Ext 2

NORTH SYDNEY BOYS HIGH SCHOOL

2012
ASSESSMENT TASK 2

Mathematics
Extension 2

General Instructions

- Reading time – 5 minutes
- Working time – 55 minutes
- Write in the booklet provided
- Write using blue or black pen
- Board approved calculators may be used
- All necessary working should be shown in every question
- Each new question is to be started on a **new page.**
- Attempt all questions

Class Teacher:
(Please tick or highlight)
- O Mr Weiss
- O Mr Ireland
- O Mr Fletcher

Student Number:

(To be used by the exam markers only.)

Question No	1	2	3	4	5	6	Total	Total
Mark	$\overline{4}$	$\overline{2}$	$\overline{9}$	$\overline{13}$	$\overline{5}$	$\overline{7}$	$\overline{40}$	$\overline{100}$

Question 1 (4 marks) **Marks**

Question 1 is the **objective response** section of the paper. Each question is worth **1 mark**.
You must select the correct response to each of the following four questions and shade the appropriate box on the supplied answer sheet.

(i) An ellipse has eccentricity $e = \frac{\sqrt{5}}{3}$. Its equation could be:

(A) $\frac{x^2}{75} + \frac{y^2}{50} = 1$ (B) $\frac{x^2}{45} + \frac{y^2}{20} = 1$ (C) $\frac{x^2}{48} + \frac{y^2}{32} = 1$ (D) $\frac{x^2}{45} + \frac{y^2}{25} = 1$

(ii) A tangent is drawn to the rectangular hyperbola $xy = 4$ at the point $(2, 2)$ on it. The tangent cuts the axes at points A and B. The length of AB is:

(A) $2\sqrt{2}$ (B) $\sqrt{2}$ (C) $4\sqrt{2}$ (D) $\frac{\sqrt{2}}{2}$

(iii) The asymptotes of the hyperbola $4y^2 - x^2 = 4$ are:

(A) $y = \pm\frac{x}{4}$ (B) $y = \pm 2x$ (C) $y = \pm 4x$ (D) $y = \pm\frac{x}{2}$

(iv) The conic $\frac{x^2}{25} + \frac{y^2}{16-\lambda} = 1$ will be an ellipse with foci on the x-axis provided that λ lies in which of the following ranges of values:

(A) $\lambda < 16$ (B) $\lambda > -9$ (C) $0 < \lambda < 16$ (D) $-9 < \lambda < 16$

Question 2 (2 marks) *Now open your answer booklet and write your solutions.*

Find real numbers a, b and c such that $\frac{1}{x(4+x^2)} \equiv \frac{a}{x} + \frac{bx+c}{4+x^2}$ **2**

Question 3 (9 marks)

Consider the rectangular hyperbola $xy = 16$. A chord, ST, is formed by joining the points $S(4s, \frac{4}{s})$ and $T(4t, \frac{4}{t})$.

(i) Show that the equation of ST is $x + sty = 4(s+t)$. **2**

(ii) It is given that ST passes through the point $(8,8)$.
Show that $2st = s + t - 2$. **1**

(iii) Show that the tangents at S and T meet at $R(\frac{8st}{s+t}, \frac{8}{s+t})$. **4**

(iv) Find the equation of the locus of R as S and T vary on the hyperbola. **2**

Question 4 (13 marks) *Start a new page.*

The hyperbola $\boldsymbol{H}$ has equation $\frac{x^2}{4} - \frac{y^2}{7} = 1$.

(i) Write down the eccentricity, e, the coordinates of the foci, S and S', the equations of its directrices, and the equations of its asymptotes. **4**

(ii) Sketch the hyperbola, showing all the details of part (i). **2**

(iii) Let P be the arbitrary point $(2\sec\theta, \sqrt{7}\tan\theta)$.

Show that P lies on $\boldsymbol{H}$, and prove that the tangent to $\boldsymbol{H}$ at P has equation

$\frac{x\sec\theta}{2} - \frac{y\tan\theta}{\sqrt{7}} = 1.$ **4**

(iv) This tangent cuts the asymptotes at L and M.

Prove that $LP = PM$. **3**

Question 5 (5 marks) *Start a new page.*

An ellipse $\boldsymbol{E}$ has equation $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

(i) Show that the tangent at any point (x_1, y_1) on $\boldsymbol{E}$ is given by

$\frac{xx_1}{a^2} + \frac{yy_1}{b^2} = 1.$ **2**

(ii) Suppose that point P is an extremity of a latus rectum of $\boldsymbol{E}$.

A tangent is drawn at P, meeting the x-axis at A and the y-axis at B.

Show that $\frac{PA}{PB} = \frac{1-e^2}{e^2}$. **3**

[You may find it helpful to draw a diagram showing A, B, P, and other relevant points.]

Please turn the page for the last question →

Question 6 (7 marks) *Start a new page.*

(i) Show that the condition for the line $y = mx + c$ to be tangent to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is that $c^2 = a^2m^2 + b^2$. **3**

(ii) Hence, or otherwise, prove that the pair of tangents from the point $(3,4)$ to the ellipse $\frac{x^2}{16} + \frac{y^2}{9} = 1$ are at right angles to each other. **4**

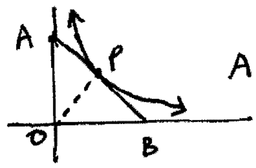
END OF EXAM

2012 – Extension 2 – Task 2

SUGGESTED SOLUTIONS

Q1

(i) B [Reason: $\frac{b^2}{a^2} = 1 - e^2 = \frac{4}{9}$]

(ii) C [Reason:  $AB = 2 \times OP$]

(iii) D [Reason: $\frac{y^2}{1} - \frac{x^2}{4} = 1$, $y = \pm \frac{b}{a}x = \pm \frac{1}{2}x$]

(iv) D [Reason: $0 < 16 - \lambda < 25 \quad \therefore -9 < \lambda < 16$]

/4

Q2

$$\frac{a}{x} + \frac{bx+c}{4+x^2} \equiv \frac{4a + ax^2 + bx^2 + cx}{x(4+x^2)}$$

$\therefore 4a = 1, \quad a = -b, \quad c = 0$

Thus, $a = \frac{1}{4}, \quad b = -\frac{1}{4}, \quad c = 0.$

/2

Q3

(i) $$m_{ST} = \frac{\frac{4}{t} - \frac{4}{s}}{4t - 4s} = \frac{\frac{4s - 4t}{st}}{4t - 4s} = \frac{-1}{st}$$

$\therefore$ chord equation is $y - \frac{4}{s} = -\frac{1}{st}(x - 4s)$

$\therefore \quad sty - 4t = -x + 4s$

$\therefore \quad x + sty = 4(s+t)$, as required.

(ii) Through $(8,8)$ $\therefore \quad 8 + 8st = 4(s+t)$

$\therefore \quad 2 + 2st = s + t$

$\therefore \quad 2st = s + t - 2$

Q3 - continued

(iii) From part (i),

let $t \to s$.

Thus tangent at S will be $x + s^2 y = 4(2s)$ ✓✓

$\therefore \; x + s^2 y = 8s \quad \text{----} (1)$

Similarly, tangent at T is:

$x + t^2 y = 8t . \quad \text{----} (2)$

[Alternatively, if $xy = 16$, $\therefore \; y \cdot 1 + x \cdot \frac{dy}{dx} = 0$

$\therefore \; \frac{dy}{dx} = -\frac{y}{x} = \frac{-\frac{4}{s}}{4s}$ at S

$= -\frac{1}{s^2} \quad (✓)$

$\therefore$ tangent at S is $y - \frac{4}{s} = -\frac{1}{s^2}(x - 4s)$

$\therefore \; x + s^2 y = 8s \quad (✓)$

Likewise at T, $x + t^2 y = 8t$.]

Solving simultaneously, $y(s^2 - t^2) = 8(s - t)$ ✓

$\therefore \; y = \frac{8}{s+t} \quad (\text{as } s \neq t)$

Now $t^2 x + t^2 s^2 y = 8st^2 \quad$ (multiplying (1) by t^2)

& $s^2 x + s^2 t^2 y = 8ts^2 \quad$ (mult. (2) by s^2)

Subtracting, $x(t^2 - s^2) = 8st(t - s)$

$\therefore \; x = \frac{8st}{s+t} \quad (\text{as } s \neq t)$ ✓

$\therefore \; R = \left(\frac{8st}{s+t}, \frac{8}{s+t} \right)$ as required.

$\boxed{Q3}$ - continued

(iv) Since $R = \left(\frac{8st}{s+t}, \frac{8}{s+t}\right)$,

we see $x = sty$.

Using part (ii), $st = \frac{s+t}{2} - 1$

$= \frac{4}{y} - 1$

$\therefore \quad x = \left(\frac{4}{y} - 1\right)y = 4 - y$

Thus $x + y - 4 = 0$ is the equation of the locus of R. ✓

Note however that since $s \neq 0$ and $t \neq 0$,

$\therefore \quad x \neq 0$.

Likewise, since $y = \frac{8}{s+t}$, $\therefore y \neq 0$.

So locus is $x + y - 4 = 0 \quad (x \neq 0, y \neq 0)$ ✓

[i.e. excluding the points $(0,4)$, $(4,0)$].

[Note also that s cannot take the value 1 or -1, otherwise $s+t=0$, i.e. the tangents do not intersect.]

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$\boxed{Q4}$

(i) $\frac{x^2}{4} - \frac{y^2}{7} = 1 \quad \therefore \quad a^2 = 4, \ b^2 = 7$

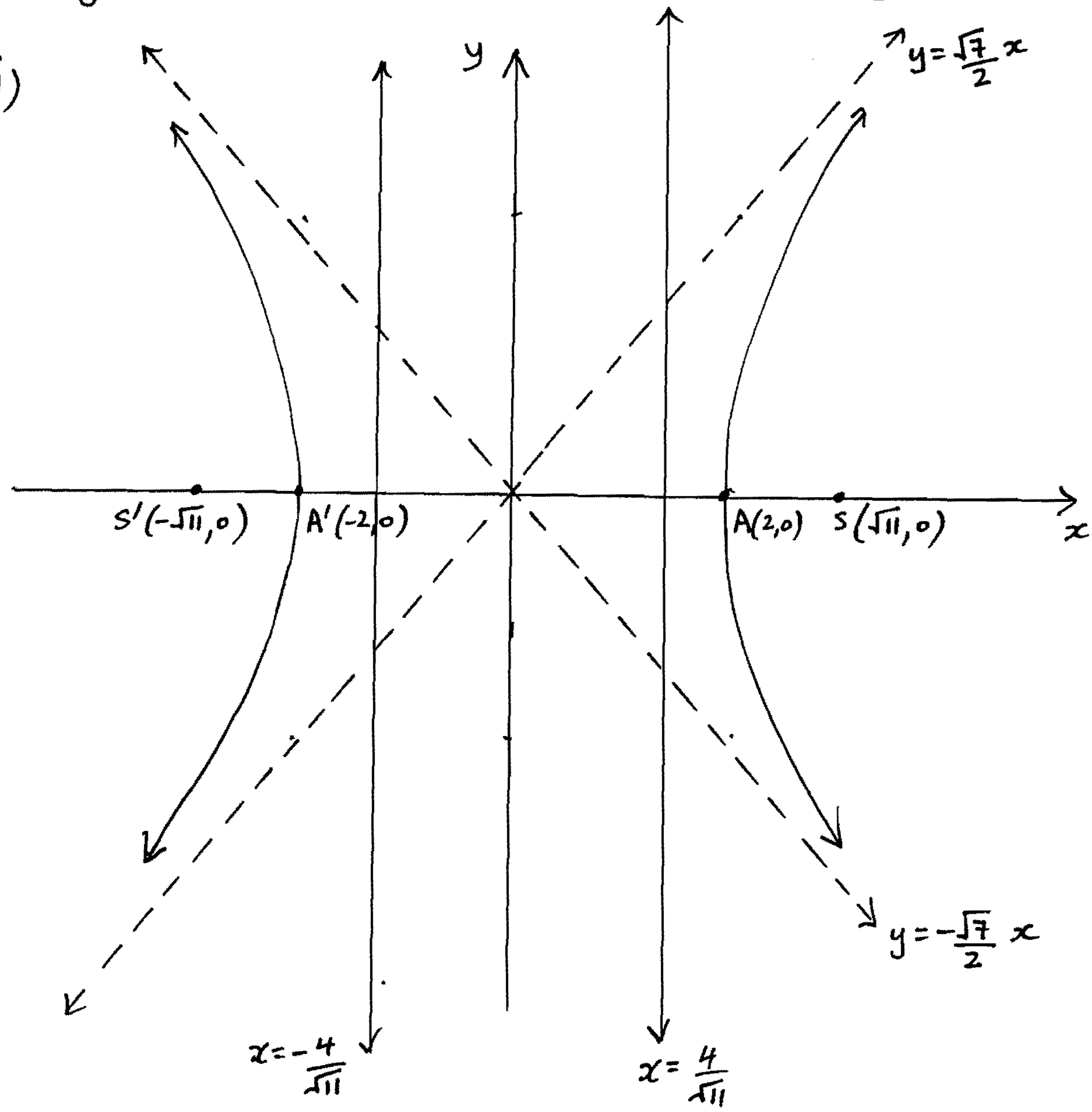
Since $b^2 = a^2(e^2 - 1) \quad \therefore \ e^2 = \frac{11}{4} \quad \therefore e = \frac{\sqrt{11}}{2}$ ✓

$S = (ae, 0) = (\sqrt{11}, 0)$, $S' = (-\sqrt{11}, 0)$ ✓

Directrices are $x = \pm \frac{a}{e} = \pm \frac{2}{\frac{\sqrt{11}}{2}} \quad \therefore \ x = \pm \frac{4}{\sqrt{11}}$ ✓

Asymptotes are $y = \pm \frac{b}{a} x \quad \therefore \quad y = \pm \frac{\sqrt{7}}{2} x$ ✓

(ii)



✓✓

$\boxed{Q4}$ - continued

(iii) At P, $\frac{x^2}{4} - \frac{y^2}{7} = \frac{\cancel{4}\sec^2\theta}{\cancel{4}} - \frac{\cancel{7}\tan^2\theta}{\cancel{7}}$

$= \sec^2\theta - \tan^2\theta$

$= 1 \qquad \therefore P$ is on H

Differentiating, $\frac{2x}{4} - \frac{2y}{7} \cdot \frac{dy}{dx} = 0$

$\therefore \frac{dy}{dx} = \frac{7x}{4y}$

$= \frac{\sqrt{7}}{2} \frac{\sec\theta}{\tan\theta}$ at P.

$\therefore$ tangent at P is:

$$y - \sqrt{7}\tan\theta = \frac{\sqrt{7}\sec\theta}{2\tan\theta}(x - 2\sec\theta)$$

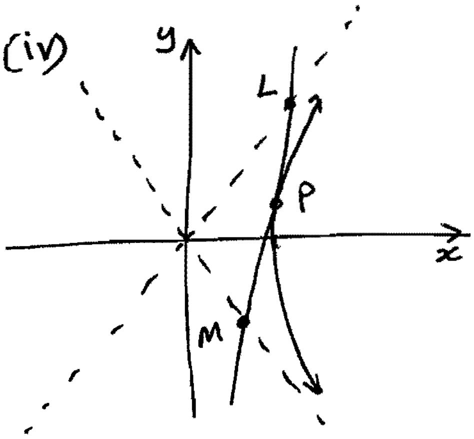
$\therefore \quad 2\tan\theta\, y - 2\sqrt{7}\tan^2\theta = \sqrt{7}\sec\theta\, x - 2\sqrt{7}\sec^2\theta$

$\therefore \quad \sqrt{7}\sec\theta\, x - 2\tan\theta\, y = 2\sqrt{7}(\sec^2\theta - \tan^2\theta)$

$= 2\sqrt{7}$

$\therefore \quad \frac{\sec\theta}{2}x - \frac{\tan\theta}{\sqrt{7}}y = 1$, as required.

(iv)



At L: $\frac{x\sec\theta}{2} - \frac{y\tan\theta}{\sqrt{7}} = 1$

and $y = \frac{\sqrt{7}}{2}x$.

$\therefore \frac{x\sec\theta}{2} - \frac{\tan\theta}{\sqrt{7}} \cdot \frac{\sqrt{7}}{2}x = 1$

$\therefore x = \frac{2}{\sec\theta - \tan\theta}$

$= \frac{2(\sec\theta + \tan\theta)}{\sec^2\theta - \tan^2\theta}$

$= 2(\sec\theta + \tan\theta)$

$\therefore y = \frac{\sqrt{7}}{2}x = \sqrt{7}(\sec\theta + \tan\theta)$ for L

Q4 - continued

(iv) - continued At M, $y = -\frac{\sqrt{7}}{2}x$

$$\therefore \quad \frac{x\sec\theta}{2} - \frac{\tan\theta}{\cancel{\sqrt{7}}} \cdot -\frac{\cancel{\sqrt{7}}}{2}x = 1$$

$$\therefore \quad x = \frac{2}{\sec\theta + \tan\theta} = 2(\sec\theta - \tan\theta)$$

$$\therefore \quad y = -\sqrt{7}(\sec\theta - \tan\theta)$$ ✓ for M

To prove $LP = PM$, just show P is the midpoint of LM:

midpoint of LM has x-coordinate:

$$x = \frac{2(\sec\theta + \tan\theta) + 2(\sec\theta - \tan\theta)}{2}$$

$$= 2\sec\theta$$

It has y coordinate:

$$y = \frac{\sqrt{7}(\sec\theta + \tan\theta) - \sqrt{7}(\sec\theta - \tan\theta)}{2}$$

$$= \sqrt{7}\tan\theta$$

i.e. midpoint of $LM = (2\sec\theta, \sqrt{7}\tan\theta) = P$ ✓

$$\therefore \quad LP = PM$$

[Alternatively, candidates may use the distance formula to calculate LP and PM].

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Q5 (i) $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$

$\therefore \dfrac{2x}{a^2}+\dfrac{2y}{b^2}\cdot\dfrac{dy}{dx}=0$

$\therefore \dfrac{dy}{dx}=-\dfrac{b^2x}{a^2y}$

$\therefore$ at $P(x_1,y_1)$, $\dfrac{dy}{dx}=-\dfrac{b^2x_1}{a^2y_1}$ ✓

$\therefore$ tangent is:

$y-y_1=-\dfrac{b^2x_1}{a^2y_1}(x-x_1)$

$a^2y_1y-a^2y_1^2=-b^2x_1x+b^2x_1^2$

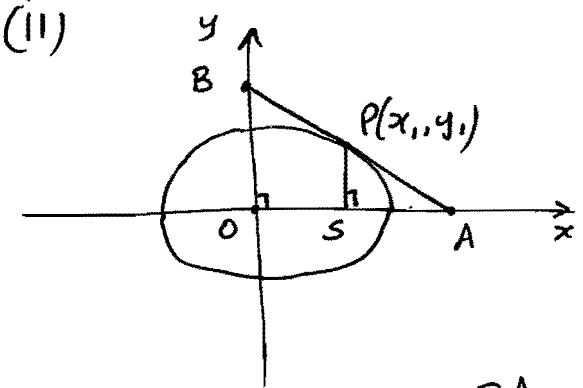
$\therefore b^2x_1x+a^2y_1y=b^2x_1^2+a^2y_1^2$

$\therefore \dfrac{x_1x}{a^2}+\dfrac{y_1y}{b^2}=\dfrac{x_1^2}{a^2}+\dfrac{y_1^2}{b^2}$

$=1$ (as P is on E) ✓

... as required.

(ii)



At A, $y=0$

$\therefore x=\dfrac{a^2}{x_1}$

$=\dfrac{a^2}{ae}$ as P on Latus Rectum

$\therefore x=\dfrac{a}{e}$ ✓

Then $\dfrac{PA}{PB}=\dfrac{SA}{OS}$ (ratios of intercepts on // lines) ✓

$=\dfrac{\frac{a}{e}-ae}{ae}=\dfrac{1}{e^2}-1=\dfrac{1-e^2}{e^2}$ ✓

... as required.

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$\boxed{Q5}$ – ALTERNATIVELY ... A LONGER SOLUTION is :–

At P, $x = ae$ $\quad \therefore \dfrac{y^2}{b^2} = 1 - \dfrac{x^2}{a^2} = 1 - e^2$

$$\therefore y^2 = b^2(1-e^2)$$
$$= a^2(1-e^2)\cdot(1-e^2)$$
$$\therefore y = \pm a(1-e^2)$$

$$\therefore P = [ae,\ a(1-e^2)].$$

At A, $y = 0$ $\quad \therefore x = \dfrac{a^2}{x_1} = \dfrac{a^2}{ae} = \dfrac{a}{e}$

$$\therefore A = \left(\tfrac{a}{e}, 0\right)$$

At B, $x = 0$ $\quad \therefore y = \dfrac{b^2}{y_1} = \dfrac{b^2}{a(1-e^2)} = \dfrac{a^2(1-e^2)}{a(1-e^2)}$

$$= a$$

$$\therefore B = (0, a).$$

Then $PA = \sqrt{\left(ae - \frac{a}{e}\right)^2 + a^2(1-e^2)^2}$

$$= a(1-e^2)\sqrt{\frac{1}{e^2} + 1}$$
$$= \frac{a(1-e^2)}{e}\sqrt{1+e^2}$$

And $PB = \sqrt{a^2e^2 + (a - a(1-e^2))^2}$

$$= \sqrt{a^2e^2 + a^2e^4}$$
$$= ae\sqrt{1+e^2}$$

$$\therefore \frac{PA}{PB} = \frac{\dfrac{a(1-e^2)\sqrt{1+e^2}}{e}}{ae\sqrt{1+e^2}} = \frac{1-e^2}{e^2}$$

... as required.

$\boxed{\text{Q6}}$ (i) Where line meets ellipse, we have

$$\frac{x^2}{a^2} + \frac{(mx+c)^2}{b^2} = 1$$

$$\therefore \quad b^2x^2 + a^2(mx+c)^2 = a^2b^2$$

$$b^2x^2 + a^2m^2x^2 + 2mca^2x + a^2c^2 = a^2b^2$$

$$\therefore (b^2+a^2m^2)x^2 + 2a^2mc\,x + (a^2c^2 - a^2b^2) = 0$$

For tangent, $\Delta = 0$

$$\therefore \quad 4a^4m^2c^2 - 4(b^2+a^2m^2)(a^2c^2-a^2b^2) = 0$$

$$\therefore \quad a^2m^2c^2 - (b^2+a^2m^2)(c^2-b^2) = 0 \qquad \text{as } a \neq 0$$

$$\therefore \quad a^2m^2c^2 - (b^2c^2 - b^4 + a^2m^2c^2 - a^2m^2b^2) = 0$$

$$\therefore \quad -c^2 + b^2 + a^2m^2 = 0 \qquad \text{as } b^2 \neq 0$$

$$\therefore \quad c^2 = a^2m^2 + b^2 \qquad \text{.... as required.}$$

(ii) From (i), $y = mx + c$ is a tangent to

$\dfrac{x^2}{16} + \dfrac{y^2}{9} = 1$ if $c^2 = 16m^2 + 9$

ie. $c = \pm\sqrt{16m^2+9}$

$\therefore$ tangents are $y = mx \pm \sqrt{16m^2+9}$

If tangents pass through $(3,4)$ we have:

$$4 = 3m \pm \sqrt{16m^2+9}$$

$$\therefore \quad 4 - 3m = \pm\sqrt{16m^2+9}$$

$$\therefore \quad 16 - 24m + 9m^2 = 16m^2 + 9$$

$$\therefore \quad 7m^2 + 24m - 7 = 0$$

P.T.O.

$\boxed{Q6}$ - continued

The equation $7m^2 - 24m - 7 = 0$

has 2 solutions for m, corresponding to the 2 tangents.

The <u>product</u> of the solutions $= \frac{-7}{7} = -1$ ✓

i.e. $m_1 \times m_2 = -1$

i.e. the tangents are perpendicular.

[Alternatively, candidates can solve the equation, getting $m = \dfrac{24 \pm \sqrt{576+196}}{14}$

$$= \frac{24 \pm \sqrt{772}}{14}$$

$$\therefore m_1 \cdot m_2 = \frac{24 + \sqrt{772}}{14} \cdot \frac{24 - \sqrt{772}}{14}$$

$$= \frac{24^2 - 772}{196}$$

$$= -1$$

$\therefore$ tangents are perpendicular.]

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