NORTH SYDNEY BOYS HIGH SCHOOL

2009 YEAR 12 HSC
ASSESSMENT TASK 2

Mathematics
Extension 2

General Instructions

- Working time – **55 minutes**
- Write on the lined paper in the booklet provided
- Write using blue or black pen
- Board approved calculators may be used
- All necessary working should be shown in every question
- Each new question is to be started on a **new page.**
- Attempt all questions

Class Teacher:
(Please tick or highlight)

- O Mr Barrett
- O Mr Fletcher
- O Mr Weiss

Student Number:

Question No	1	2	3	4	Total	Total
Mark	/13	/9	/11	/7	/40	/100

QUESTION 1 **(13 marks)** **Marks**

The ellipse E has equation $\frac{x^2}{4} + \frac{y^2}{3} = 1$

(i) Calculate the eccentricity 1

(ii) Write down the coordinates of the foci S and S' 2

(iii) Write down the equation of each directrix 2

(iv) Sketch E, showing all essential features 2

(v) Show that the tangent at the point P (x_1, y_1) on the ellipse is 3

$$\frac{xx_1}{4} + \frac{yy_1}{3} = 1$$

(vi) The tangent at P intersects the directrix at the point T, where T has coordinates (m,n) and m > 0. Show that PT subtends a right angle at S. 3

QUESTION 2 **(9 marks) Start a new page**

(a) Use the method of mathematical induction to show that 4

$4^n > 2n + 1$ where n is a positive integer.

(b) Resolve the following into partial fractions :

(i) $\frac{2}{(x-1)(x+3)}$ 2

(ii) $\frac{4x+2}{(x+3)(x^2+1)}$ 3

QUESTION 3 **(11 marks) Start a new page** **Marks**

(i) Sketch the graph of the hyperbola $9x^2 - 16y^2 = 144$,

showing clearly the foci, the directrices, and the asymptotes. 4

(ii) Show that the normal at P ($4\sec\theta$, $3\tan\theta$) has equation 3

$$\frac{4x}{\sec\theta} + \frac{3y}{\tan\theta} = 25$$

(iii) A line through P parallel to the y-axis meets the asymptote 2

in the first quadrant at Q. The normal at P meets the x-axis at G.

Find the coordinates of Q and G.

(iv) Show that GQ is perpendicular to OQ. 2

QUESTION 4 **(7 marks) Start a new page**

The points P ($2p, \frac{2}{p}$) and Q ($2q, \frac{2}{q}$) lie on the rectangular hyperbola $xy = 4$

M is the midpoint of PQ. P and Q move on the hyperbola such that the chord PQ always passes through the point (4, 2).

(i) Show that PQ has equation $x + pqy = 2(p + q)$ 3

(ii) Show that $pq = p + q - 2$ 1

(iii) Find the equation of the locus of M. 3

SOLUTIONS TO 4UNIT ASSESSMENT 2

Q1 (i) $a = 2, \quad b = \sqrt{3}$

$$e^2 = 1 - \frac{b^2}{a^2}$$
$$= \frac{1}{4}$$
$$\therefore e = \frac{1}{2}$$ 1

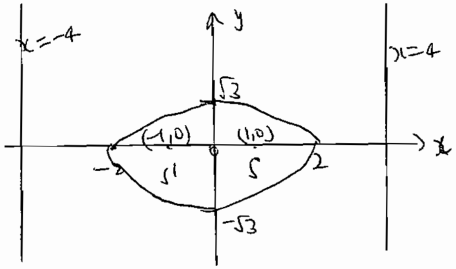
(ii) S is $(1, 0)$ 1

S' is $(-1, 0)$ 1

(iii) $x = 4$ 1

$x = -4$ 1

(iv)



2

−1 for each missing feature

(v) $$\frac{x^2}{4} + \frac{y^2}{3} = 1$$

Differentiate both sides with respect to x

$$\Rightarrow \frac{2x}{4} + \frac{2y}{3} \cdot \frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = \frac{-3x}{4y}$$ 1

At P, $\frac{dy}{dx} = \frac{-3x_1}{4y_1}$

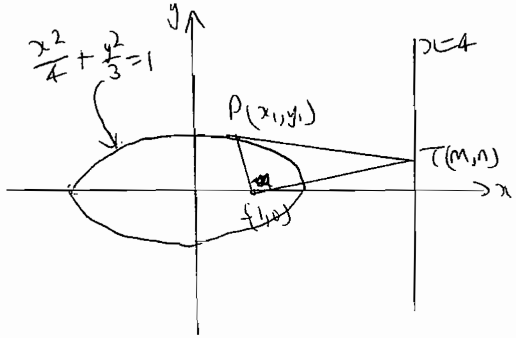
equation of tangent at P: $y - y_1 = -\frac{3x_1}{4y_1}(x - x_1)$

$$4yy_1 - 4y_1^2 = -3xx_1 + 3x_1^2$$ 1
$$3xx_1 + 4yy_1 = 3x_1^2 + 4y_1^2$$
$$\frac{xx_1}{4} + \frac{yy_1}{3} = \frac{x_1^2}{4} + \frac{y_1^2}{3}$$

But $\frac{x_1^2}{4} + \frac{y_1^2}{3} = 1$ because $P(x_1, y_1)$ lies on ellipse 1

$$\therefore \frac{xx_1}{4} + \frac{yy_1}{3} = 1$$

(vi)



To find coordinates of T

sub $x = 4$ into equation of PT

$$\Rightarrow x_1 + \frac{yy_1}{3} = 1$$
$$y = \frac{3}{y_1}(1 - x_1)$$

i.e. T is $\left(4, \frac{3}{y_1}(1 - x_1)\right)$

gradient $SP = \frac{y_1}{x_1 - 1}$

gradient $ST = \frac{\frac{3}{y_1}(1 - x_1)}{3} = \frac{1 - x_1}{y_1}$

gradient $SP \times$ gradient $ST = \frac{y_1}{x_1 - 1} \times \frac{1 - x_1}{y_1}$

$$= -1$$

$\therefore SP \perp ST$

Q2 (a) When $n=1$, LHS $= 4^1 = 4$

RHS $= 2\times1+1 = 3$

LHS $>$ RHS 1

$\therefore$ true for $n=1$

Assume true for $n=k$, where k is a positive integer

i.e. $4^k > 2k+1$

Try to prove true for $n=k+1$

i.e. required to prove $4^{k+1} > 2(k+1)+1$

i.e. $4^{k+1} > 2k+3$

proof: $4^{k+1} = 4\times 4^k > 4(2k+1)$ from assumption 1

i.e. $4^{k+1} > 8k+4$ 1

$> 2k+3$ because $k>0$

$\therefore$ true for $n=k+1$ if true for $n=k$

But already proved true for $n=1$

$\therefore$ true for $n=2,3,4,\ldots$ etc 1

and $\therefore$ true for all positive integers n

(b) (i) $\dfrac{2}{(x-1)(x+3)} = \dfrac{a}{x-1} + \dfrac{b}{x+3}$

$2 = a(x+3) + b(x-1)$

sub. $x=1 \Rightarrow 4a=2 \Rightarrow a=\frac12$ 1

sub. $x=-3 \Rightarrow -4b=2 \Rightarrow b=-\frac12$ 1

then $\dfrac{2}{(x-1)(x+3)} = \dfrac{1}{2(x-1)} - \dfrac{1}{2(x+3)}$

(ii) $\dfrac{4x+2}{(x+3)(x^2+1)} = \dfrac{a}{x+3} + \dfrac{bx+c}{x^2+1}$

$4x+2 = a(x^2+1) + (x+3)(bx+c)$ 1

sub. $x=-3 \Rightarrow 10a=-10 \Rightarrow a=-1$ 1

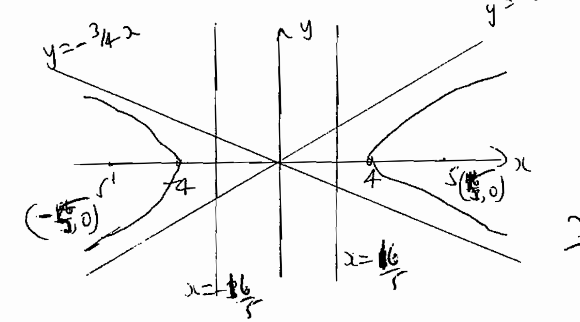
equate coefficients of $x^2 \Rightarrow a+b=0 \Rightarrow b=1$ 1

equate constants $\Rightarrow a+3c=2 \Rightarrow c=1$

then $\dfrac{4x+2}{(x+3)(x^2+1)} = \dfrac{-1}{x+3} + \dfrac{x+1}{x^2+1}$

Q3

(i)



4

-1 for each part missing

$\dfrac{x^2}{16} - \dfrac{y^2}{9} = 1$ $\quad a=4,\ b=3$

$e = \sqrt{1+\frac{b^2}{a^2}} = \frac54$

(ii) $x = 4\sec\theta$

$\dfrac{dx}{d\theta} = 4\sec\theta\tan\theta$

$y = 3\tan\theta$

$\dfrac{dy}{d\theta} = 3\sec^2\theta$

$\dfrac{dy}{dx} = \dfrac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \dfrac{3\sec^2\theta}{4\sec\theta\tan\theta}$

$= \dfrac{3\sec\theta}{4\tan\theta}$ 1

gradient of normal at $P = \dfrac{-4\tan\theta}{3\sec\theta}$

OR

$9x^2 - 16y^2 = 144$

Differentiate both sides with respect to x

$\Rightarrow 18x - 32y\dfrac{dy}{dx} = 0$

$\dfrac{dy}{dx} = \dfrac{18x}{32y} = \dfrac{9x}{16y}$ 1

gradient of normal at P

$= -\dfrac{16y_1}{9x_1}$

$= -\dfrac{48\tan\theta}{36\sec\theta}$

$= -\dfrac{4\tan\theta}{3\sec\theta}$

equation of normal at P is

$y - 3\tan\theta = \dfrac{-4\tan\theta}{3\sec\theta}(x - 4\sec\theta)$ 1

$3y\sec\theta - 9\tan\theta\sec\theta = -4x\tan\theta + 16\sec\theta\tan\theta$

$4x\tan\theta + 3y\sec\theta = 25\tan\theta\sec\theta$

$\dfrac{4x}{\sec\theta} + \dfrac{3y}{\tan\theta} = 25$ 1

OR

equation of normal at (x_1, y_1) is

$y - y_1 = \dfrac{-16y_1}{9x_1}(x - x_1)$ 1

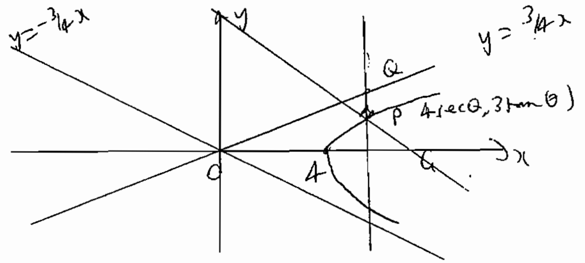
$9x_1 y - 9x_1 y_1 = -16xy_1 + 16x_1 y_1$

$16xy_1 + 9yx_1 = 25x_1 y_1$

$\dfrac{16x}{x_1} + \dfrac{9y}{y_1} = 25$

sub. $x_1 = 4\sec\theta,\ y_1 = 3\tan\theta$

(iii)



To find coordinates of Q: sub. $x = 4\sec\theta$ into $y = \frac{3}{4}x$

$\Rightarrow \quad y = 3\sec\theta$

i.e. Q is $(4\sec\theta, 3\sec\theta)$ 1

To find coordinates of G: sub. $y=0$ into $\frac{4x}{\sec\theta} + \frac{3y}{\tan\theta} = 25$

$\Rightarrow \quad x = \frac{25\sec\theta}{4}$

i.e. G is $\left(\frac{25}{4}\sec\theta, 0\right)$ 1

(iv) gradient $OQ = \frac{3}{4}$ 1

gradient $GQ = \frac{-3\sec\theta}{\frac{9}{4}\sec\theta} = \frac{-12}{9} = -\frac{4}{3}$

gradient OQ × gradient GQ = −1 1

$\therefore OQ \perp GQ$

Q4

(i) gradient $PQ = \dfrac{\frac{2}{q} - \frac{2}{p}}{2q-2p} = \dfrac{\frac{p-q}{pq}}{q-p} = -\dfrac{1}{pq}$ 1

equation PQ: $y - \frac{2}{p} = -\frac{1}{pq}(x - 2p)$ 1

$pqy - 2q = -x + 2p$

i.e. $x + pqy = 2(p+q)$ 1

(ii) PQ passes through $(4, 2)$

sub. $x=4, y=2 \Rightarrow 4 + 2pq = 2(p+q)$

$2 + pq = p + q$ 1

i.e. $pq = p + q - 2$

(iii) Coords. of M are $\left(\frac{2p+2q}{2}, \frac{\frac{2}{p}+\frac{2}{q}}{2}\right)$

$= \left(p+q, \frac{1}{p}+\frac{1}{q}\right)$ 1

$x = p + q \quad \ldots (1)$

$y = \frac{1}{p} + \frac{1}{q} \quad \ldots (2)$

$y = \frac{q+p}{pq} = \frac{p+q}{p+q-2}$ 1

i.e. $y = \frac{x}{x-2}$ 1