Ext 2

NORTH SYDNEY BOYS HIGH SCHOOL

2010 HSC ASSESSMENT TASK 2

Mathematics
Extension 2

General Instructions

- Working time – **60 minutes**
- Write in the booklet provided on both sides of the paper
- Write using blue or black pen
- Board approved calculators may be used
- All necessary working should be shown in every question
- Each new question is to be started on a **new page.**

- Attempt all questions

Class Teacher:
(Please tick or highlight)

- O Mr Barrett
- O Mr Trenwith
- O Mr Weiss

Student Number:________________________

(To be used by the exam markers only.)

Question No	1	2	3	4	5	6	7	Total	Total %
Mark	$\overline{5}$	$\overline{15}$	$\overline{4}$	$\overline{5}$	$\overline{6}$	$\overline{4}$	$\overline{6}$	$\overline{45}$	$\overline{100}$

Question 1 (5 marks)

Sketch a graph of $\frac{x^2}{9} - \frac{y^2}{16} = 1$, showing the coordinates of the vertices and foci, and the equations of the directrices and asymptotes. **5**

Question 2 (15 marks)

(a) Sketch a graph of $x = 2t$, $y = \frac{2}{t}$, showing the coordinates of the vertices and foci, and the equations of the directrices. **3**

(b) Show that the equation of the tangent to this curve at the point P where $t = p$ is $x + p^2y = 4p$. **2**

(c) This tangent intersects the x-axis at A, and the y-axis at B, and O is the origin. Show that the area of triangle OAB is independent of the position of P. **3**

(d) Q is the point on the curve where $t = q$. **3**
Show that the tangents at P and Q intersect at $T\left(\frac{4pq}{p+q}, \frac{4}{p+q}\right)$.

(e) The chord PQ has a gradient of $-\frac{1}{2}$. **2**
Show that $pq = 2$.

(f) Hence find the Cartesian equation for the locus of T. **2**

Question 3 (4 marks)

(a) Write down **in integral form** an expression for the area enclosed by the ellipse $\frac{x^2}{9} + \frac{y^2}{4} = 1$. **2**

(b) **Hence**, find the area enclosed by the ellipse. **2**

Question 4 (5 marks)

$P(a\cos\theta,\ b\sin\theta)$ is a point on the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$.

(a) Show that the equation of the normal to P is: **3**

$$ax\sin\theta - by\cos\theta = \left(a^2 - b^2\right)\sin\theta\cos\theta$$

(b) G is the point where this normal meets the x-axis, N is the foot of the perpendicular from P to the x-axis, and O is the origin. **2**

Show that $\dfrac{OG}{ON} = e^2$.

Question 5 (6 marks)

(a) Decompose $\dfrac{x^2 - x + 2}{(x+1)\left(x^2+1\right)}$ into partial fractions. **3**

(b) Hence find $\displaystyle\int \frac{x^2 - x + 2}{(x+1)\left(x^2+1\right)}\,dx$, writing your answer in the form $\ln[f(x)] + c$. **3**

Question 6 (4 marks)

It can be shown that $1^2 + 2^2 + 3^2 + \ldots + n^2 = \dfrac{1}{6}n(n+1)(2n+1)$.

[DO NOT prove this result.]

Prove by mathematical induction (using the result above) that:

$$1^2 + (1^2 + 2^2) + (1^2 + 2^2 + 3^2) + \ldots + \sum_{k=1}^{n} n^2 = \frac{n}{12}(n+1)^2(n+2)$$

Question 7 (6 marks)

(a) Show that if $y = mx + k$ is a tangent to the hyperbola $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$, then $a^2m^2 - b^2 = k^2$. **3**

(b) Hence find the equation of the tangents from the point $(1, 3)$ to the hyperbola $\dfrac{x^2}{4} - \dfrac{y^2}{15} = 1$, and the coordinates of their points of contact. **3**

Question 1

$$\frac{x^2}{9} - \frac{y^2}{16} = 1$$

5

$b^2 = a^2(e^2-1)$

$16 = 9(e^2-1)$

$e^2 - 1 = \frac{16}{9}$

$e^2 = \frac{25}{9}$

$\boxed{e = \frac{5}{3}}$ – [1]

$ae = 3 \times \frac{5}{3}$

$= 5$

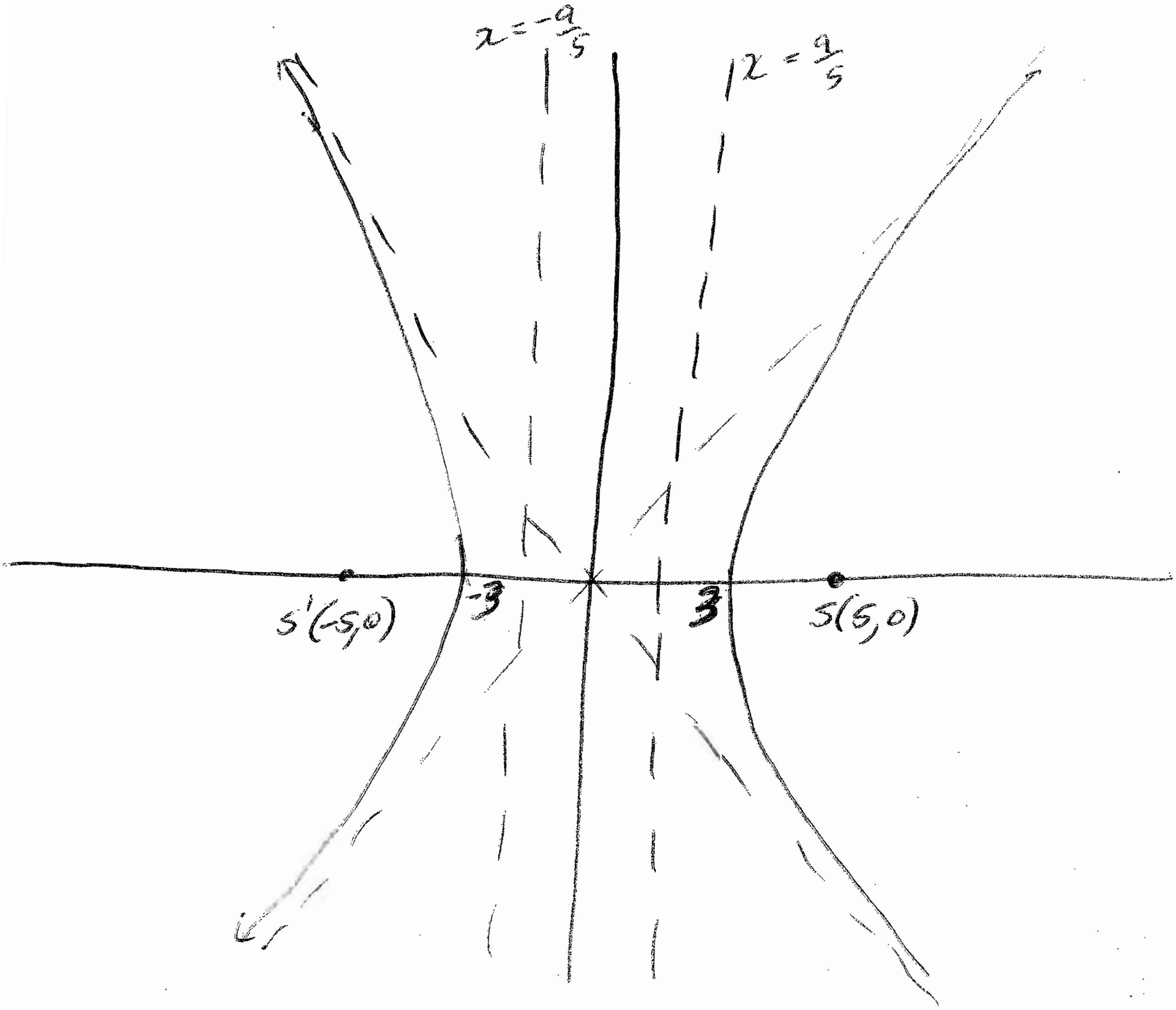
$\therefore$ $\boxed{\text{Foci } (\pm 5, 0)}$ – [1]

$\frac{a}{e} = 3 \times \frac{3}{5}$

$= \frac{9}{5}$

$\boxed{\text{directrices: } x = \pm \frac{9}{5}}$ – [1]

$\boxed{\text{asymptotes: } y = \pm \frac{4}{3}x}$ – [1]

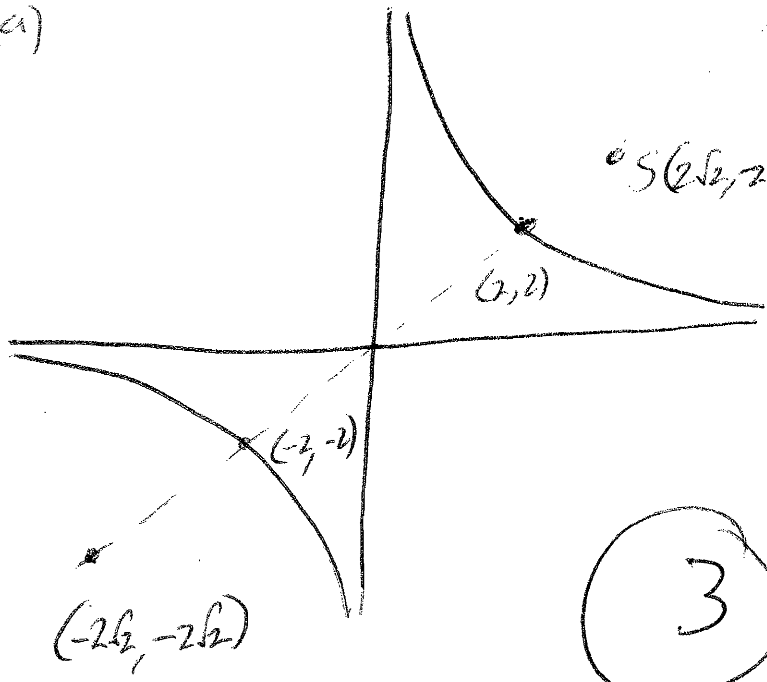


– [1] for correct shape & correct vertices (don't penalise if they write an x-int of 3 instead of (3,0))

Question 2

15

(a)



$xy = 4$

$V(\pm 2, \pm 2)$ [1]

$S(\pm 2\sqrt{2}, \pm 2\sqrt{2})$ [1]

directrices : $x + y = \pm 2\sqrt{2}$

[1]

3

(b) $x = 2t \qquad y = \dfrac{2}{t}$

$\dfrac{dx}{dt} = 2 \qquad \dfrac{dy}{dt} = \dfrac{-2}{t^2}$

$\dfrac{dy}{dx} = \dfrac{-2}{t^2} \cdot \dfrac{1}{2}$

$= -\dfrac{1}{t^2}$

$\therefore m_T = -\dfrac{1}{p^2}$ [1]

$y - \dfrac{2}{p} = -\dfrac{1}{p^2}(x - 2p)$

$p^2 y - 2p = -x + 2p$

$x + p^2 y = 4p$ [1]

2

(c) x int $(y=0) \Rightarrow x = 4p$ — [1]

y int $(x=0) \Rightarrow y = \frac{4}{p}$ — [1]

3

$\therefore$ Area $= \frac{1}{2} \times 4p \times \frac{4}{p}$

$= 8$ units^2 (which is indep of p) [1]

(d) $x + p^2y = 4p$

$x + q^2y = 4q$

$\ominus$: $(p^2 - q^2)y = 4(p-q)$ [1]

$y = \frac{4}{p+q}$ [1]

$x + \frac{4p^2}{p+q} = 4p$

$x = \frac{4p(p+q) - 4p^2}{p+q}$

$x = \frac{4pq}{p+q}$ [1]

$\therefore T\left[\frac{4pq}{p+q}, \frac{4}{p+q}\right]$

3

(e) $m_{PQ} = -\frac{1}{2}$

$\frac{\frac{2}{p} - \frac{2}{q}}{2p - 2q} = -\frac{1}{2}$ [1]

cross multiply: $\frac{4}{p} - \frac{4}{q} = 2q - 2p$

2

$\times pq$: $4q - 4p = 2pq(q-p)$

$4(q-p) = 2pq(q-p)$

$pq = 2$ [1]

(f) $pq = 2 \Rightarrow x = \dfrac{8}{p+q}, \quad y = \dfrac{4}{p+q}$

$\Downarrow$

$p+q = \dfrac{8}{x} \Rightarrow y = 4 \times \dfrac{x}{8}$

[1] $\quad \boxed{y = \dfrac{x}{2}}$ [1]

2

Question 3

(a) $\dfrac{x^2}{9} + \dfrac{y^2}{4} = 1$

$\dfrac{y^2}{4} = 1 - \dfrac{x^2}{9}$

$= \dfrac{9-x^2}{9}$

$y^2 = \dfrac{4}{9}(9-x^2)$

$y = \pm \dfrac{2}{3}\sqrt{9-x^2}$

[1]

2

$\therefore A = 4\int_0^3 \dfrac{2}{3}\sqrt{9-x^2}\,dx$

$\boxed{A = \dfrac{8}{3}\int_0^3 \sqrt{9-x^2}\,dx}$

[1]

OR $x = \pm \dfrac{3}{2}\sqrt{4-y^2}$

$\therefore A = 6\int_0^2 \sqrt{4-y^2}\,dy$

(b) $A = \dfrac{8}{3} \times \dfrac{\pi}{4} \times 3^2$

$A = 6\pi$ units^2 (2)

Question 4

(a) $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$

$\dfrac{2x}{a^2} + \dfrac{2y}{b^2} \cdot \dfrac{dy}{dx} = 0$

$\dfrac{dy}{dx} = \dfrac{-b^2x}{a^2y}$

at P, $m_T = \dfrac{-b\cos\theta}{a\sin\theta}$

$\therefore\ m_N = \dfrac{a\sin\theta}{b\cos\theta}$ [2]

$\Rightarrow y - b\sin\theta = \dfrac{a\sin\theta}{b\cos\theta}(x - a\cos\theta)$

$by\cos\theta - b^2\sin\theta\cos\theta = ax\sin\theta - a^2\sin\theta\cos\theta$

$\boxed{ax\sin\theta - by\cos\theta = (a^2-b^2)\sin\theta\cos\theta}$

[1]

3

(b) $(y=0)$ $ax\sin\theta = (a^2-b^2)\sin\theta\cos\theta$

$x_G = \dfrac{a^2-b^2}{a}\cos\theta$ [1]

$\dfrac{OG}{ON} = \dfrac{\frac{a^2-b^2}{a}\cos\theta}{a\cos\theta}$

$= \dfrac{a^2-b^2}{a^2}$

$= 1 - \dfrac{b^2}{a^2}$

$= e^2$ [1]

2

5

Question 5

(a) $\dfrac{x^2-x+2}{(x+1)(x^2+1)} = \dfrac{a}{x+1} + \dfrac{bx+c}{x^2+1}$

$x^2-x+2 = a(x^2+1) + (bx+c)(x+1)$

$(x=-1)$: $4 = 2a$

$\underline{a=2}$ [1]

$(x=0)$: $2 = a+c$

$\underline{c=0}$ [1]

$(x=1)$: $2 = 2a + 2(b+c)$

$2 = 4 + 2b$

$\underline{b=-1}$ [1]

$\therefore \dfrac{x^2-x+2}{(x+1)(x^2+1)} = \dfrac{2}{x+1} - \dfrac{x}{x^2+1}$

3

(b) $\displaystyle\int \frac{x^2-x+2}{(x+1)(x^2+1)}\,dx = \int\left(\frac{2}{x+1} - \frac{x}{x^2+1}\right)dx$

$= 2\ln(x+1) - \frac{1}{2}\ln(x^2+1) + C$ [2]

$= \ln\left[\dfrac{(x+1)^2}{\sqrt{x^2+1}}\right] + C$ [1]

3

Question 6

Prove $1^2 + (1^2+2^2) + \ldots + \sum_{k=1}^{n} k^2 = \frac{n}{12}(n+1)^2(n+2)$

Test $n=1$: $LHS = 1^2 = 1$ $\quad RHS = \frac{1}{12}(2)^2(3)$
$= 1$

$\therefore$ true for $n=1$ [1]

Assume true for $n = \cancel{k}$

Prove true for $n+1$

$$LHS = 1^2 + (1^2+2^2) + \ldots + \sum_{k=1}^{n} k^2 + \sum_{k=1}^{n+1} k^2 \quad [1]$$

$$= \frac{n}{12}(n+1)^2(n+2) + \sum_{k=1}^{n+1} k^2 \quad \text{(by assumption)}$$

$$= \frac{n}{12}(n+1)^2(n+2) + \frac{1}{6}(n+1)(n+2)(2n+3) \quad [1]$$

$$= \frac{1}{12}(n+1)(n+2)\left[n(n+1) + 2(2n+3)\right]$$

$$= \frac{1}{12}(n+1)(n+2)(n^2 + 5n + 6)$$

$$= \frac{1}{12}(n+1)(n+2)(n+2)(n+3)$$

$$= \frac{n+1}{12}(n+2)^2(n+3) \quad [1]$$

$\therefore$ BLAH BLAH BLAH

4

Question 7

(a) $\frac{x^2}{a^2} - \frac{(mx+k)^2}{b^2} = 1$

$b^2x^2 - a^2(mx+k)^2 = a^2b^2$

$b^2x^2 - a^2m^2x^2 - 2a^2mkx - a^2k^2 = a^2b^2$

$(b^2-a^2m^2)x^2 - 2a^2mkx - a^2(b^2+k^2) = 0$ [1]

$\Delta = 0$

$4a^4m^2k^2 + 4a^2(b^2-a^2m^2)(b^2+k^2) = 0$ [1]

$(\div 4a^2)$ $a^2m^2k^2 + b^4 + b^2k^2 - a^2b^2m^2 - a^2m^2k^2 = 0$

$b^2(b^2 - a^2m^2 + k^2) = 0$

$b \neq 0$, $\therefore k^2 = a^2m^2 - b^2$ [1] 3

(b) $y = mx + k$, $(1,3) \Rightarrow 3 = m + k$ [1] $\Rightarrow k = 3 - m$

$a^2 = 4$, $b^2 = 15 \Rightarrow 4m^2 - 15 = k^2$ [1]

$4m^2 - 15 = (3-m)^2$

$4m^2 - 15 = 9 - 6m + m^2$

$3m^2 + 6m - 24 = 0$

$m^2 + 2m - 8 = 0$

$(m+4)(m-2) = 0$

$m = -4, 2$

$k = 7, 1$ [1]

3

$\therefore$ $\boxed{y = -4x + 7 \;\&\; y = 2x + 1}$