2 Q(x) + R(x)$

i) Explain why $R(x)$ is a linear polynomial **(1)**

ii) When $P(x)$ and $P'(x)$ are both divided by $(x+2)$, the remainder in each case is 6. Find $R(x)$ **(3)**

Question 7 continued on next page

b) The diagram below shows the curve $f(x) = e^x$.
The area bounded by the curve, the coordinate axes and the line $x = 1$ can be approximated using upper and lower rectangles.
Some of these rectangles have been drawn on the diagram.
There are n upper rectangles and n lower rectangles, each of width $\frac{1}{n}$.



Let A_k be the area of the rectangle with height $e^{\frac{k}{n}}$, where $0 \le k \le n$ and k is an integer.

i) Show that the sum of the upper rectangles can be given by $\sum_{k=1}^{k=n} A_k = \frac{1}{n}\left[\frac{e^{\frac{1}{n}}(e-1)}{e^{\frac{1}{n}}-1}\right]$ **(3)**

ii) Find the exact value of the area. **(1)**

iii) Use parts i) and ii) to show that $e^{\frac{1}{n}} < \frac{n}{n-1}$ **(3)**

iv) Use the sum of the lower rectangles and part ii) to show that $\frac{n+1}{n} < e^{\frac{1}{n}}$ **(2)**

v) If $\left(\frac{11}{10}\right)^p < e^{300} < \left(\frac{10}{9}\right)^p$, then find an integer value of p. **(2)**

Question 8 (15 marks)

a) i) Use factorisation to show that $\left(1-x^2\right)^{n-1}-\left(1-x^2\right)^n=x^2\left(1-x^2\right)^{n-1}$ **(2)**

ii) If $I_n=\int_0^1\left(1-x^2\right)^n dx$ for $n\geq 0$, then use integration by parts to show that **(3)**

$$I_n=\frac{2n}{2n+1}I_{n-1} \text{ for } n\geq 1$$

b) i) Sketch the function $f(x)=\frac{e^x+e^{-x}}{2}$ and the line $y=x$ on the same set of axes. **(2)**

ii) Find the largest domain containing positive values of x such that $f(x)$ has an inverse function $f^{-1}(x)$. **(1)**

iii) Sketch the inverse function $y=f^{-1}(x)$ on the same diagram as part i) **(1)**

iv) Find the inverse function $f^{-1}(x)$ **(3)**

v) Find the coordinates of the point P on the curve $f(x)=\frac{e^x+e^{-x}}{2}$ such that the distance between $f(x)$ and $f^{-1}(x)$ is a minimum. **(3)**

STANDARD INTEGRALS

$$\int x^n \, dx = \frac{1}{n+1} x^{n+1}, \quad n \neq -1; x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x} \, dx = \ln x \ , \quad x > 0$$

$$\int e^{ax} \, dx = \frac{1}{a} e^{ax}, \quad a \neq 0$$

$$\int \cos ax \, dx = \frac{1}{a} \sin ax, \quad a \neq 0$$

$$\int \sin ax \, dx = -\frac{1}{a} \cos ax, \quad a \neq 0$$

$$\int \sec^2 ax \, dx = \frac{1}{a} \tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax \, dx = \frac{1}{a} \sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2 + x^2} \, dx = \frac{1}{a} \tan^{-1} \frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2 - x^2}} \, dx = \sin^{-1} \frac{x}{a}, \quad a > 0, \ -a < x < a$$

$$\int \frac{1}{\sqrt{x^2 - a^2}} \, dx = \ln\left(x + \sqrt{x^2 - a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2 + a^2}} \, dx = \ln\left(x + \sqrt{x^2 + a^2}\right)$$

Note $\ln x = \log_e x, \quad x > 0$

QUESTION 1

a) $\frac{d}{dx}\left(\sin x - x\cos x + c\right) = \cos x - (\cos x - x\sin x)$ ✓✓

$= x\sin x$

$\therefore \int x\sin x\,dx = \sin x - x\cos x$

(or use integration by parts)

b) let $u = x^2$ and $v' = \cos x$

$du = 2x$ $v = \sin x$ ✓

$I = \left[x^2\sin x\right]_0^{\frac{\pi}{2}} - 2\int_0^{\pi/2} x\sin x\,dx$ ✓

$= \frac{\pi^2}{4} - 2\left[\sin x - x\cos x\right]_0^{\frac{\pi}{2}}$ ✓

$= \frac{\pi^2}{4} - 2$ ✓

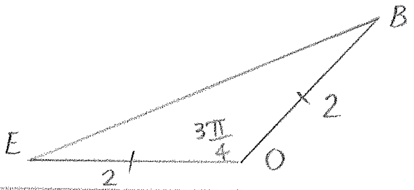
c) i) $w = 2\operatorname{cis}\frac{\pi}{4}$ ✓

ii) $w = \sqrt{2} + i\sqrt{2}$ ✓

iii) α) rotation through $\frac{\pi}{2}$ gives D ✓

β) F ✓

iv)



$\arg(w+2) = \angle BEO = \angle EBO$ ✓

Hence, $\angle BEO = \frac{\pi}{8}$ ($\angle$ sum of ΔEBO)

v) $EB^2 = |w+2|^2 = 2^2 + 2^2 - 2(2)(2)\cos\frac{3\pi}{4} = 8 + 4\sqrt{2} = 4(2+\sqrt{2})$ ✓✓

vi) $\cos\frac{\pi}{8} = \frac{EB^2 + EO^2 - BO^2}{2\,EB\cdot EO} = \frac{8+4\sqrt{2}}{4\,EB} = \frac{2+\sqrt{2}}{EB}$

$\therefore \cos^2\frac{\pi}{8} = \frac{(2+\sqrt{2})^2}{EB^2} = \frac{(2+\sqrt{2})^2}{4(2+\sqrt{2})} = \frac{2+\sqrt{2}}{4}$ ✓✓

vii) $z^8 = 2^8$

$z^8 - 256 = 0$ ✓

QUESTION 2

a) i) $(\text{cis}\,\theta)^3 = \cos^3\theta + 3i\cos^2\theta\sin\theta - 3\cos\theta\sin^2\theta - i\sin^3\theta$

$= \cos^3\theta - 3\cos\theta\sin^2\theta + i(3\cos^2\theta\sin\theta - \sin^3\theta)$ ✓

Also $(\text{cis}\,\theta)^3 = \cos 3\theta + i\sin 3\theta$ (De Moivre's Theorem) ✓

Equating imaginary parts $\sin 3\theta = 3\sin\theta(1-\sin^2\theta) - \sin^3\theta$ ✓

$= 3\sin\theta - 4\sin^3\theta$

---

ii) $\int 3\sin\theta\cos\theta - 4\sin^3\theta\cos\theta \, d\theta = \dfrac{3\sin^2\theta}{2} - \sin^4\theta + c$ ✓✓

---

b) i) $4x - 3 \equiv B(x-3) + C(x-1)$

$x = 1 \longrightarrow -2B = 1 \longrightarrow B = -\frac{1}{2}$ ✓

$x = 3 \longrightarrow 2C = 9 \longrightarrow C = \frac{9}{2}$ ✓

---

ii) $I = \int \dfrac{9}{2(x-3)} - \dfrac{1}{2(x-1)} \, dx = \frac{9}{2}\log|x-3| - \frac{1}{2}\log|x-1|$ ✓

$= \log \dfrac{|x-3|^{9/2}}{|x-1|^{1/2}} + c$ ✓

) i) $t = \tan\frac{x}{2}$

$dt = \frac{1}{2}\sec^2\frac{x}{2}\,dx$

$\dfrac{2dt}{1+t^2} = dx$

$x = 0,\ t = 0$

$x = \frac{\pi}{2},\ t = 1$ ✓

$I = \int_0^1 \dfrac{1}{2 + \frac{1-t^2}{1+t^2}} \cdot \dfrac{2dt}{1+t^2}$

$= \int_0^1 \dfrac{2}{3+t^2}\,dt$ ✓

$= \dfrac{2}{\sqrt{3}}\left[\tan^{-1}\dfrac{t}{\sqrt{3}}\right]_0^1$

$= \dfrac{\sqrt{3}\pi}{9}$ ✓

') i) $x = \sec\theta$

$dx = \sec\theta\tan\theta\,d\theta$

$\dfrac{dx}{x\sqrt{x^2-1}} = d\theta$ ✓

$I = \int d\theta = \theta + c$ ✓ $= \cos^{-1}\left(\frac{1}{x}\right) + c$

since $\frac{1}{x} = \cos\theta$ and $\theta = \cos^{-1}\left(\frac{1}{x}\right)$ ✓

---

ii) $I = \cos^{-1}\left(\frac{1}{3}\right) - \cos^{-1}\left(\frac{1}{2}\right)$

Let $\alpha = \cos^{-1}\left(\frac{1}{3}\right)$, $\beta = \cos^{-1}\left(\frac{1}{2}\right)$

$\cos\alpha = \frac{1}{3}$, $\cos\beta = \frac{1}{2}$ ✓

$\sin\alpha = \frac{2\sqrt{2}}{3}$, $\sin\beta = \frac{\sqrt{3}}{2}$

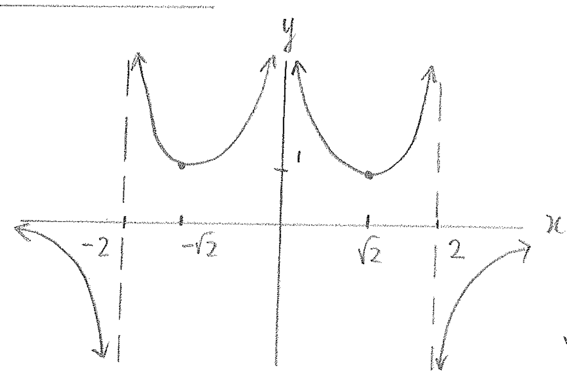
$\cos(\alpha - \beta) = \frac{1}{3}\times\frac{1}{2} + \frac{2\sqrt{2}}{3}\times\frac{\sqrt{3}}{2}$ ✓

$= \dfrac{1+2\sqrt{6}}{6}$

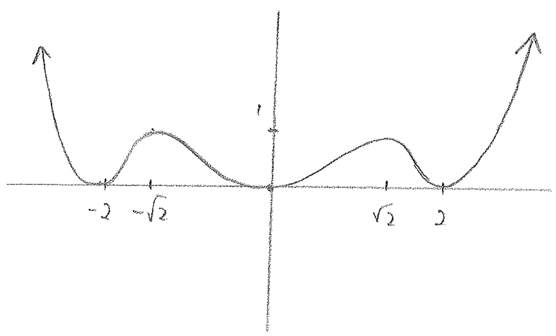
$\alpha - \beta = \cos^{-1}\left(\dfrac{1+2\sqrt{6}}{6}\right)$

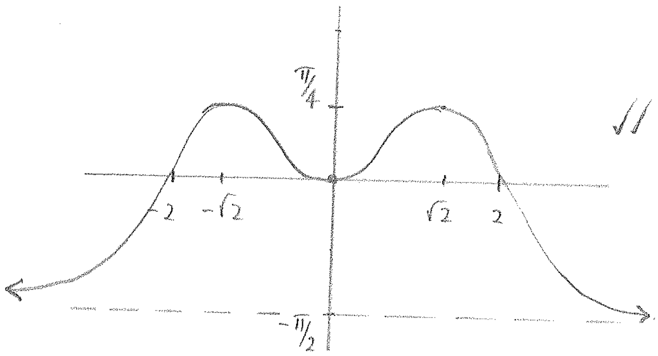
QUESTION 3

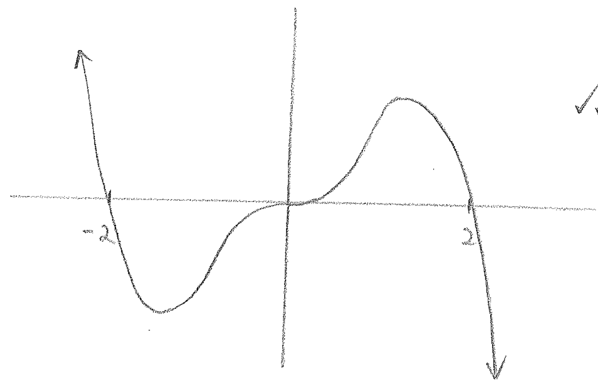
i)



ii)







b) i) $12y^2 \frac{dy}{dx} - 3\frac{dy}{dx} = 3x^2 + 1 \longrightarrow \frac{dy}{dx} = \frac{3x^2+1}{12y^2-3}$

ii) at $(0,-1)$ $\frac{dy}{dx} = \frac{1}{9}$

$y + 1 = \frac{1}{9}(x - 0) \longrightarrow x - 9y - 9 = 0$

iii) when $x = 0$, $4y^3 - 3y + 1 = 0 \ldots (1)$

from ii) we know $y = -1$ is a root of (1)

$\therefore 4y^3 - 3y + 1 = (y+1)(4y^2 - 4y + 1)$

$= (y+1)(2y-1)^2$

$\therefore$ Double root when $y = \frac{1}{2}$, $x = 0$

$\therefore$ Vertical tangent at $(0, \frac{1}{2})$

Equation of this tangent is $x = 0$ (y-axis)

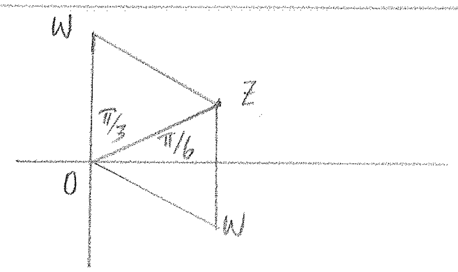
QUESTION 4

a) i) $z = \text{cis}\,\frac{\pi}{6}$ and $\bar{z} = \text{cis}\left(-\frac{\pi}{6}\right)$ ✓

$z + \bar{z} = 2\cos\frac{\pi}{6} = \sqrt{3}$ and $z\bar{z} = \text{cis}\,0 = 1$ ✓

$\therefore$ z and $\bar{z}$ are roots of $x^2 - \sqrt{3}x + 1 = 0$ ✓

ii)



$w = \bar{z} = \text{cis}\left(-\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2} - \frac{i}{2}$ ✓

OR

$w = i$ ✓

b) i) $(x-1)(x-3) > 0 \longrightarrow D : \{x < 1 \text{ OR } x > 3\}$

ii) $I = \int \frac{1}{\sqrt{(x-2)^2 - 1}}\,dx = \log\left| x - 2 + \sqrt{(x-2)^2 - 1} \right| + c$ ✓✓

c) i) $\alpha\beta\gamma = \frac{-d}{a} = 16 \longrightarrow 8\gamma = 16 \longrightarrow \gamma = 2$ ✓

sub. $\gamma = 2 \longrightarrow 8 - 8 + 2b - 16 = 0 \longrightarrow b = 8$ ✓

ii) $x^3 - 2x^2 + 8x - 16 = 0$

$(x^2 + 8)(x - 2) = 0$

$\therefore x = 2,\ \pm 2\sqrt{2}i$ $\longrightarrow$ $\alpha = 2\sqrt{2}i$ ✓, $\beta = -2\sqrt{2}i$ ✓

iii) Roots are $\alpha^2, \beta^2, \gamma^2 = -8, -8, 4$ ✓

$\therefore$ Equation is $(x+8)^2(x-4) = 0$

ie $x^3 + 12x^2 - 256 = 0$ ✓

iv) $\alpha^3 + \beta^3 + \gamma^3 = 2(\alpha^2 + \beta^2 + \gamma^2) - 8(\alpha + \beta + \gamma) + 48$ ✓

$= 2(-12) - 8(2) + 48$ ✓

$= 8$

OR simply $2^3 + (2\sqrt{2}i)^3 + (-2\sqrt{2}i)^3 = 8$

Question 5

a) i) $x = 1$ ✓

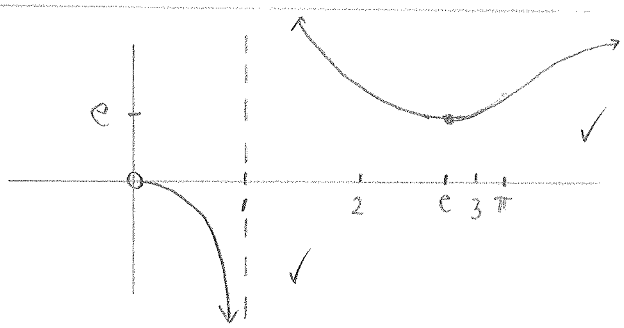
ii) $f'(x) = \dfrac{\log_e x - 1}{(\log_e x)^2}$ and $f'(e) = \dfrac{\log_e e - 1}{(\log_e e)^2} = 0$ ✓

x	2	e	3
$f'(x)$	$-$	0	$+$

and $f(e) = \dfrac{e}{\log_e e} = e \rightarrow (e, e)$ is a min turn. pt. ✓

iii) $f'(x) = \dfrac{1}{\log_e x} - \dfrac{1}{(\log_e x)^2} \longrightarrow$ as $x \to 0^+$, $f'(x) \longrightarrow 0^-$ ✓

iv)



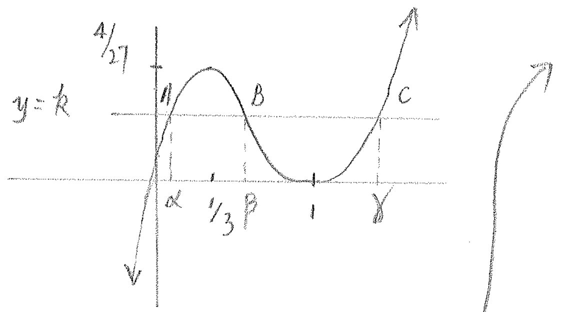
✓ ✓

v) since (e, e) is a min turn. pt then $f(e) < f(\pi)$ ✓

$\therefore \dfrac{e}{\log_e e} < \dfrac{\pi}{\log_e \pi}$ ✓

vi) $e \log \pi < \pi \log e \longrightarrow \log \pi^e < \log e^\pi \longrightarrow \pi^e < e^\pi$ ✓✓

b) i)



$y = x(x-1)^2$

and $\dfrac{dy}{dx} = 3x^2 - 4x + 1$

$\dfrac{dy}{dx} = 0 \longrightarrow (3x-1)(x-1) = 0$ ✓

stat. at $\left(\frac{1}{3}, \frac{4}{27}\right)$ and $(1, 0)$

$\therefore$ for three distinct points then $0 < k < \dfrac{4}{27}$ ✓

ii) B is the midpoint of AC $\longrightarrow \beta = \dfrac{\alpha + \gamma}{2} \longrightarrow \alpha + \gamma = 2\beta$

Now α, β, γ are roots of $x^3 - 2x^2 + x - k = 0$ .. ① ✓

Sum of roots $= 3\beta = 2 \longrightarrow \beta = \dfrac{2}{3}$

Substitution in ① gives $k = \dfrac{2}{27}$ ✓

iii) All cubics have point symmetry, $y = k$ is a horizontal line

If $AB = BC$ then C reflects $180°$ to A around B.

Hence, symmetry about B and B is the inflexion point ✓

Question 6

a) i) $(z+2)(z^2-2z+4)$ ✓

ii) $(z+2)\left[(z-1)^2-(\sqrt{3}i)^2\right] = (z+2)(z-1-\sqrt{3}i)(z-1+\sqrt{3}i)$ ✓✓

b) i) $z^3+8=0 \longrightarrow z=-2$ OR $z^2-2z+4=0$

since w is complex $\longrightarrow w^2-2w+4=0$ ✓

$w^2+4=2w$

ii) $(w^2+4)^3 = (2w)^3 = 8w^3 = 8\times-8 = -64$ ✓

c) $\dfrac{x}{\sqrt{x^2}} = \begin{cases} 1 & \text{if } 0<x\le 1 \\ -1 & \text{if } -1\le x<0 \\ \text{undefined} & \text{if } x=0 \end{cases}$ ✓

d) i) $f'(x) = \dfrac{1}{\sqrt{1-(1-x^2)}}\cdot\dfrac{-x}{\sqrt{1-x^2}}$ ✓

$= \dfrac{-x}{\sqrt{x^2}\sqrt{1-x^2}}$

$= \begin{cases} \dfrac{-1}{\sqrt{1-x^2}} & \text{if } 0<x<1 \\ \dfrac{1}{\sqrt{1-x^2}} & \text{if } -1<x<0 \\ \text{und.} & \text{if } x=0,\pm1 \end{cases}$ ✓ ✓

ii) as $x\to0^-$, $f'(x)\to1$ ✓

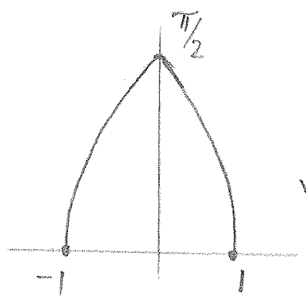
as $x\to0^+$, $f'(x)\to-1$

$f(0)=\sin^{-1}(1)=\frac{\pi}{2}$

Since at $(0,\frac{\pi}{2})$ ✓

$m_1\times m_2=-1$ then branches meet at 90°

iii)



vertical tangents at $(\pm1,0)$ ✓

✓

iv) $V = \pi\int_0^{\pi/2}\cos^2 y\,dy$ ✓

$= \frac{\pi}{2}\int_0^{\pi/2} 1+\cos 2y\,dy$ ✓

$= \frac{\pi}{2}\left[y+\frac{1}{2}\sin 2y\right]_0^{\frac{\pi}{2}}$

$V = \frac{\pi}{2}\times\frac{\pi}{2}$

$V = \frac{\pi^2}{4}$ units^3 ✓

Question 7

a) i) $P(x)$ is divided by a quadratic polynomial. Hence, the remainder $R(x)$ is linear, since $\deg(R(x)) < \deg(x+2)^2$. ✓

ii) $P(x) = (x+2)^2 Q(x) + Ax + B$ from i)

$P'(x) = (x+2)^2 Q'(x) + 2(x+2)Q(x) + A$ ✓

Now $P'(-2) = A = 6$ and $P(-2) = 6(-2) + B = 6 \rightarrow B = 18$ ✓

$\therefore R(x) = 6x + 18$ ✓

b) i) $A_1 = \frac{1}{n} e^{\frac{1}{n}}$, $A_2 = \frac{1}{n} e^{\frac{2}{n}}$, $A_3 = \frac{1}{n} e^{\frac{3}{n}}$ ✓

Now $\frac{A_2}{A_1} = \frac{A_3}{A_2} = e^{\frac{1}{n}}$ and the series is geometric

$\therefore A_1 + A_2 + \ldots + A_n = \frac{a(r^n - 1)}{r-1}$ where $a = A_1$, $r = e^{\frac{1}{n}}$ ✓

i.e. $\sum_{k=1}^{k=n} A_k = \frac{\frac{1}{n} e^{\frac{1}{n}}\left[(e^{\frac{1}{n}})^n - 1\right]}{e^{1/n} - 1}$ ✓

$= \frac{1}{n}\left[\frac{e^{1/n}(e-1)}{e^{1/n} - 1}\right]$

ii) $A = \int_0^1 e^x \, dx = \left[e^x\right]_0^1 = e - 1$ ✓

iii) $e - 1 < \frac{1}{n}\left[\frac{e^{1/n}(e-1)}{e^{1/n} - 1}\right]$ ✓ $\rightarrow n(e^{\frac{1}{n}} - 1) < e^{1/n}$ ✓ $\rightarrow$

$\rightarrow ne^{1/n} - e^{1/n} < n \rightarrow e^{1/n}(n-1) < n$ ✓ $\rightarrow e^{1/n} < \frac{n}{n-1}$

iv) Sum of lower rectangles $= A_0 + A_1 + \ldots + A_{n-1}$ where $A_0 = \frac{1}{n}$

Now $\sum_{k=0}^{n-1} A_k = \frac{1}{n}\left[\frac{1(e-1)}{e^{1/n} - 1}\right] < e - 1$ ✓ using ii) and i)

i.e. $\frac{1}{n} < e^{1/n} - 1$ ✓ $\rightarrow \frac{n+1}{n} < e^{1/n}$

v) $\frac{n+1}{n} < e^{1/n} < \frac{n}{n-1}$ ✓

then $n = 10 \rightarrow \frac{11}{10} < e^{1/10} < \frac{10}{9}$ $\rightarrow$ $\left(\frac{11}{10}\right)^{3000} < e^{300} < \left(\frac{10}{9}\right)^{3000}$

$\therefore p = 3000$ ✓

Question 8

a) i) $L.H.S. = (1-x^2)^{n-1}\left[1-(1-x^2)\right]$ ✓

$= x^2(1-x^2)^{n-1}$ ✓

$= RHS$

ii) $I_n = \left[x(1-x^2)^n\right]_0^1 + 2n\int_0^1 x^2(1-x^2)^{n-1}$ ✓

let $u = (1-x^2)^n$; $v' = 1$

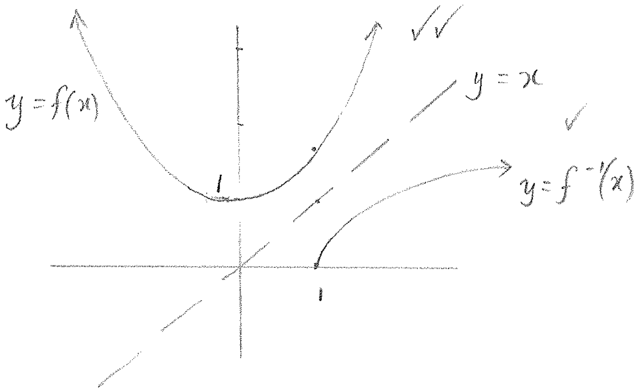
$u' = -2nx(1-x^2)^{n-1}$; $v = x$

$I_n = 2n\int_0^1 (1-x^2)^{n-1} - (1-x^2)^n\,dx$ from i) ✓

$I_n = 2n\,I_{n-1} - 2n\,I_n$

$I_n(1+2n) = 2n\,I_{n-1}$ ✓ $\longrightarrow$ $I_n = \dfrac{2n}{2n+1}I_{n-1}$

b) i) ✓✓ ✓



ii) largest domain is $x \geq 0$ ✓

iv) $2x = e^y + \dfrac{1}{e^y}$

$e^{2y} - 2xe^y + 1 = 0$ ✓

$e^y = \dfrac{2x \pm \sqrt{4x^2-4}}{2}$

$= x \pm \sqrt{x^2-1}$ ✓

$y = \log\left(x \pm \sqrt{x^2-1}\right)$

since $x \geq 1$ and $y \geq 0$ ✓

$f^{-1}(x) = \log\left(x + \sqrt{x^2-1}\right)$

v) The distance between $f(x)$ and $f^{-1}(x)$ is a minimum if the distance between $f(x)$ and $y=x$ is a minimum.

We need to find this distance D in terms of x.

Now,

$D = \dfrac{\left|x - \dfrac{e^x+e^{-x}}{2}\right|}{\sqrt{2}}$ ✓

since $\dfrac{e^x+e^{-x}}{2} > x$ (see graph) then $D = \frac{1}{\sqrt{2}}\left(\frac{e^x + e^{-x}}{2} - x\right)$

$$\frac{dD}{dx} = \frac{1}{\sqrt{2}}\left(\frac{e^x - e^{-x}}{2} - 1\right)$$

and solving $\frac{dD}{dx} = 0$ gives $e^x - \frac{1}{e^x} = 2$

ie $e^{2x} - 2e^x - 1 = 0 \quad \ldots (1)$ ✓

(1) $\rightarrow$ $e^x = \frac{2 \pm \sqrt{8}}{2}$ and $\boxed{e^x = 1 + \sqrt{2}}$ as $e^x > 0$

As D can only be a minimum as there is not a maximum then D is a minimum when $e^x = 1 + \sqrt{2}$

ie $x = \log(1 + \sqrt{2})$

Now $e^{-x} = \frac{1}{1 + \sqrt{2}} = \sqrt{2} - 1$ and $\frac{e^x + e^{-x}}{2} = \sqrt{2}$

$\therefore$ Coordinates of P are $(\log(1+\sqrt{2}), \sqrt{2})$ ✓