Moriah College
MATHEMATICS DEPARTMENT

Mathematics

Year 12

Extension 2 Pre-Trial 2009

Examiners: E Apfelbaum, G Wagner

General Instructions

- Reading time – 5 minutes
- Working time – 3 hours
- Write using blue or black pen
- Board-approved calculators may be used
- A table of standard integrals is provided.
- All necessary working should be shown in every question

Total marks (**120**)

- Attempt Questions 1–8
- All questions are of equal value
- Use a **SEPARATE** booklet for each question

STUDENT NUMBER:__________________

Question 1

a) Find $\displaystyle\int_{-5}^{1} \frac{dx}{x^2+4x+13}$ **3**

b) Find $\displaystyle\int \frac{2\,dx}{x^2-4x}$ **3**

c) Use integration by parts to find the value of $\int_1^3 x^3 \log x\,dx$ **3**

d) Find $\displaystyle\int \frac{\sin x\,dx}{1+\cos 2x}$ **3**

e) Using a suitable substitution, find **3**

$$\int \frac{dx}{x^2\sqrt{9-x^2}}$$

Question 2

a) Find all values of x and y such that $(x+iy)^2 = 4i$ **3**

b) Let $w = -2 + 2\sqrt{3}i$ **6**

i) Find $|w|$

ii) Show that $w^2 = 4\overline{w}$

iii) Prove that w is a root of the equation $z^3 - 64 = 0$.

iv) Find all roots of the equation $z^3 - 64 = 0$.

c) Draw neat sketches showing the main features of the following locii of z. **4**

i) $\frac{-\pi}{3} \leq \arg(z-2) \leq \frac{\pi}{3}$

ii) $|z-2-2i| \leq 2$

d) i) For z satisfying $|z-2-2i| \leq 2$, find the greatest value of $\arg(z)$. **2**

ii) For z satisfying $|z-2-2i| \leq 2$, find the greatest value of $\text{mod}(z)$

Question 3

a) Consider the ellipse $\frac{x^2}{4}+y^2=1$. **6**

i) Find the eccentricity, e, of this ellipse

ii) Find the coordinates of the foci of the ellipse.

iii) Use implicit differentiation to show that the derivative of the equation of the ellipse is given by $\frac{dy}{dx}=\frac{-x}{4y}$

iv) Show that the gradient of the tangent at the endpoint of the latus rectum in the first quadrant is $-e$.

b) **5**

i) Find $\frac{(2-2i)(-\sqrt{3}+i)}{2i}$ in the form $a+ib$.

ii) Express $2-2i$ in mod-arg form

iii) Express $\frac{(2-2i)(-\sqrt{3}+i)}{2i}$ in mod-arg form

iv) Hence find the exact value of $\sin 15°$

c) Let $P(x)=2x^3+3x^2+8x-5$ **4**

i) Write down the only possible rational roots of the equation $P(x)=0$

ii) Hence solve completely, the equation $P(x)=0$.

Question 4

a) Use the substitution $t = \tan \frac{x}{2}$ to show that $\int_0^{\frac{\pi}{2}} \frac{dx}{2 - \sin x} = \frac{2}{3\sqrt{3}}\pi$ **4**

b) $P(cp, \frac{c}{p})$ lies on the rectangular hyperbola $xy = c^2$. **4**

i) Show that the equation of the normal at P is $p^3x - py - cp^4 + c = 0$

ii) Hence, find the coordinates of the other point Q, where this normal cuts the hyperbola.

c) A complex number z moves in the Argand plane so that $\text{Re}(z) = 2|z-3|$ **4**

i) Explain why the locus defined by the above equation is an ellipse with one focus at (3,0).

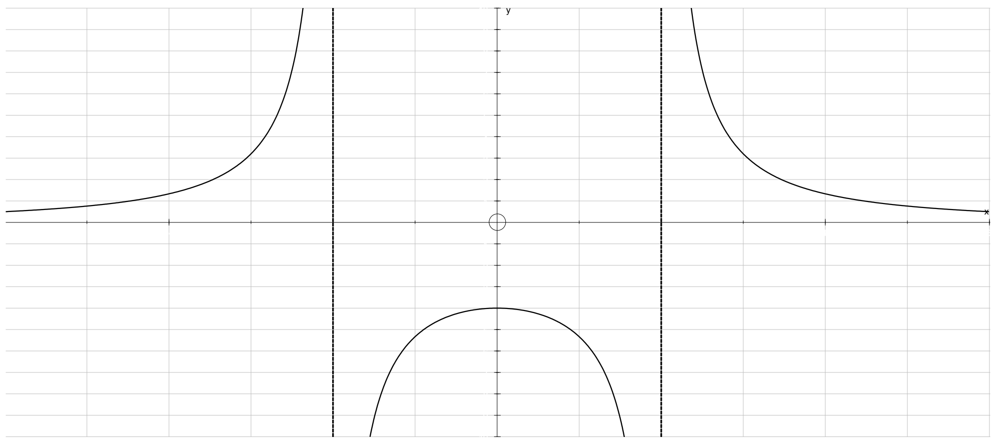
ii) Find the centre of the ellipse and the length of its major axis.

d) Solve $x^4 + x^3 - 3x^2 - 5x - 2 = 0$ given that it has a root of multiplicity 3 **3**

Question 5

a) **8**

The graph of $f(x) = \dfrac{1}{(x-4)(x+4)}$ is sketched below.



Draw neat sketches showing the main features of:

i) $y = \dfrac{1}{(4-x)(x+4)}$

ii) $y^2 = \dfrac{1}{(x-4)(x+4)}$

iii) $y = \dfrac{x}{(x-4)(x+4)}$

iv) $y = \tan^{-1}\left(\dfrac{1}{(x-4)(x+4)}\right)$

b) i) Show that the equation $x^3 - 6x^2 + 9x - 5 = 0$ has only one real root, α **7**

ii) Determine the two consecutive integers between which α lies.

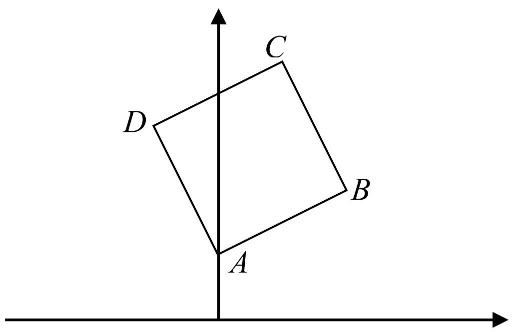
iii) Express the modulus of each of the complex roots in terms of α .

iv) Deduce that the value of this modulus lies between 1 and $\dfrac{\sqrt{5}}{2}$

Question 6

a) Find all the solutions of the equation $\tan 3x = \cot 2x$ **4**

b) **4**



The diagram shows the square *ABCD* in the complex plane.
A represents the complex number *i* and *B* represents the complex number *z*.

i) Find the complex number represented by the vector $\overrightarrow{AD}$.

ii) Hence, or otherwise, find the complex number represented by the point *C*.

c) i) Show that the graph of $y = e^x \sin x$ crosses the *x*-axis when $x = k\pi$ (*k* is an integer). **7**

ii) Draw a sketch of $y = e^x \sin x$ in the domain $0 \le x \le 2\pi$.

iii) Let $I = \int_{(k-1)\pi}^{k\pi} e^x \sin x dx$ where *k* is an integer.

Show that $2I = \dfrac{(-1)^{k-1} e^{k\pi} \left(e^{\pi} + 1\right)}{e^{\pi}}$

iv) Find the area bounded by $y = e^x \sin x$, the *x*-axis and the lines $x = 0$ and $x = 2\pi$

Question 7

a) i) Show that the equation of the tangent to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \ (a > b > 0)$ at the point $P(a\sec\theta, b\tan\theta)$ is: **8**

$bx\sec\theta - ay\tan\theta = ab$

ii) If this tangent passes through a focus of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, show it must be parallel to $y = x$ or $y = -x$.

iii) Show also that its point of contact with the hyperbola lies on a directrix of the ellipse.

b) **7**

i) If $I_n = \int_0^{\frac{\pi}{2}} x^n \cos x dx$, show that $I_n = \left(\frac{\pi}{2}\right)^n - n(n-1)I_{n-2}$

ii) Hence find the value of I_4

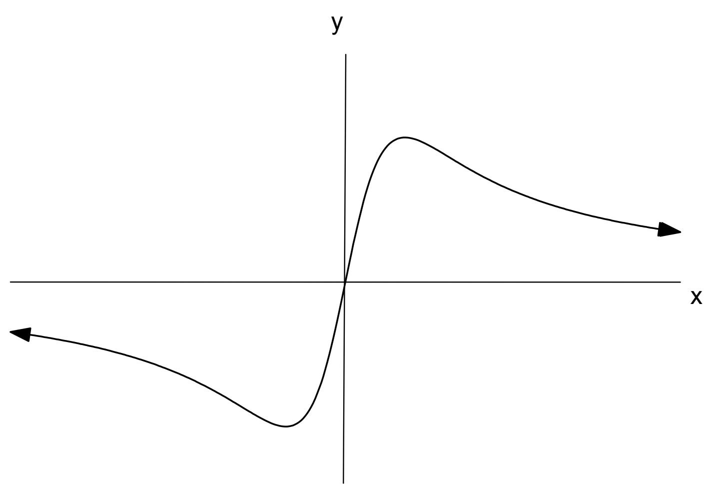
iii) Find the value of I_3

Question 8

a) i) If w is a complex cube root of unity, show that $1+w+w^2=0$ **4**

ii) Find the cubic equation whose roots are $1, 1+w, 1+w^2$.

b) The graph of the function $f(x)=\dfrac{2x}{1+x^2}$ is shown below. **4**



i) Sketch the curve $y=|f(x)|$

ii) Find the acute angle between the two branches of the curve $y=|f(x)|$, where they meet at the origin.

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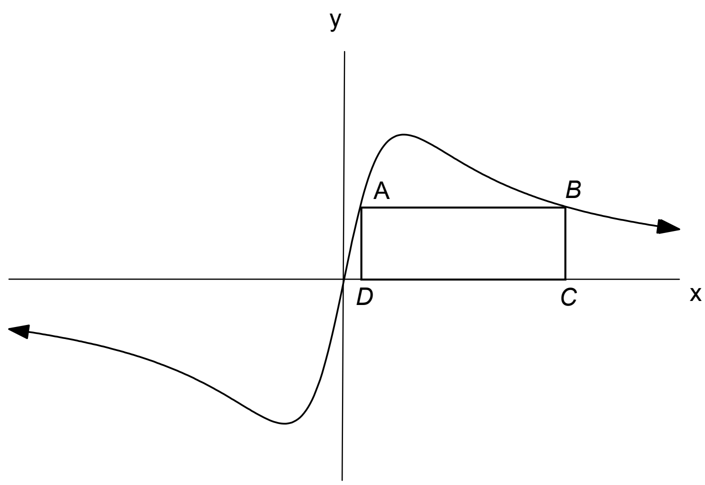
c) **7**

The graph of the function $f(x)=\dfrac{2x}{1+x^2}$ is again shown below.
In this diagram a rectangle $ABCD$ has been drawn. The inscribed rectangle has its base CD lying on the x-axis and its height given by the function.



i) Show that $f(x)=f\left(\dfrac{1}{x}\right)$

ii) The area below the curve and above the x-axis, in the first quadrant, is unlimited. Show that this is true by considering the definite integral $\int_0^a \dfrac{2x}{1+x^2}\,dx$.

iii) Let C be the variable point $(x,0)$. Using part i) or otherwise, find an expression for the area of the inscribed rectangle $ABCD$ in terms of x.

iv) Hence, find the limit of the area of the inscribed rectangle

Question one

a) $\int_{-5}^{1}\frac{dx}{x^2+4x+13}=\int_{-5}^{1}\frac{dx}{(x+2)^2+9}$

$=\left[\frac{1}{3}\tan^{-1}\frac{x+2}{3}\right]_{-5}^{1}$

$=\frac{1}{3}\left[\tan^{-1}1-\tan^{-1}-1\right]=\frac{\pi}{6}$

b) Let $\frac{2}{x(x-4)}=\frac{A}{x-4}+\frac{B}{x}$

$Ax+B(x-4)=2$

$x=0\Rightarrow B=\frac{-1}{2}$, $x=4\Rightarrow A=\frac{1}{2}$

So, integral becomes

$\frac{1}{2}\int\frac{1}{x-4}-\frac{1}{x}dx=\frac{1}{2}\ln\left|\frac{x-4}{x}\right|+C$

c) $\int_1^3 x^3\log x\,dx$

$u'=x^3\Rightarrow u=\frac{x^4}{4}$

$v=\log x\Rightarrow v'=\frac{1}{x}$

So, $\int_1^3 x^3\log x\,dx=\left[\frac{x^4\log x}{4}\right]_1^3-\int_1^3\frac{x^3}{4}dx$

$=\frac{81\ln 3}{4}-\left[\frac{x^4}{16}\right]_1^3=\frac{81\ln 3}{4}-5$

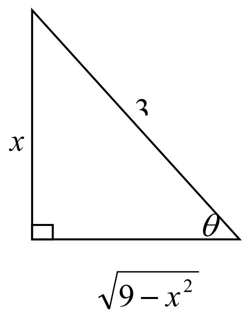
d) =

$\int\frac{\sin x\,dx}{2\cos^2 x}=\frac{1}{2}\int\sin x\cos^{-2}x\,dx=-\frac{1}{2}\frac{(\cos x)^{-1}}{-1}=\frac{1}{2}\sec x+C$

e) Let $x=3\sin\theta$, $\frac{dx}{d\theta}=3\cos\theta$. Integral becomes;

$\int\frac{3\cos\theta\,d\theta}{9\sin^2\theta\sqrt{9-9\cos^2\theta}}=\int\frac{d\theta}{9\sin^2\theta}=\frac{1}{9}\int\operatorname{cosec}^2\theta\,d\theta$

$=\frac{-\cot\theta}{9}=\frac{-\sqrt{9-x^2}}{9x}+C$



Question two

a) $x^2-y^2+2ixy=4i$

$\Rightarrow x^2-y^2=0\ \&\ xy=2$

$\therefore x=\pm\sqrt{2},\ y=\pm\sqrt{2}$

b)i) $|w|=4$

ii) LHS = $w^2=-8-8\sqrt{3}i$

RHS = $4\overline{w}=4(-2-2\sqrt{3}i)=-8-8\sqrt{3}i$

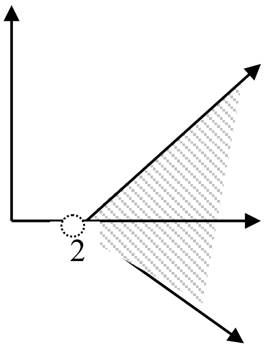
= LHS

iii) $\arg w=\frac{2\pi}{3}$.

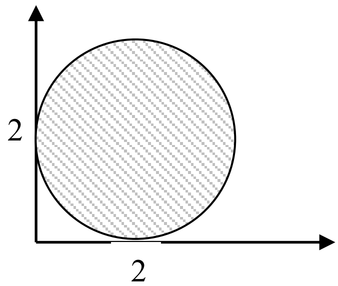
So $w=4cis(\frac{2\pi}{3})$ and $w^3=64cis(2\pi)=64$.

iv) Since coefficients are real, roots are $w,\overline{w}$, and 4, i.e. $-2\pm2\sqrt{3}i,4$.

c) i)



ii)



c)

i) Max value arg $z=\frac{\pi}{2}$

ii) greatest value of $|z|=2\sqrt{2}+2$

Question three

a)

i) $e^2 = 1 - \frac{b^2}{a^2} = 1 - \frac{1}{4} \Rightarrow e = \frac{\sqrt{3}}{2}$

ii) Focii are $(\pm ae, 0) = (\pm\sqrt{3}, 0)$

iii) $\frac{x^2}{4} + y^2 = 1$ - differentiate both sides:

$\frac{2x}{4} + 2y\frac{dy}{dx} = 0$

$2y\frac{dy}{dx} = \frac{-x}{2} \Rightarrow \frac{dy}{dx} = \frac{-x}{4y}$

iv) Find end point of the latus rectum;

$x = \sqrt{3} \Rightarrow y = \sqrt{1 - \frac{3}{4}} = \frac{1}{2}.$

$\therefore \frac{dy}{dx} = \frac{-x}{4y} = \frac{-\sqrt{3}}{4 \times \frac{1}{2}} = \frac{-\sqrt{3}}{2} = -e$

b)i)

$= \frac{-2\sqrt{3} + 2i + 2i\sqrt{3} + 2}{2i} \times \frac{2i}{2i}$

$= \frac{2(1-\sqrt{3}) + 2i(1+\sqrt{3})}{-4} \times 2i$

$= \frac{-(1+\sqrt{3}) + i(1-\sqrt{3})}{-1} = \sqrt{3} + 1 + i(\sqrt{3} - 1)$

ii) $2 - 2i = 2\sqrt{2}cis(\frac{-\pi}{4})$

iii) $= \frac{2\sqrt{2}(\frac{-\pi}{4})2cis(\frac{5\pi}{6})}{2cis(\frac{\pi}{2})} = 2\sqrt{2}cis(\frac{\pi}{12})$

iv) $Im(\sqrt{3} + 1 + i(\sqrt{3} - 1)) = 2\sqrt{2}\sin 15^\circ$

$\therefore \sin 15^\circ = \frac{\sqrt{3} - 1}{2\sqrt{2}}$

c) i)the only possible rational roots are:

$\pm\frac{1}{2}, \pm\frac{5}{2}, \pm 1, \pm 5$

ii) $P(\frac{1}{2}) = 0$

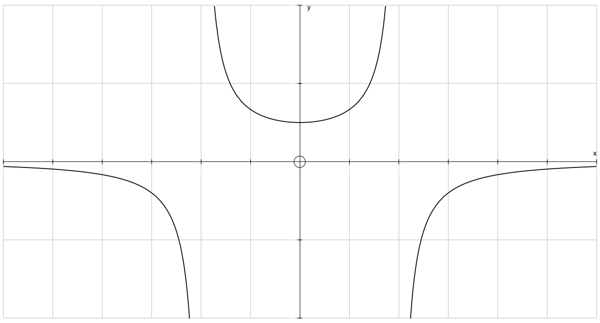
So, $P(x) = (2x-1)(x^2 + 2x + 5)$

And the roots are $\frac{1}{2}, -1 \pm 2i$

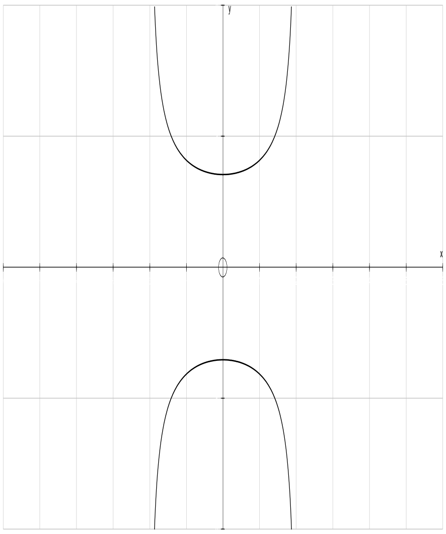
Question four	**Question four**
a) Let $t = \tan\frac{x}{2}$. When $x = 0$, $t=0$; $x = \frac{\pi}{2}, t = 1$ $\frac{dt}{dx} = \frac{1}{2}\sec^2\frac{x}{2} = \frac{1}{2}(\tan^2\frac{x}{2} + 1) = \frac{1}{2}(t^2 + 1) \Rightarrow$ $dx = \frac{2dt}{t^2 + 1}$ $\sin x = \frac{2t}{1 + t^2}$, so intergral becomes: $\int_0^1 \frac{2dt}{(1 + t^2)(2 - \frac{2t}{1 + t^2})} = \int_0^1 \frac{dt}{1 + t^2 - t}$ Complete the square; integral becomes: $\int_0^1 \frac{dt}{(t - \frac{1}{2})^2 + \frac{3}{4}} = \left(\frac{2}{\sqrt{3}}\tan^{-1}\frac{(2t-1)}{\sqrt{3}}\right)_0^1$ $= \frac{2}{\sqrt{3}}(\frac{\pi}{6} - \frac{-\pi}{6}) = \frac{2\pi}{3\sqrt{3}}$ b) i) $y = c^2 x^{-1} \Rightarrow y' = -c^2 x^{-2}$ At P, m tangent is $\frac{-1}{p^2}$, m normal is p^2. Equation of normal: $(y - \frac{c}{p}) = p^2(x - cp)$ $py - c = p^3(x - cp)$ $py - c = p^3 x - cp^4$ $p^3 x - py - cp^4 + c = 0.$ ii) Let Q be $(cq, \frac{c}{q})$, $mQP = p^2$. $mQP = \frac{-1}{pq} = p^2 \Rightarrow q = \frac{-1}{p^3}$. So Q is: $(\frac{-c}{p^3}, -cp^3)$.	c)i) $\lvert z - 3\rvert$- distance of z from (3,0) $= \frac{1}{2}$ its distance from the y axis. $\Rightarrow$ellipse with focus (3,0) and directrix the y axis. ii) It follows from i) that the ellipse has centre (4,0), major axis has length 4 units. d) If multiple root is α , $P(\alpha) = P'(\alpha) = P''(\alpha) = 0.$ So, $12x^2 + 6x - 6 = 0$ has solution α . $\alpha = -1, \frac{1}{2}.$ $\Rightarrow$roots are -1,-1,-1, and 2.

Question five

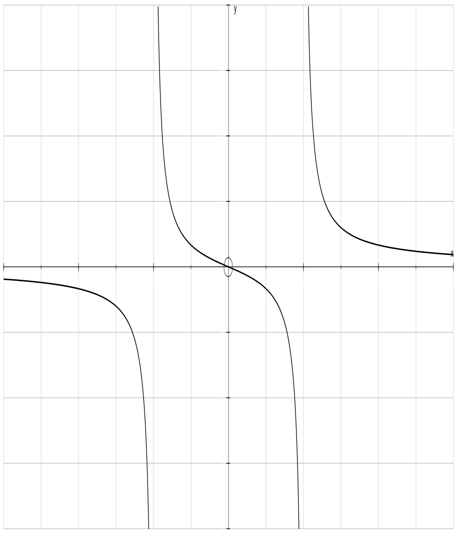
a) i)



ii)



iii)



b)i) Consider the graph of the function,

$f(x) = x^3 - 6x^2 + 9x - 5 = 0$.

Find the turning points. They are: (1,-1) (max) and (3,-5) (min). Since they are both below the *x* axis, the graph of $y = f(x)$ only crosses the *x* axis once, and hence the equation only has one real root, α.

ii) $f(4) = -1 < 0$ and $f(5) = 15 > 0$, so $4 < \alpha < 5$.

iii) Let the complex roots be $z \,\&\, \bar{z}$.

Product of the roots:

$\alpha z\bar{z} = 5$

$\therefore \alpha|z|^2 = 5$

$\therefore |z| = \sqrt{\dfrac{5}{\alpha}}$

$4 < \alpha < 5$

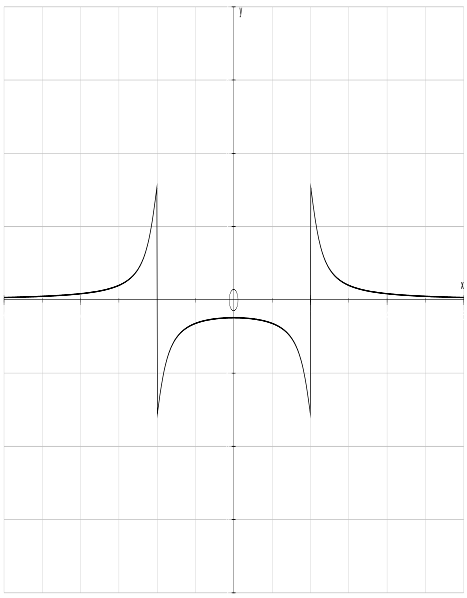
$\therefore \dfrac{1}{5} < \dfrac{1}{\alpha} < \dfrac{1}{4}$

$1 < \dfrac{5}{\alpha} < \dfrac{5}{4}$

$1 < \sqrt{\dfrac{5}{\alpha}} < \dfrac{\sqrt{5}}{2}$

$1 < |z| < \dfrac{\sqrt{5}}{2}$

a) iv)



Question six a) $\tan 3x = \cot 2x$

$\tan 3x = \tan(\frac{\pi}{2} - 2x)$

$3x = k\pi + \frac{\pi}{2} - 2x$

$5x = k\pi + \frac{\pi}{2}$

$x = \frac{k\pi}{5} + \frac{\pi}{10} = \frac{\pi}{10}(2k+1)$ $(\cos 3x \neq 0)$

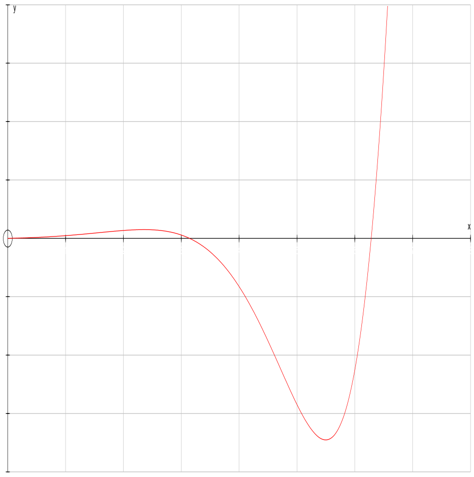
b) i) $\overrightarrow{AB} = z - i$

$\Rightarrow \overrightarrow{AD} = i(z-i) = 1 + iz$

ii) Let this number be u.

$u - i = z - i + 1 + iz \Rightarrow u = 1 + z + iz$

c) i) $x = k\pi, y = e^{k\pi}\sin k\pi = e^{k\pi} \times 0 = 0.$



iii) $I = \int_{(k-1)\pi}^{k\pi} e^x \sin x\,dx$ - integrate by parts.

$u = e^x, u' = e^x$ & $v' = \sin x, v = -\cos x$

$I = \left[-e^x \cos x\right] + \int_{(k-1)\pi}^{k\pi} e^x \cos x\,dx$, Use parts again:

$u = e^x \quad u' = e^x$

$v' = \cos x \quad v = \sin x$

$I = \left[-e^x \cos x\right] + \left[e^x \sin x\right] - \int_{(k-1)\pi}^{k\pi} e^x \sin x\,dx$

$I = \left[-e^x \cos x\right] + \left[e^x \sin x\right] - I$.........so,

$2I = \left[e^x(\sin x - \cos x)\right]_{(k-1)\pi}^{k\pi}$

$e^{k\pi}(\sin k\pi - \cos k\pi) - e^{(k-1)\pi}(\sin(k-1)\pi - \cos(k-1)\pi)$

$= e^{k\pi}(-\cos(k\pi) + e^{(k-1)\pi}\cos(k-1)\pi)$

k even: =

$e^{k\pi}(-1) + e^{(k-1)\pi}(-1) = e^{(k-1)\pi}(-1)(e^{\pi}+1)$

k odd: =

$e^{(k-1)\pi}(e^{\pi}+1) \Rightarrow 2I = \frac{(-1)^{k-1}e^{k\pi}(e^{\pi}+1)}{e^{\pi}}$

Question six (cont'd)

Area = $\int_0^{\pi} e^x \sin x dx + \left|\int_{\pi}^{2\pi} e^x \sin x dx\right|$

$= \frac{e^{\pi}(e^{\pi}+1)}{2e^{\pi}} + \frac{e^{2\pi}(e^{\pi}+1)}{2e^{\pi}} = \frac{(e^{\pi}+1)^2}{2}u^2$

Question seven

i) Find $\frac{dy}{dx}$ (implicitly):

$\frac{2x}{a^2} - \frac{2y}{b^2}\frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = \frac{b^2 x}{a^2 y} = \frac{b^2 a\sec\theta}{a^2 b\tan\theta} = \frac{b\sec\theta}{a\tan\theta}$ at P.

$\therefore$ Equation tangent is:

$y - b\tan\theta = \frac{b\sec\theta}{a\tan\theta}(x - a\sec\theta)$

$a\tan\theta y - ab\tan^2\theta = b\sec\theta x - ab\sec^2\theta$

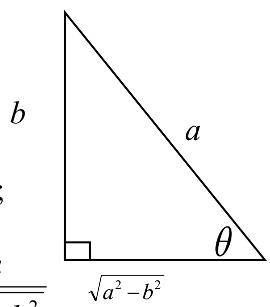
$b\sec\theta x - a\tan\theta y = ab(\sec^2\theta - \tan^2\theta) = ab$

ii) Focus of the ellipse: $\left(a\sqrt{\frac{a^2-b^2}{a^2}}, 0\right) = \left(\pm\sqrt{a^2-b^2}, 0\right)$ lies on the above tangent. So, substituting:

$\pm b\sec\theta\sqrt{a^2-b^2} = ab$

$\sec\theta = \frac{\pm a}{\sqrt{a^2-b^2}}$

$\tan\theta = \frac{b}{\sqrt{a^2-b^2}}$



$\therefore$ gradient of the tangent is;

$\frac{b\sec\theta}{a\tan\theta} = \frac{\pm ab}{\sqrt{a^2-b^2}} \div \frac{ba}{\sqrt{a^2-b^2}}$

$= \pm 1$

And the result follows.

iii) Point of contact with the hyperbola is:

$(a\sec\theta, b\tan\theta) = ((\frac{a^2}{\sqrt{a^2-b^2}}, b\tan\theta)$

$= (a \div \frac{\sqrt{a^2-b^2}}{a}$,........)

$= (\frac{a}{'e'}, ...)$

$\therefore$ lies on a directrix of the ellipse.

Question seven (cont'd)

i) Using parts:

$u = x^n, u' = nx^{n-1}; v' = \cos x, v = \sin x$

$$I_n = \left[x^n \sin x\right]_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} nx^{n-1}\sin x\,dx$$

$$= \left(\frac{\pi}{2}\right)^n - \int_0^{\frac{\pi}{2}} nx^{n-1}\sin x\,dx \rightarrow \text{ use parts:}$$

$u = x^{n-1}, u' = (n-1)x^{n-2}; v' = \sin x, v = -\cos x$, so

$$I_n = \left(\frac{\pi}{2}\right)^n - n\left(\left[-x^{n-1}\cos x\right]_0^{\frac{\pi}{2}} + (n-1)\int_0^{\frac{\pi}{2}} x^{n-2}\cos x dx\right)$$

$$= \left(\frac{\pi}{2}\right)^n - n(0) - n(n-1)I_{n-2}$$

As required.

ii)

$$I_4 = \left(\frac{\pi}{2}\right)^4 - 12I_2$$

$$= \left(\frac{\pi}{2}\right)^4 - 12\left(\frac{\pi}{2}\right)^2 + 12\times 2I_0$$

$$I_0 = \int_0^{\frac{\pi}{2}} \operatorname{coss} x dx = 1 \therefore I_4 = \left(\frac{\pi}{2}\right)^4 - 12\left(\frac{\pi}{2}\right)^2 + 24$$

iii) $I_3 = \left(\frac{\pi}{2}\right)^3 - 6I_1 = \left(\frac{\pi}{2}\right)^3 - 6\int_0^{\frac{\pi}{2}} x\cos x dx$

Use parts: $u = x, u' = 1; v' = \cos x, v = \sin x$

$$\left[x\sin x\right]_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} \sin x dx = \frac{\pi}{2} - 1$$

$$\therefore I_3 = \left(\frac{\pi}{2}\right)^3 - 6\frac{\pi}{2} + 6$$

Question eight

a) i) $w^3 = 1 \Rightarrow w^3 - 1 = 0 \Rightarrow (w-1)(w^2+w+1) = 0$

$w \neq 1, \therefore w^2 + w + 1 = 0$

ii) Sum of roots: 2+ $w^2 + w + 1$=2

In pairs

$= 1 + w + 1 + w^2 + 1 + w + w^2 + w^3 = 1 + w^3 = 2$

Product of roots; $= ((1+w)(1+w^2) = 1$

$\therefore$Equation is:

$x^3 - 2x^2 + 2x - 1 = 0$

b)

Question eight (cont'd)

b)ii) $f'(x) = \dfrac{2-2x^2}{(1+x^2)^2}$

$$f'(0) = 2 \Rightarrow \tan\theta = \left|\frac{2--2}{1+2\times -2}\right| = \frac{4}{3} \Rightarrow \theta = 53^\circ 8'$$

c) i) $f(\frac{1}{x}) = \dfrac{\frac{2}{x}}{1+\frac{1}{x^2}} \times \dfrac{x^2}{x^2}$

$$= \frac{2x}{1+x^2} = f(x)$$

ii)

$$\int_0^a \frac{2x}{1+x^2} dx = \left[\ln(1+x^2)\right]_0^a = \ln(1+a^2) \rightarrow \infty as\ a \leftarrow \infty$$

(iii) D=$(\frac{1}{x},0)$

Area (x)=$(x - \frac{1}{x})(\frac{2x}{1+x^2}) = \dfrac{2(x^2-1)}{1+x^2}$

iv) As $x \rightarrow \infty, A(x) \rightarrow 2$