# Ext 2

# NORTH SYDNEY BOYS HIGH SCHOOL

**2011**
**ASSESSMENT TASK 2**

# Mathematics
## Extension 2

### General Instructions

- Working time – 55 minutes
- Write in the booklet provided
- Write using blue or black pen
- Board approved calculators may be used
- All necessary working should be shown in every question
- Each new question is to be started on a **new page.**

- Attempt all questions

**Class Teacher:**
(Please tick or highlight)
- O Mr Fletcher
- O Mr Ireland
- O Mrs Collins/Mr Rezcallah

Student Number:

(To be used by the exam markers only.)

| Question No | 1 | 2 | 3 | Total | Total |
|---|---|---|---|---|---|
| Mark | /15 | /15 | /8 | /38 | /100 |

## Question 1 (15 marks)

**Marks**

(a) For the hyperbola $\dfrac{x^2}{144}-\dfrac{y^2}{25}=1$ find

(i) The eccentricity. 1

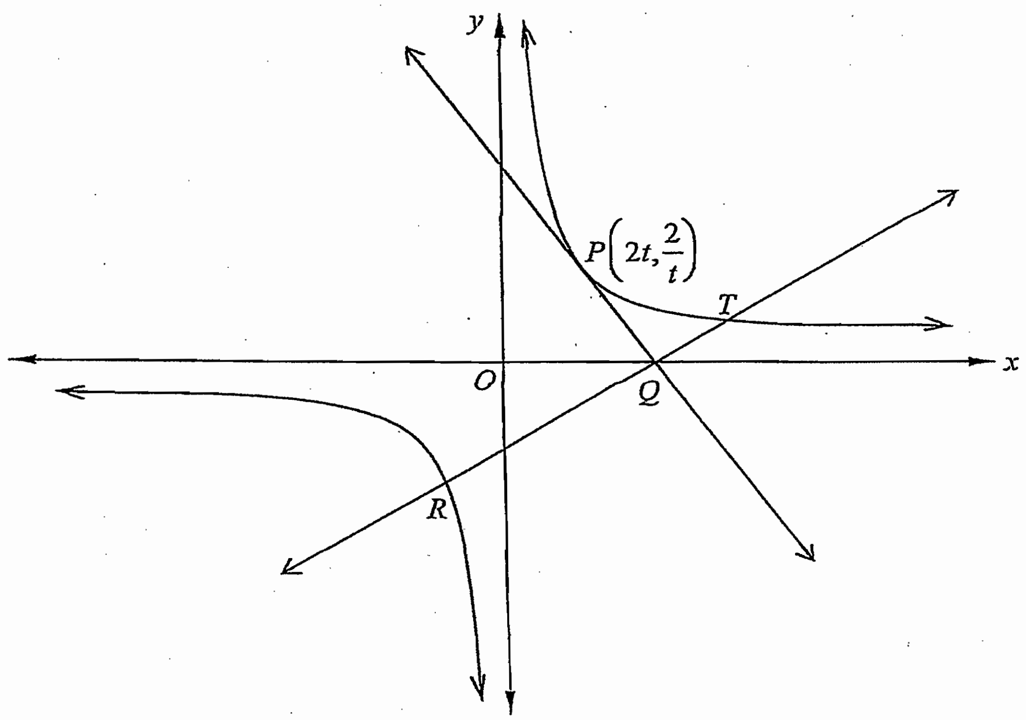
(ii) The co-ordinates of the foci. 1

(iii) The equations of the asymptotes. 1

(iv) The equations of the directrices. 1

Sketch the graph of the hyperbola showing the above information. 2

(b) $P\left(2t,\dfrac{2}{t}\right)$ is a point on the rectangular hyperbola $xy=4$. The tangent at $P$ cuts the $x$-axis at $Q$ and the line through $Q$, perpendicular to the tangent at $P$, cuts the hyperbola at the points $R$ and $T$ as shown.




(i) Show that the equation of the tangent at $P$ is $x+t^2y=4t$. 2

(ii) Show that the line through $Q$, perpendicular to the tangent at $P$, has equation $t^2x-y=4t^3$. 3

(iii) If $M$ is the midpoint of $RT$, show that $M$ has coordinates $\left(2t,\,-2t^3\right)$. 3

(iv) Find the equation of the locus of $M$, as $P$ moves on the curve $xy=4$. 1

**Question 2 (15 marks)** **Marks**

(a) Consider the ellipse $\frac{x^2}{16}+\frac{y^2}{12}=1$

(i) Use differentiation to derive the equations of the tangent and normal to the ellipse at the point $P(2, 3)$ **3**

(ii) The tangent and normal to the ellipse at $P$ cut the $y$ axis at $A$ and $B$ respectively. Find the co-ordinates of $A$ and $B$. **2**

(iii) Show that $AB$ subtends a right angle at the focus $S$ of the ellipse **2**

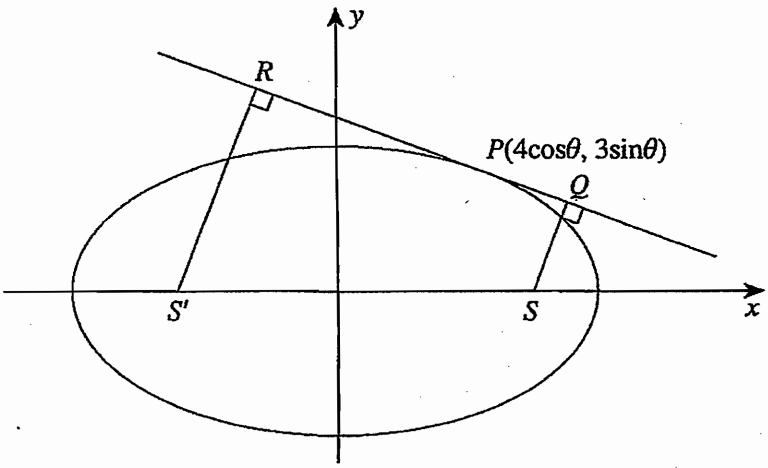
(iv) Show that $A,P,S$ and $B$ are concyclic **1**

(v) Find the centre and radius of the circle which passes through the points $A,P,S$ and $B$. **2**

(b) Consider the ellipse $\frac{x^2}{16}+\frac{y^2}{9}=1$.

A tangent to this ellipse is drawn at point $P(4\cos\theta, 3\sin\theta)$

Perpendiculars are drawn from each focus of the ellipse to meet the tangent at $Q$ and $R$ as shown on the diagram.




(i) Prove that the equation of the tangent at $P$ is $\frac{x\cos\theta}{4}+\frac{y\sin\theta}{3}=1$. **2**

(ii) Show that $QS \times RS' = 9$. **3**

## Question 3 (8 marks)

**Marks**

(a) Resolve $\frac{1}{1-x^4}$ into partial fractions **3**

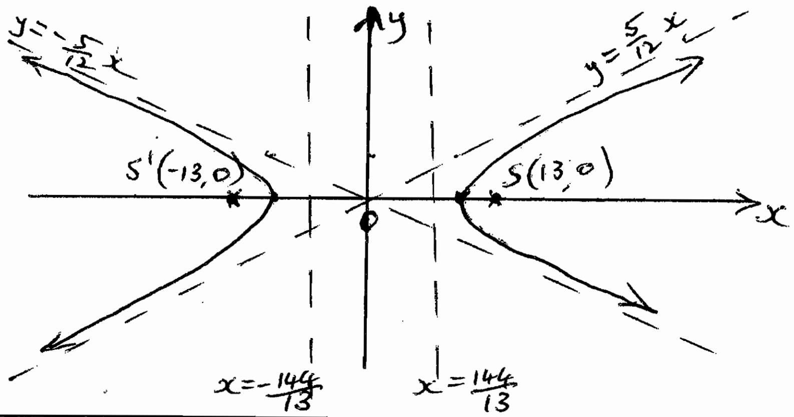
(b) (i) Prove by Mathematical Induction that if $n$ is a positive integer, **4**

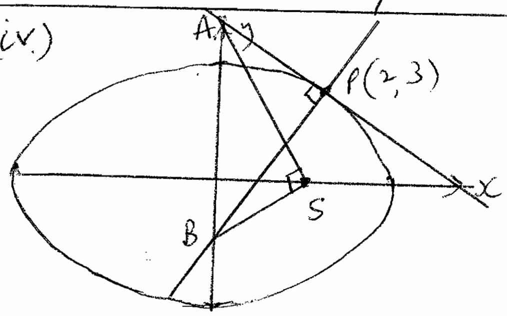
then $2^{(n+4)} > (n+4)^2$

(ii) By choosing a suitable substitution, or otherwise, show that **1**

if $a$ is a positive integer, then $2^{3(a+2)} > 9(a+2)^2$

SOLUTIONS EXTENSION 2 TASK 2 2011

| SAMPLE ANSWERS | CRITERIA | MARKS |
|---|---|---|
| 1 (a) $\frac{x^2}{144} - \frac{y^2}{25} = 1$ $\quad a = 12,\ b = 5$ | | |
| (i) $b^2 = a^2(e^2-1)$ <br> $e^2 = \frac{25}{144} + 1$ <br> $= \frac{169}{144}$ <br> $e = \frac{13}{12}$ $\quad e > 0$ | Finds eccentricity | 1 |
| (ii) foci $(\pm ae, 0)$ $\quad ae = 12 \times \frac{13}{12}$ <br> $\therefore$ foci $(\pm 13, 0)$ | Finds coords of foci | 1 |
| (iii) Asymptotes : $y = \pm \frac{b}{a}x$ <br> $y = \pm \frac{5}{12}x$ | Finds equations of asymptotes | 1 |
| (iv) Directrices $x = \pm \frac{a}{e}$ <br> $x = \pm \frac{144}{13}$ $\left(= \pm 11\frac{1}{13}\right)$ | Finds equations of directrices | 1 |
|  | Graphs correct shape determined by correctly labelled asymptotes | 1 |
| | Indicates correctly directrices & foci | 1 |
| (b) (i) $y = 4x^{-1}$ $\quad \frac{dy}{dx} = -4x^{-2}$ <br> At $x = 2t$, $m = -\frac{1}{t^2}$ <br> $\therefore$ Equation of tangent is <br> $y - \frac{2}{t} = -\frac{1}{t^2}(x - 2t)$ | Finds correct gradient or makes significant progress towards solution. | 1 |
| $t^2y - 2t = -x + 2t$ <br> $x + t^2y = 4t$ | Shows correct. equation | 1 |
| (ii) At Q, $y = 0$ $\therefore$ from (i) $x = 4t$ <br> ie. $Q(4t, 0)$ | Finds correct coords of Q | 1 |
| Line through Q $\perp$ tangent at P has <br> $m = t^2$ <br> $\therefore$ Equation of line is <br> $y - 0 = t^2(x - 4t)$ <br> $y = t^2x - 4t^3$ | Finds correct gradient or makes significant progress towards solution | 1 |
| $\therefore t^2x - y = 4t^3$ | Shows correct equation | 1 |
| (iii) R and T lie on the hyperbola and the line through Q. Solve $xy = 4$ and $t^2x - y = 4t^3$ <br> $x(t^2x - 4t^3) = 4$ | Establishes correct equation | 1 |
| $t^2x^2 - 4t^3x - 4 = 0$ Let roots be $\alpha, \beta$. <br> then $\alpha + \beta = \frac{4t^3}{t^2} = 4t$ | Finds x or y coordinate | 1 |
| $M(x, y)$ is halfway between the roots <br> $\therefore x = \frac{\alpha+\beta}{2} = 2t$ <br> As y lies on RT $y = t^2(2t) - 4t^3 = -2t^3$ <br> $\therefore M(2t, -2t^3)$ | Finds the midpoint | 1 |

| SAMPLE ANSWERS | CRITERIA | MARKS |
|---|---|---|
| ① (iv) At $M$, $x = 2t$ $y = -2t^3$<br>$\therefore y = -2\left(\frac{x}{2}\right)^3 = -\frac{x^3}{4}$<br>Thus the locus of $M$ is the curve $y = -\frac{x^3}{4}$, $x \neq 0, y \neq 0$ | Finds the locus | 1 |
| ② (a) (i) $\frac{x^2}{16} + \frac{y^2}{12} = 1 \Rightarrow \frac{2x}{16} + \frac{2y}{12}\frac{dy}{dx} = 0$.<br>$\frac{dy}{dx} = \frac{-x}{8} \div \frac{y}{6} = \frac{-3x}{4y}$ | Obtains correct expression for $\frac{dy}{dx}$ | 1 |
| At $P(2,3)$ $m = -\frac{1}{2}$<br>$\therefore$ Equation of tangent at $P$ is:<br>$y - 3 = -\frac{1}{2}(x-2)$<br>$2y - 6 = -x + 2$<br>$x + 2y - 8 = 0$ | Finds correct equation of tangent | 1 |
| Equation of normal at $P$ is<br>$y - 3 = 2x - 4$<br>$2x - y - 1 = 0$ | Finds correct equation of normal | 1 |
| (ii) Sub $x = 0$ into equation of tangent<br>$2y - 8 = 0 \Rightarrow A(0,4)$ | Finds coords of $A$ | 1 |
| Sub $x = 0$ into equation of normal<br>$-y - 1 = 0 \Rightarrow B(0,-1)$ | Finds coords of $B$ | 1 |
| (iii) $b^2 = a^2(1-e^2) \Rightarrow e = \frac{1}{2}$<br>Foci $(\pm ae, 0) \Rightarrow S(\frac{1}{2} \times 4, 0) = (2,0)$ | Correctly finds coords of focus | 1 |
| $S(2,0)$ $A(0,4)$ $B(0,-1)$<br>$m_{AS} \cdot m_{BS} = -2 \times \frac{1}{2}$<br>$= -1$<br>$\therefore \angle ASB = 90°$ | Shows $m_{AS} \cdot m_{BS} = -1$ | 1 |
| (iv)   AP is tangent at P<br>BP is normal at P<br>$\therefore \angle APB = 90°$<br>$\therefore$ AB subtends equal angles of 90° at P and S<br>$\therefore$ APSB are concyclic | Shows A, P, S and B are concyclic. | 1 |
| (v) Diameter is $AB$, so centre is $(0, \frac{3}{2})$ | Finds centre | 1 |
| Radius is $2\frac{1}{2} = \frac{5}{2}$ | Finds radius | 1 |

| Sample Answers | Criteria | Marks |
|---|---|---|
| (2) (b) (i) $x = 4\cos\theta$ $y = 3\sin\theta$<br>$\frac{dx}{d\theta} = -4\sin\theta$ $\frac{dy}{d\theta} = 3\cos\theta$<br>$\frac{dy}{dx} = -\frac{3\cos\theta}{4\sin\theta}$ | Correctly finds $\frac{dy}{dx}$ | 1 |
| Equation of tangent at $P$ is:<br>$y - 3\sin\theta = -\frac{3\cos\theta}{4\sin\theta}(x - 4\cos\theta)$<br>$4y\sin\theta - 12\sin^2\theta = -3x\cos\theta + 12\cos^2\theta$<br>$3x\cos\theta + 4y\sin\theta = 12(\sin^2\theta + \cos^2\theta)$<br>$\frac{3x\cos\theta}{12} + \frac{4y\sin\theta}{12} = 1$<br>$\frac{x\cos\theta}{4} + \frac{y\sin\theta}{3} = 1$ as required. | Uses correct process to find required equation. | 1 |
| (ii) $a = 4$, $b = 3$ $e = \sqrt{1 - \frac{9}{16}} = \frac{\sqrt{7}}{4}$<br>$\therefore S$ and $S'$ have coords $(\sqrt{7}, 0)$ and $(-\sqrt{7}, 0)$ respectively | Correctly finds $e$ and coords of $S$ and $S'$ | 1 |
| Equation of tangent is<br>$3x\cos\theta + 4y\sin\theta - 12 = 0$<br>So, $d_{QS} = \frac{\lvert 3\sqrt{7}\cos\theta - 12\rvert}{\sqrt{9\cos^2\theta + 16\sin^2\theta}}$<br>$d_{RS'} = \frac{\lvert -(3\sqrt{7}\cos\theta + 12)\rvert}{\sqrt{9\cos^2\theta + 16\sin^2\theta}}$ | Correctly finds expressions for $d_{QS}$ and $d_{RS'}$ | 1 |
| $\therefore QS \times RS' = \frac{\lvert -(63\cos^2\theta - 144)\rvert}{9\cos^2\theta + 16\sin^2\theta}$<br>$= \frac{\lvert 9(16 - 7\cos^2\theta)\rvert}{9\cos^2\theta + 16(1 - \cos^2\theta)}$<br>$= \frac{\lvert 9(16 - 7\cos^2\theta)\rvert}{16 - 7\cos^2\theta}$<br>$= 9$ as required. | Simplifies the product of $d_{QS}$ and $d_{RS'}$ correctly to give the desired result. | 1 |

| SAMPLE ANSWERS | CRITERIA | MARKS |
|---|---|---|
| (3) (a) $\frac{1}{1-x^4} = \frac{A}{1-x} + \frac{B}{1+x} + \frac{Cx+D}{1+x^2}$ $= \frac{A(1+x)(1+x^2)+B(1-x)(1+x^2)+(1-x^2)(Cx+D)}{1-x^4}$ | Correctly factorises $1-x^4$ and forms sum of 3 algebraic fractions | 1 |
| $= \frac{A(1+x+x^2+x^3)+B(1-x+x^2-x^3)+Cx+D-Cx^3-Dx^2}{1-x^4}$ $= \frac{(A-B-C)x^3+(A+B-D)x^2+(A-B+C)x+(A+B+D)}{1-x^4}$ | Correctly forms equations | 1 |
| $A-B-C=0$ — (1); $A+B-D=0$ — (2); $A-B+C=0$ — (3); $A+B+D=1$ — (4). (4)−(2) $\Rightarrow$ $2D=1$, $D=\frac{1}{2}$. (3)−(1) $\Rightarrow$ $2C=0$, $C=0$. $\therefore A-B=0$ — (5); $A+B=\frac{1}{2}$ — (6) $\Rightarrow A=\frac{1}{4}, B=\frac{1}{4}$ | Finds correct values for $A, B, C, D$ | 1 |
| $\therefore \frac{1}{1-x^4} = \frac{1}{4(1-x)} + \frac{1}{4(1+x)} + \frac{1}{2(1+x^2)}$ | $\frac{1}{1-x^4} = \frac{1}{2(1-x^2)} + \frac{1}{2(1+x^2)}$ | Max 2 |
| (b) (i) RTP $2^{(n+4)} > (n+4)^2$. For $n=1$, $2^5 > 5^2$, $32 > 25$ which is true | Correctly shows result is true for $n=1$ | 1 |
| Assume the result is true for $n=k$ i.e. $2^{(k+4)} > (k+4)^2$. Consider $n=k+1$, then $2^{k+1+4} = 2(2^{k+4})$ ie RTP $2^{k+5} > 2(k+4)^2$ | Uses correct process to arrive at $2^{k+5} > (k+5)^2$ | 1 |
| $= 2k^2+16k+32$ $= k^2+10k+25+k^2+6k+7$ $= (k+5)^2+k^2+6k+7$ | Shows $k^2+6k+7>0$ since $k>0$. Shows $2k^2+16k+32 > k^2+10k+25$ | 1 |
| Now, since $k$ is a positive integer, $k^2+6k+7>0$ $\therefore 2^{k+5} > (k+5)^2$ i.e. $2^{[(k+1)+4]} > [(k+1)+4]^2$. $\therefore$ If the result is true for $n=k$, it is also true for $n=k+1$. Now, the result is true for $n=1$, hence it is true for $n=2$ and $n=3$ and so on for all positive integers $n$. | Correctly shows result is true for $n=k+1$, given it is true for $n=k$ and Concludes by induction result is true for all $n$ i.e, must show true for $n=1, n=2, \ldots$ | 1 |
| (ii) Now let $3(a+2) = n+4$ $\therefore 3a+2 = n$. Sub. $n=3a+2$ in $2^{n+4} > (n+4)^2$: $2^{3a+2+4} > (3a+2+4)^2$, $2^{3a+6} > [3(a+2)]^2$, $\therefore 2^{3(a+2)} > 9(a+2)^2$ | Uses substitution $n=3a+2$ to arrive at required result | 1 |
| | Numerical substitution | 0 |
| | Correct Proof by Mathematical Induction | 1 |