Total Marks – 100
Attempt Questions 1-4

Question 1 *(25 marks)* Use a SEPARATE writing booklet. **Marks**

(a) Evaluate $\int_0^4 \frac{x\,dx}{\sqrt{16-x^2}}$. **2**

(b) Find $\int \sec^3 x \tan x\, dx$. **2**

(c) Evaluate $\int_0^2 \frac{dx}{x^2-4x+8}$ **3**

(d) (i) Prove $2\sin A\cos B = sin(A-B)+\sin(A+B)$. **1**

(ii) Hence, evaluate $\int_0^{\frac{\pi}{12}} \sin 4x \cos 2x\, dx$. **2**

(e) (i) Find real numbers A, B, C such that

$$\frac{2x}{(1+x)(x^2+1)} = \frac{A}{1+x} + \frac{Bx+C}{x^2+1}.$$ **3**

(ii) Hence find $\int \frac{2x}{(1+x)(x^2+1)}dx$. **2**

(f) (i) Show that $\displaystyle\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$. **2**

(ii) Hence, evaluate $\displaystyle\int_0^{\pi} \frac{x\sin x}{1+\cos^2 x}\,dx$ **3**

(g) (i) If $I_n = \displaystyle\int \frac{dx}{\left(x^2+1\right)^n}$ prove that

$$I_n = \frac{1}{2(n-1)}\left[\frac{x}{(x^2+1)^{n-1}} + (2n-3)I_{n-1}\right].$$ **3**

(ii) Hence evaluate $\displaystyle\int_0^1 \frac{dx}{\left(x^2+1\right)^2}$. **2**

Question 2 *(25 marks)* Use a SEPARATE writing booklet. **Marks**

(a) $z_1 = 1 + i$ and $z_2 = \sqrt{3} - i$.

(i) Find $\dfrac{z_1}{z_2}$ in the form $a+ib$ where a and b are real. **1**

(ii) Write z_1 and z_2 in modulus – argument form. **2**

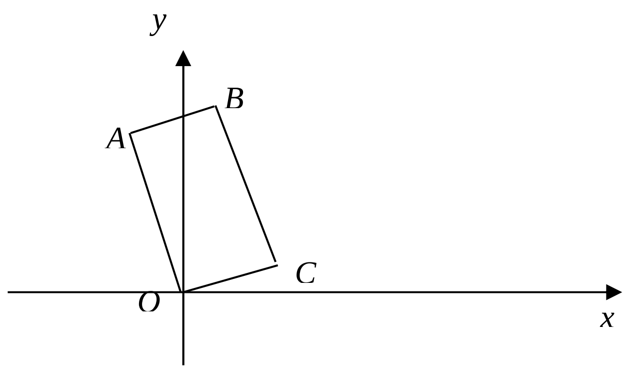
(iii) By equating equivalent expressions for $\dfrac{z_1}{z_2}$, write $\cos\dfrac{5\pi}{12}$ as a surd. **2**

(b) (i) Express in modulus argument form, $1-\sqrt{3}i$. **1**

(ii) Hence evaluate $\left(1-\sqrt{3}i\right)^5$ in the form $x+iy$. **3**

(c) Given $|z+i| \le 2$ and $0 \le arg(z+1) \le \dfrac{\pi}{4}$. Sketch the region in an Argand diagram which contains the point P representing z. **3**

(d)



The points $OABC$ are the vertices of a rectangle on the Argand diagram with $|OA| = 2|OC|$. If OC represents the complex number $p+iq$, write down the complex numbers represented by:

(i) $\overrightarrow{OA}$ **1**

(ii) $\overrightarrow{OB}$ **1**

(iii) $\overrightarrow{BC}$ **1**

(iv) $\overrightarrow{AC}$ **1**

(e) Consider the five 5th roots of unity.

(i) Solve $z^5 - 1 = 0$ over the complex field giving your answers in modulus-argument form. **3**

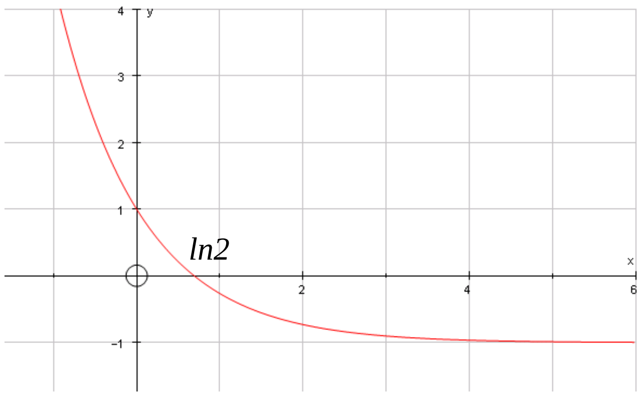
(ii) Hence express $z^5 - 1$ as the product of real linear and quadratic factors. **3**

(iii) Write down the complex roots of $z^4 + z^3 + z^2 + z + 1 = 0$ giving your answers in modulus-argument form. **1**

(iv) Hence prove that $\cos\frac{2\pi}{5} + \cos\frac{4\pi}{5} = -\frac{1}{2}$. **2**

Question 3 *(15 marks)* Use a SEPARATE writing booklet. **Marks**

(a)



The diagram shows the graph of $f(x)=2e^{-x}-1$. On separate diagrams sketch the following graphs, showing the intercepts on the axes and the equations of any asymptotes:

(i) $y=|f(x)|$. **2**

(ii) $y=\{f(x)\}^2$. **2**

(iii) $y=\dfrac{1}{f(x)}$. **2**

(iv) $y=ln\ \{\text{f(x)}\}$. **2**

(v) $y=xf(x)$. **2**

(b) Consider the curve $y^2=x^4(4+x)$

(i) Sketch the curve. **2**

(ii) Find the area of the loop of the curve from $x=-4$ to $x=0$. **3**

Question 4 *(15 marks)* Use a SEPARATE writing booklet. **Marks**

(a) (i) Show that $(2x+1)$ is a factor of $P(x)=2x^3-3x^2+8x+5$ **1**

(ii) Hence solve $P(x)=0$ over the complex field. **3**

(b) Consider the polynomial $P(x)=(x+2)^2Q(x)+R(x)$

(i) Explain why $R(x)$ is a linear polynomial **1**

(ii) When $P(x)$ and $P'(x)$ are both divided by $(x+2)$, the remainder in each case is 6. Find $R(x)$. **3**

(c) The roots of $x^4+x^3+2x^2+3x+1=0$ are $\alpha,\beta,\gamma,\delta$

(i) Find the polynomial with roots $\frac{1}{\alpha},\frac{1}{\beta},\frac{1}{\gamma},\frac{1}{\delta}$. **2**

(ii) Hence, or otherwise, show that

$$\left(\alpha+\frac{1}{\alpha}\right)+\left(\beta+\frac{1}{\beta}\right)+\left(\gamma+\frac{1}{\gamma}\right)+\left(\delta+\frac{1}{\delta}\right)=-4$$ **1**

(d) (i) Explain why $1+x+\frac{x^2}{2!}+\frac{x^3}{3!}$ has no double zeros. **2**

(note: $3!=3\times2\times1$)

(ii) Find the relationship between a,b and c if ax^4+bx^2+c has a double zero. **2**

End of paper

Q1

(a)

$$\int_0^4 \frac{x}{\sqrt{16-x^2}}\,dx$$

$$= -\left[\sqrt{16-x^2}\right]_0^4$$

$$= -(0-4)$$

$$= 4$$

(b)

$$\int \sec^3 x \tan x \, dx$$

$$= \int \sec^2 x \cdot \sec x \tan x \, dx$$

$$= \frac{\sec^3 x}{3} + C$$

Using Standard Integral for $\int \sec x \tan x \, dx = \sec$

(c)

$$\int_0^2 \frac{dx}{x^2-4x+8}$$

$$= \int_0^2 \frac{dx}{(x-2)^2+4}$$

$$= \frac{1}{2}\left[\tan^{-1}\frac{(x-2)}{2}\right]_0^2$$

$$= \frac{1}{2}\left(0 - \tan^{-1}(-1)\right)$$

$$= \frac{\pi}{8}$$

1(d) (i) Prove $2\sin A\cos B = \sin(A-B)+\sin(A+B)$

$$\sin(A-B) = \sin A\cos B - \cos A\sin B$$
$$\sin(A+B) = \sin A\cos B + \cos A\sin B$$

$\therefore$ $\sin(A-B)+\sin(A+B) = 2\sin A\cos B$ ✓

(ii) $$\int_0^{\pi/12} \sin 4x\cos 2x\,dx$$

$$= \frac{1}{2}\int_0^{\pi/12}\left[\sin(4x-2x)+\sin(4x+2x)\right]dx$$

$$= \frac{1}{2}\int_0^{\pi/12}\left[\sin 2x + \sin 6x\right]dx$$

$$= -\frac{1}{2}\left[\frac{\cos 2x}{2}+\frac{\cos 6x}{6}\right]_0^{\pi/12}$$

$$= -\frac{1}{2}\left[\frac{\cos\frac{\pi}{6}}{2}+\frac{\cos\frac{\pi}{2}}{6}-\frac{1}{2}-\frac{1}{6}\right]$$

$$= -\frac{1}{2}\left[\frac{\sqrt{3}}{4}-\frac{2}{3}\right]$$

$$= \frac{8-3\sqrt{3}}{24}$$

$(1+x)(x^2+1) \qquad 1+x \qquad x+1$

$$2x \equiv A(x^2+1) + (1+x)(Bx+C)$$

let $x = 0$

$\therefore 0 = A + C$

let $x = -1$

$-2 = 2A$

$A = -1$ ✓

$\therefore C = 1$ ✓

let $x = 1$

$\therefore 2 = -2 + 2(B+1)$

$\therefore B+1 = 2$

$B = 1$ ✓

$$\therefore \frac{2x}{(1+x)(x^2+1)} \equiv \frac{-1}{1+x} + \frac{x+1}{x^2+1}$$

(ii) $\int \frac{2x}{(1+x)(x^2+1)}\,dx = -\ln|1+x| + \int \frac{x}{x^2+1}\,dx + \int \frac{1}{1+}$

$= -\ln|1+x| + \frac{1}{2}\ln|x^2+1| + \tan^{-1}x$

1 (5) (i) $\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$ let $u = a - x$

$\frac{du}{dx} = -1$

$$\therefore RHS = -\int_a^0 f(u)\cdot\frac{du}{dx}\,dx$$ ✓

$$= -\int_a^0 f(u)\,du$$

$$= \int_0^a f(x)\,dx$$ ✓

$= LHS$

(ii) $$\int_0^{\pi} \frac{x\sin x}{1+\cos^2 x}\,dx$$

$$= \int_0^{\pi} \frac{(\pi - x)\sin(\pi - x)}{1 + (\cos(\pi - x))^2}\,dx$$

$$= \int_0^{\pi} \frac{(\pi - x)\sin x}{1+\cos^2 x}\,dx$$ ✓

$$\therefore 2\int_0^{\pi} \frac{x\sin x}{1+\cos^2 x}\,dx = \pi\int_0^{\pi} \frac{\sin x}{1+\cos^2 x}\,dx$$ ✓

$$\int_0^{\pi} \frac{x\sin x}{1+\cos^2 x}\,dx = -\frac{\pi}{2}\Big[\tan^{-1}(\cos x)$$

$$= \frac{-\pi}{2}\Big[\;\left(\frac{-\pi}{4}\right) - \frac{\pi}{\ldots}$$

$$= \frac{\pi^2}{4}$$ ✓

$$I_n = \int \frac{dx}{(x^2+1)^n}$$

$$I_n = \int \frac{\frac{d(x)}{dx} \cdot dx}{(x^2+1)^n} \qquad u = x \,,\, V = (x^2+1)^{-n}$$

$$= \frac{x}{(x^2+1)^n} + 2n \int \frac{x^2}{(x^2+1)^{n+1}}\, dx \qquad \text{as } \frac{d[(x^2+1)}{dx} = -\frac{2n}{(x^2+}$$

$$= \frac{x}{(x^2+1)^n} + 2n \int \frac{x^2+1}{(x^2+1)^{n+1}} - 2n \int \frac{1}{(x^2+1)^{n+1}}\, dx \checkmark$$

$$= \frac{x}{(x^2+1)^n} + 2n I_n - 2n I_{n+1}$$

$$\therefore 2n I_{n+1} = \frac{x}{(x^2+1)^n} + (2n-1) I_n$$

$$\therefore I_{n+1} = \frac{1}{2n}\left[\frac{x}{(x^2+1)^n}\right] + \frac{2n-1}{2n} I_n \checkmark$$

let $n = N-1$

$$\therefore I_N = \frac{1}{2(N-1)}\left[\frac{x}{(x^2+1)^{N-1}} + (2N-3) I_{N-1}\right]$$

$$\therefore I_n = \frac{1}{2(n-1)}\left[\frac{x}{(x^2+1)^{n-1}} + (2n-3) I_{n-1}\right]$$

(ii) $\int_0^1 \frac{dx}{(x^2+1)^2} = I_2$

$I = \frac{1}{} \left[\frac{x}{} \right]^1 \ldots I$

Q2 Solutions

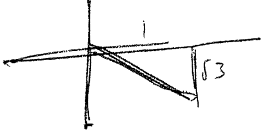
(a) (i)
$$\frac{z_1}{z_2} = \frac{1+i}{\sqrt{3}-i} \times \frac{\sqrt{3}+i}{\sqrt{3}+i}$$
$$= \frac{\sqrt{3}-1}{4} + \frac{(1+\sqrt{3})i}{4}$$

(ii)
$$z_1 = \sqrt{2}\,\text{cis}\,\frac{\pi}{4}$$
$$z_2 = 2\,\text{cis}\left(-\frac{\pi}{6}\right)$$

(iii)
$$\therefore \frac{z_1}{z_2} = \frac{\sqrt{2}\,\text{cis}\left(\frac{\pi}{4}\right)}{2\,\text{cis}\left(-\frac{\pi}{6}\right)}$$
$$= \frac{1}{\sqrt{2}}\,\text{cis}\,\frac{5\pi}{12}$$

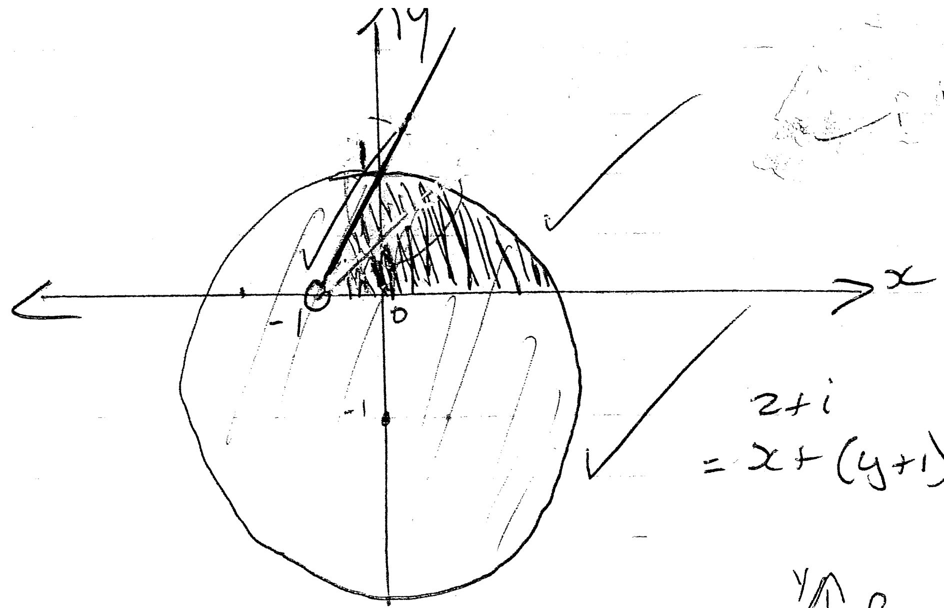
$\therefore$ Equating real parts from (i) & (iii)
$$\frac{1}{\sqrt{2}}\cos\frac{5\pi}{12} = \frac{\sqrt{3}-1}{4}$$
$$\cos\frac{5\pi}{12} = \frac{\sqrt{6}-\sqrt{2}}{4}$$

(b) (i) $1-\sqrt{3}i = 2\,\text{cis}\left(-\frac{\pi}{3}\right)$



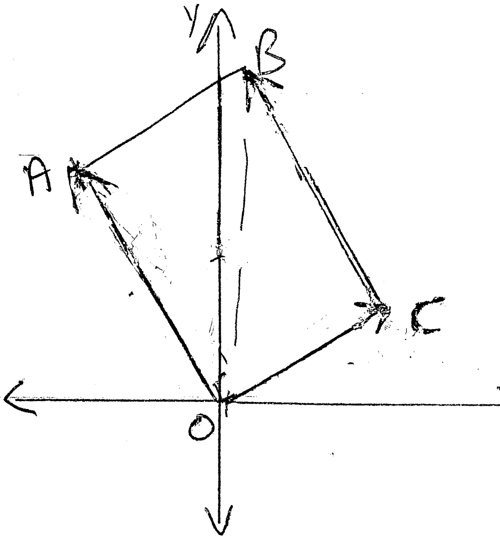
(ii)
$$(1-\sqrt{3}i)^5 = 2^5\,\text{cis}\left(-\frac{5\pi}{3}\right)$$
$$= 2^5\,\text{cis}\,\frac{\pi}{3}$$
$$= 32\left(\frac{1}{2} + \frac{\sqrt{3}}{2}i\right)$$
$$= 16 + 16\sqrt{3}\,i$$

Q2 (c)



$z+i$
$= x+(y+1)i$

(d) Assuming $|OA| = 2|OC|$



(i) If OC is $p+iq$

$\therefore \ \vec{OA} = 2\times(p+iq)\times i$
$= -2q+2ip$ ✓

(ii) $\vec{OB} = \vec{OA} + \vec{OC}$
$= (p-2q) + (2p+q)i$ ✓

(iii) $\vec{BC} = 2q - 2ip$ ✓

(iv) $\vec{AC}$

$\vec{OC} + \vec{CA} = \vec{OA}$
$\vec{OC} - \vec{AC} = \vec{OA}$
$\vec{AC} = \vec{OC} - \vec{OA}$
$= p+iq - (-2q+2$
$= (p+2q)+(q-2p)$ ✓

2(e)(i) let $z = \cos\theta + i\sin\theta$

$z^5 = \cos 5\theta + i \sin 5\theta$

let $\cos 5\theta + i \sin \theta = 1$

$\therefore \cos 5\theta = 0$

$\therefore 5\theta = 0, 2\pi, 4\pi, 6\pi, 8\pi$

$\theta = 0, \frac{2\pi}{5}, \frac{4\pi}{5}, \frac{6\pi}{5}, \frac{8\pi}{5}$

(iii)

$\therefore$ 5 roots are $\text{cis}\,0, \text{cis}\frac{2\pi}{5}, \text{cis}\frac{4\pi}{5}, \text{cis}\left(-\frac{2\pi}{5}\right), \text{cis}\left(-\frac{4\pi}{5}\right)$

(ii) $z^5 - 1 = (z-z_1)(z-z_2)(z-z_3)(z-z_4)(z-z_5)$

$= (z-1)\left(z^2 - 2\cos\frac{2\pi}{5}z + 1\right)\left(z^2 - 2\cos\frac{4\pi}{5}z + 1\right)$

as Sum of conjugate root $= 2\cos\frac{2\pi}{5}, 2\cos\frac{4\pi}{5}$

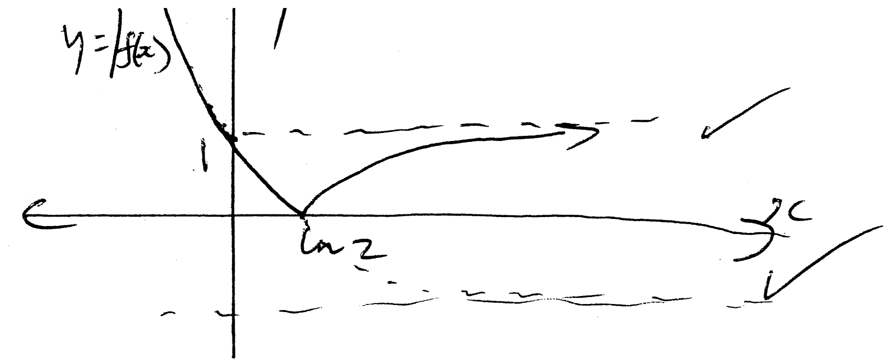
(iii) Sum of roots $= 0$.

$\therefore 1 + 2\cos\frac{2\pi}{5} + 2\cos\frac{4\pi}{5} = 0$ ✓

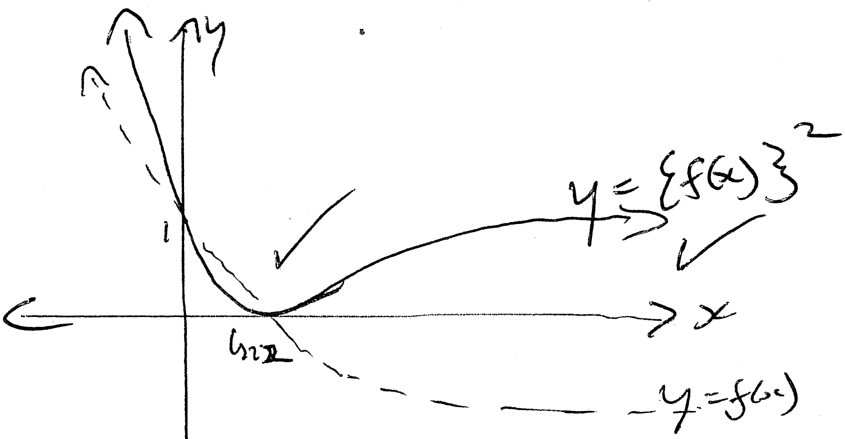
$\therefore \cos\frac{2\pi}{5} + \cos\frac{4\pi}{5} = -\frac{1}{2}$

(a)

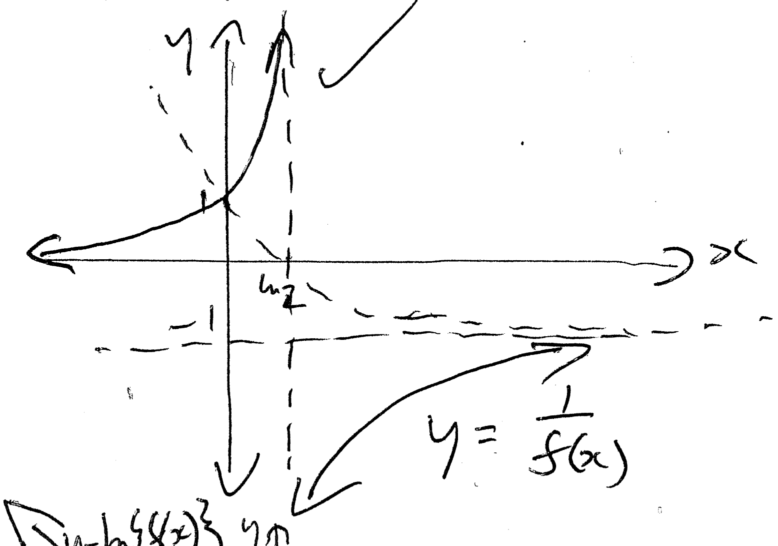
(i)



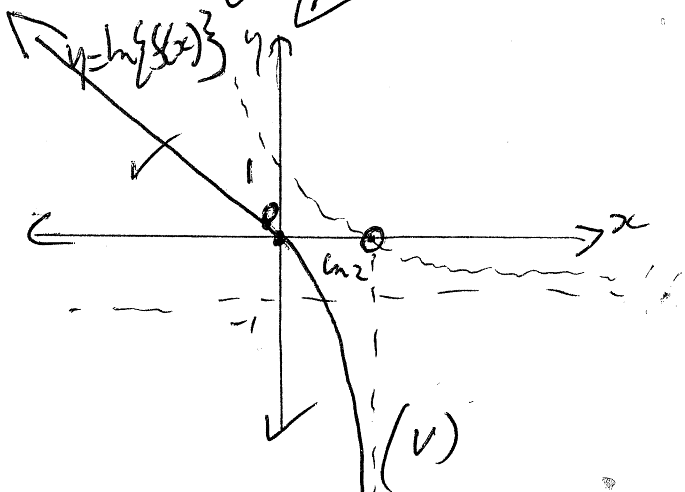
(ii)



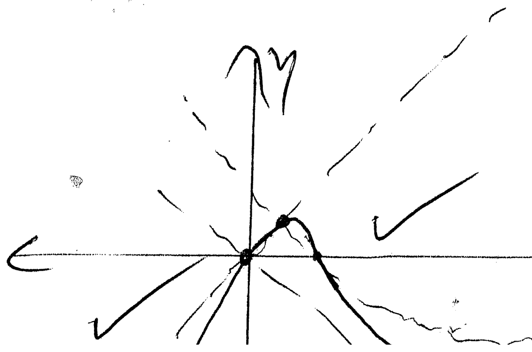
(iii)



(iv)

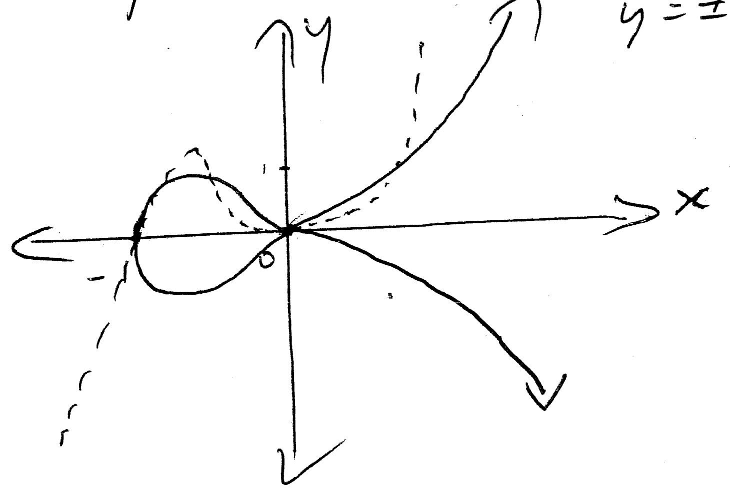


(v)



3(b) (i) $y^2 = x^4(4+x)$

$y = \pm\sqrt{x^4(4+x)}$



(ii) $\text{Area of loop} = 2\int_{-4}^{0} x^2\sqrt{4+x}\,dx$

let $u = 4+x$ $\therefore \frac{du}{dx} = 1$

$\therefore \text{Area} = 2\int_{0}^{4} (u-4)^2\sqrt{u}\,du$

$= 2\int_{0}^{4} (u^2 - 8u + 16)\,u^{\frac{1}{2}}\,du$

$= 2\int_{0}^{4} \left(u^{5/2} - 8u^{3/2} + 16u^{\frac{1}{2}}\right) du$

$= 2\left[\frac{2}{7}u^{7/2} - \frac{16}{5}u^{5/2} + \frac{32}{3}u^{3/2}\right]_0^4$

$= \frac{4096}{105}$

$= 39\frac{1}{105}\ u^2$ #

Q4 (a) (i) $P(-\frac{1}{2}) = -\frac{2}{8} - \frac{3}{4} - \frac{8}{2} + 5$

$= 0$

$\therefore 2x+3$ is a factor

(ii)

$$\begin{array}{r} x^2 - 2x + 5 \\ 2x+1 \enclose{longdiv}{2x^3 - 3x^2 + 8x + 5} \\ \underline{2x^3 + x^2} \\ -4x^2 + 8x \\ \underline{-4x^2 - 2x} \\ 10x + 5 \\ \underline{10x + 5} \end{array}$$

✓

$\therefore P(x) = (2x+1)(x^2 - 2x + 5)$

$= (2x+1)[(x-1)^2 + 4]$

$= (2x+1)[(x-1)^2 - (4i)^2]$

$= (2x+1)[x - 1 - 2i][x - 1 + 2i]$

$\therefore P(x) = 0$

$x = -\frac{1}{2}, \; 1+2i, \; 1-2i$ #

4(b) $P(x) = (x+2)^2 Q(x) + R(x)$

(i) $P(x)$ is divided by a quadratic $(x+2)^2$, therefore the remainder must be in the form $ax+b$

(ii) $P'(x) = 2(x+2)Q(x) + Q'(x)(x+2)^2 + R'(x)$

$\therefore$ as $R(x) = ax+b$

$R'(x) = a \quad \therefore a = 6$

Now as $P(-2) = 6 \quad \therefore -2a+b = 6$

$\therefore b = 18$

$\therefore R(x) = 6x+18$

(c) $x^4 + x^3 + 2x^2 + 3x + 1 = 0 \quad \ldots (*)$

(i) let $\alpha = \frac{1}{x}$

$\therefore (\frac{1}{x})^4 + 3(\frac{1}{x})^3 + 2(\frac{1}{x})^2 + 3(\frac{1}{x}) + 1 = 0$

$\therefore x^4 + 3x^3 + 2x^2 + x + 1 = 0$ has roots $\frac{1}{\alpha}, \frac{1}{\beta}, \frac{1}{\gamma}, \frac{1}{\delta}$

(ii) Sum of roots $(*) = -1$

Sum of roots $(\otimes) = -3$

$\therefore (\alpha + \frac{1}{\alpha}) + (\beta + \frac{1}{\beta}) + (\gamma + \frac{1}{\gamma}) + (\delta + \frac{1}{\delta}) = -4$

$4(\ldots)$ $\quad 2! \quad 3!$

$$f'(x) = 1 + \frac{x}{1!} + \frac{x^2}{2!}$$

let $x = \alpha$ be the double root

$$\therefore \cancel{1} + \cancel{\alpha} + \cancel{\frac{\alpha^2}{2}} + \frac{\alpha^3}{8} = \cancel{1} + \cancel{\alpha} + \cancel{\frac{\alpha^2}{2}}$$

$$\therefore \alpha = 0$$

Now $f(0) \neq 0$ $\therefore$ No double root

(ii) let $f(x) = ax^4 + bx^2 + c$ --- (1)

$$\therefore f'(x) = 4ax^3 + 2bx \quad \text{--- (2)}$$

from (2) $4ax^3 + 2bx = 0$

$$2x(2ax^2 + b) = 0$$

as $x \neq 0$ $\therefore x^2 = -\frac{b}{2a}$

$\therefore$ Sub into (1)

$$a\left(-\frac{b}{2a}\right)^2 + b\left(-\frac{b}{2a}\right) + c = 0$$

$$\frac{\cancel{a}b^2}{4a^{\cancel{2}}} - \frac{b}{2a} + c = 0$$

$$b^2 - 2b^2 + 4ac = 0$$

$$4ac - b^2 = 0$$