NORTH SYDNEY BOYS HIGH SCHOOL

100 Years 1912 ~ 2012

MATHEMATICS (EXTENSION 2)

2013 HSC Course Assessment Task 2

March 18, 2013

General instructions

- Working time – 1 hour. (plus 5 minutes reading time)
- Write using blue or black pen. Where diagrams are to be sketched, these may be done in pencil.
- Board approved calculators may be used.
- Attempt **all** questions.
- At the conclusion of the examination, bundle the booklets + answer sheet used in the correct order within this paper and hand to examination supervisors.

SECTION I

- Mark your answers on the answer sheet provided (numbered as page 5)

SECTION II

- Commence each new question on a new page. Write on both sides of the paper.
- All necessary working should be shown in every question. Marks may be deducted for illegible or incomplete working.

STUDENT NUMBER: .... **# BOOKLETS USED:** .....

Class (please ✔)

◯ 12M4A – Mr Fletcher ◯ 12M4B – Mr Lam ◯ 12M4C – Ms Ziaziaris

Marker's use only.

QUESTION	1-4	5	6	**Total**	%
MARKS	$\overline{4}$	$\overline{18}$	$\overline{18}$	$\overline{40}$	

Section I: Objective response

Mark your answers on the multiple choice sheet provided. **Marks**

1. $P(z)$ is a polynomial in z of degree 4 with real coefficients. Which one of the following statements must be false? **1**

(A) $P(z)$ has four real roots.

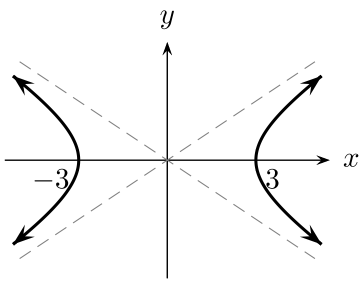
(B) $P(z)$ has two real roots and two non-real roots.

(C) $P(z)$ has one real root and three non-real roots.

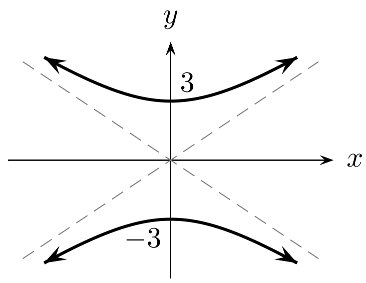
(D) $P(z)$ has no real roots.

2. Which of the following is the graph of $16y^2 - 9x^2 = 144$? **1**

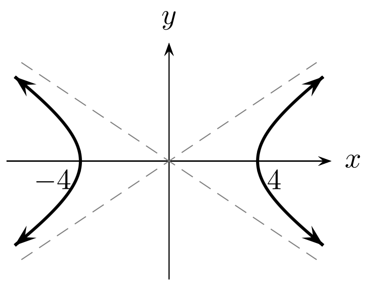
(A)



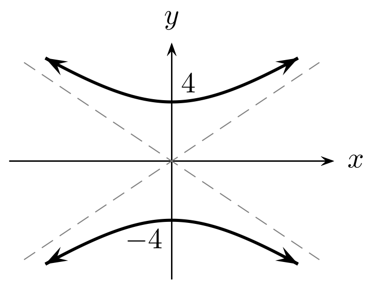
(C)



(B)



(D)



3. The equation $x^3 + 2x + 1 = 0$ has roots α, β and γ. Which of the following equations has roots 2α, 2β and 2γ? **1**

(A) $x^3 + 8x + 8 = 0$

(B) $x^3 + 16x + 8 = 0$

(C) $2x^3 + 4x + 2 = 0$

(D) $8x^3 + 4x + 1 = 0$

4. The points $P(a\cos\theta, b\sin\theta)$ and $Q(a\cos\phi, b\sin\phi)$ lie on the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$. The chord PQ subtends a right angle at $(0, 0)$. Which of the following is the correct expression? **1**

(A) $\tan\theta\tan\phi = -\dfrac{b^2}{a^2}$

(B) $\tan\theta\tan\phi = -\dfrac{a^2}{b^2}$

(C) $\tan\theta\tan\phi = \dfrac{b^2}{a^2}$

(D) $\tan\theta\tan\phi = \dfrac{a^2}{b^2}$

End of Section I
Examination continues overleaf...

Section II: Short answer

Question 5 (18 Marks) Commence a NEW page. **Marks**

(a) The equation of a hyperbola is given by $4x^2 - 9y^2 = 36$.

i. Find the coordinates of S and S', the foci of the hyperbola. **2**

ii. Find the equations of the directrices M and M'. **1**

iii. Find the equations of the asymptotes. **1**

iv. Let P be any point on the hyperbola. Show that $|PS - PS'| = 6$. **2**

(b) The line $x = 8$ is the directrix, and $(2, 0)$ is the corresponding focus of the ellipse with equation **2**

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad a > b > 0$$

Find the value of a and b.

(c) The point $P(a\cos\theta, b\sin\theta)$ lies on the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ and the ellipse meets the x axis at the points A and A'.

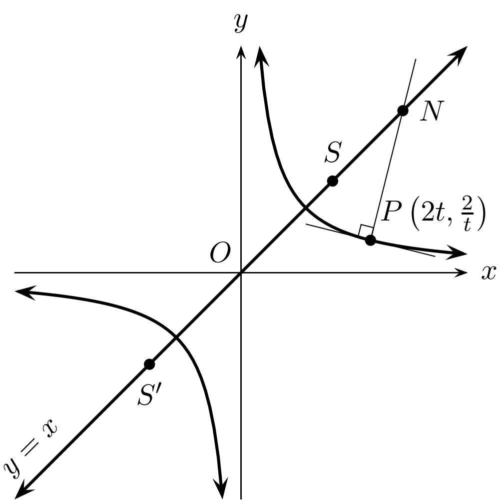
i. Prove that the tangent at P has the equation **2**

$$\frac{x\cos\theta}{a} + \frac{y\sin\theta}{b} = 1$$

ii. The tangent at P meets the tangents from A and A' at points Q and Q' respectively. Find the coordinates of Q and Q'. **2**

iii. The points A, A', Q' and Q form a trapezium. Prove that the product of lengths of the parallel lines is independent of the position of P. **2**

(d) The diagram shows the hyperbola $xy = 4$.



i. What are the coordinates of the foci S and S'? **1**

Question 5 continues overleaf...

Question 5 continued from previous page...

The point $P\left(2t, \frac{2}{t}\right)$ lies on the curve, where $t \neq 0$. The normal at P intersects the straight line $y = x$ at N. O is the origin.

Given the equation of the normal at P is

$$y = t^2 x + \frac{2}{t} - 8$$

ii. Find the coordinates of N. **1**

iii. Show that $\triangle OPN$ is isosceles. **2**

Question 6 (18 Marks) Commence a NEW page. **Marks**

(a) The polynomial equation $x^3 - 6x^2 + 3x - 2 = 0$ has roots α, β and γ. **2**

Evaluate $\alpha^3 + \beta^3 + \gamma^3$.

(b) $x^3 + px^2 + qx + r = 0$ has roots α, β and γ, where $\alpha = \beta + \gamma$. **3**

Show that $p^3 - 4pq + 8r = 0$.

(c) Given $-2 - i$ is a zero of $P(x) = x^4 + 6x^3 + 14x^2 + 14x + 5$, find all zeros of $P(x)$. **3**

(d) i. The polynomial equation $P(x) = 0$ has a double root at $x = \alpha$. **2**

Show that $x = \alpha$ is also a root of the equation $P'(x) = 0$.

ii. $y = mx$ is a tangent to the curve $y = 3 - \dfrac{1}{x^2}$. Show that the equation $mx^3 - 3x^2 + 1 = 0$ has a double root. **1**

iii. Hence find the equation of any such tangents. **3**

(e) Given that $\cos 5\theta = 16\cos^5\theta - 20\cos^3\theta + 5\cos\theta$,

i. Find the roots of $16x^5 - 20x^3 + 5x = 0$. **2**

ii. Hence show that **2**

$$\cos^2\frac{\pi}{10} + \cos^2\frac{3\pi}{10} = \frac{5}{4}$$

End of paper.

Answer sheet for Section I

Mark answers to Section I by fully blackening the correct circle, e.g "●"

STUDENT NUMBER: ....

Class (please ✔)

◯ 12M4A – Mr Fletcher ◯ 12M4B – Mr Lam ◯ 12M4C – Ms Ziaziaris

1 – Ⓐ Ⓑ Ⓒ Ⓓ

2 – Ⓐ Ⓑ Ⓒ Ⓓ

3 – Ⓐ Ⓑ Ⓒ Ⓓ

4 – Ⓐ Ⓑ Ⓒ Ⓓ

Suggested Solutions

Section I

1. (C) **2.** (C) **3.** (A) **4.** (B)

Section II

Question 5 (Fletcher)

(a) i. (2 marks)

✓ [1] for correct eccentricity.

✓ [1] for correct foci.

$$4x^2 - 9y^2 = 36$$
$$\frac{x^2}{9} - \frac{y^2}{4} = 1$$
$$\therefore a = 3 \qquad b = 2$$

Finding the eccentricity,

$$b^2 = a^2\left(e^2 - 1\right)$$
$$4 = 9\left(e^2 - 1\right)$$
$$e^2 = 1 + \frac{4}{9} = \frac{13}{9}$$
$$\therefore e = \frac{\sqrt{13}}{3}$$

Coordinates of foci: $S(\pm ae, 0)$

$$S\left(\sqrt{13}, 0\right) \qquad S'\left(-\sqrt{13}, 0\right)$$

ii. (1 mark)

$$x = \pm\frac{a}{e} = \pm\frac{3}{\frac{\sqrt{13}}{3}} = \pm\frac{9}{\sqrt{13}}$$

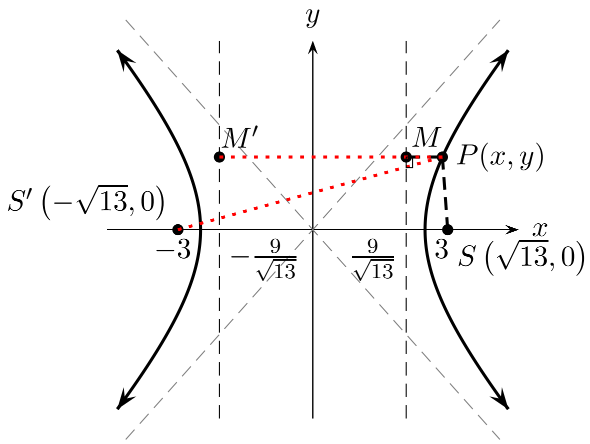
iii. (1 mark)

$$y = \pm\frac{b}{a}x = \pm\frac{2}{3}x$$

iv. (2 marks)

✓ [1] for using $PS = ePM$ and $PS' = ePM'$.

✓ [1] for final answer.



$$\text{Since}\frac{PS}{PM} = e \quad \text{and} \quad \frac{PS'}{PM'} = e$$
$$\therefore \begin{cases} PS = ePM & (1) \\ PS' = ePM' & (2) \end{cases}$$

Equation (1) subtract equation (2):

$$\left|PS - PS'\right| = e\left|PM - PM'\right|$$
$$= e\left|MM'\right|$$
$$= \frac{\cancel{\sqrt{13}}}{3} \times 2 \times \frac{9}{\cancel{\sqrt{13}}}$$
$$= 6$$

(b) (2 marks)

✓ [1] for finding a and e.

✓ [1] for final answer.

$$x = \frac{a}{e} = 8$$
$$(2, 0) = (ae, 0)$$
$$\begin{cases} \frac{a}{e} = 8 & (1) \\ ae = 2 & (2) \end{cases}$$

(1) × (2):

$$a^2 = 16$$
$$\therefore a = 4$$
$$\therefore e = \frac{1}{2}$$
$$b^2 = a^2\left(1 - e^2\right)$$
$$= 16\left(1 - \frac{1}{4}\right)$$
$$= 16 \times \frac{3}{4} = 12$$
$$\therefore b = 2\sqrt{3}$$

(c) i. (2 marks)

$$\begin{cases} x = a\cos\theta \\ y = b\sin\theta \end{cases}$$

$$\frac{dx}{d\theta} = -a\sin\theta$$

$$\frac{dy}{d\theta} = b\cos\theta$$

$$\therefore \frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = -\frac{b\cos\theta}{a\sin\theta}$$

Equation of the tangent at $P\left(a\cos\theta, b\sin\theta\right)$ is

$$y - b\sin\theta = -\frac{b\cos\theta}{a\sin\theta}\left(x - a\cos\theta\right)$$

$$ay\sin\theta - ab\sin^2\theta = -bx\cos\theta + ab\cos^2\theta$$

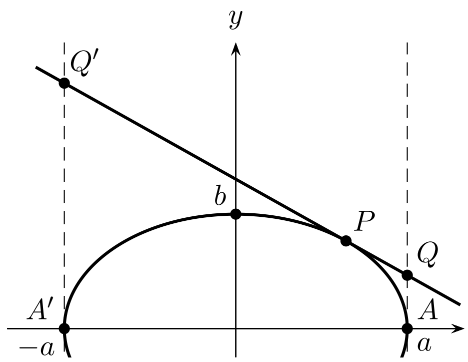
$$\underbrace{bx\cos\theta + ay\sin\theta}_{\div ab} = \underbrace{ab\cos^2\theta + ab\sin^2\theta}_{\div ab}$$

$$\therefore \frac{x\cos\theta}{a} + \frac{y\sin\theta}{b} = 1$$

ii. (2 marks)

✓ [1] for coordinates of Q

✓ [1] for coordinates of Q'



As the equation of the tangent is

$$\frac{x\cos\theta}{a} + \frac{y\sin\theta}{b} = 1$$

The coordinates of Q: when $x = a$,

$$\frac{\cancel{a}\cos\theta}{\cancel{a}} + \frac{y\sin\theta}{b} = 1$$

$$y\frac{\sin\theta}{b} = 1 - \cos\theta$$

$$y = \frac{b(1-\cos\theta)}{\sin\theta}$$

Similarly, the coordinates of Q': when $x = -a$

$$y = \frac{b(1+\cos\theta)}{\sin\theta}$$

$$\therefore Q\left(a, \frac{b(1-\cos\theta)}{\sin\theta}\right)$$

$$Q'\left(-a, \frac{b(1+\cos\theta)}{\sin\theta}\right)$$

iii. (2 marks)

✓ [1] for identifying parallel lines in trapezium and multiplying their lengths.

✓ [1] for final answer.

Parallel lines are AQ and AQ'. Hence

$$AQ \times AQ' = \frac{b(1-\cos\theta)}{\sin\theta} \times \frac{b(1+\cos\theta)}{\sin\theta}$$

$$= \frac{b^2\cancel{(1-\cos^2\theta)}}{\cancel{\sin^2\theta}}$$

$$= b^2$$

(d) i. (1 mark)

$$xy = 4 = c^2$$

$$\therefore c = 2$$

As $e = \sqrt{2}$, and $S\left(\pm c\sqrt{2}, \pm c\sqrt{2}\right)$,

$$\therefore S\left(2\sqrt{2}, 2\sqrt{2}\right) \qquad S'\left(-2\sqrt{2}, -2\sqrt{2}\right)$$

ii. (1 mark)

$$y = t^2 x + \frac{2}{t} - 8$$

Coordinates of N occur when $y = x$, i.e.

$$x = t^2 x + \frac{2}{t} - 8$$

$$x\left(1 - t^2\right) = \frac{2}{t} - 8$$

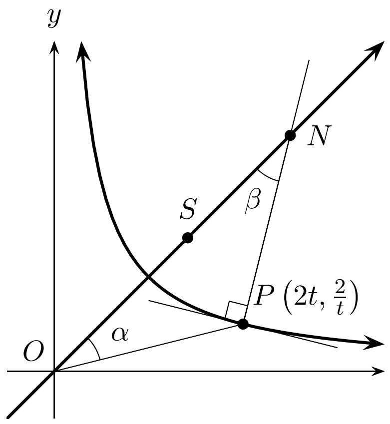
$$x = y = \frac{8t-2}{t\left(t^2-1\right)}$$

$$\therefore N\left(\frac{8t-2}{t\left(t^2-1\right)}, \frac{8t-2}{t\left(t^2-1\right)}\right)$$

iii. (2 marks)

✓ [1] for substantial progress towards proof.

✓ [1] for correct conclusion.



$$m_{OP} = \frac{\frac{2}{t}}{2t} = \frac{1}{t^2}$$

Gradient of normal:

$$m_{PM} = t^2$$

Angle between ON and OP (α):

$$\begin{aligned}\tan\alpha &= \left|\frac{m_2 - m_1}{1 + m_1 m_2}\right| \\ &= \left|\frac{1 - \frac{1}{t^2}}{1 + 1\left(\frac{1}{t^2}\right)}\right| \times \frac{t^2}{t^2} \\ &= \left|\frac{t^2 - 1}{t^2 + 1}\right|\end{aligned}$$

Angle between ON and PN (β):

$$\tan\beta = \left|\frac{1 - t^2}{1 + 1(t^2)}\right| = \left|\frac{1 - t^2}{1 + t^2}\right|$$

As $\tan\alpha = \tan\beta$ and α, $\beta < 90°$,

$$\therefore \alpha = \beta$$

and $\triangle OPN$ is isosceles.

Question 6 (Fletcher)

(a) (2 marks)

✓ [1] for $\alpha^3 + \beta^3 + \gamma^3$ in terms of α, β and γ.

✓ [1] for final answer.

If α, β and γ are the roots,

$$\begin{aligned}\alpha^3 - 6\alpha^2 + 3\alpha - 2 &= 0 \\ \beta^3 - 6\beta^2 + 3\beta - 2 &= 0 \\ \gamma^3 - 6\gamma^2 + 3\gamma - 2 &= 0\end{aligned}$$

Adding and moving terms to other side,

$$\begin{aligned}&\alpha^3 + \beta^3 + \gamma^3 \\ &= 6\left(\alpha^2 + \beta^2 + \gamma^2\right) - 3\left(\alpha + \beta + \gamma\right) + 6\end{aligned}$$

Now to find $\alpha^2 + \beta^2 + \gamma^2$:

$$\begin{aligned}(\alpha + \beta + \gamma)^2 &= (\alpha + \beta)^2 + 2(\alpha + \beta)\gamma + \gamma^2 \\ &= \alpha^2 + 2\alpha\beta + \beta^2 + 2\alpha\gamma + 2\beta\gamma + \gamma^2\end{aligned}$$

$$\therefore \alpha^2 + \beta^2 + \gamma^2 = (\alpha + \beta + \gamma)^2 - 2\left(\alpha\beta + \alpha\gamma + \beta\gamma\right)$$

$$\begin{aligned}\therefore \alpha^3 + \beta^3 + \gamma^3 &= 6\left[(\alpha + \beta + \gamma)^2 - 2\left(\alpha\beta + \alpha\gamma + \beta\gamma\right)\right] \\ &\quad - 3\left(\alpha + \beta + \gamma\right) + 6 \\ &= 6\left[(-6)^2 - 2\left(3\right)\right] - 3(-6) + 6 \\ &= 168\end{aligned}$$

(b) (3 marks)

✓ [2] (max) for significant progress towards final answer.

✓ [1] for final answer.

$$x^3 + px^2 + qx + r = 0$$

with roots α, β, γ and $\alpha = \beta + \gamma$.

- Using the sum of roots,

$$\begin{aligned}\alpha + \beta + \gamma = 2\alpha &= \frac{-p}{1} \\ \therefore \alpha &= -\frac{p}{2}\end{aligned}$$

- Using the pairs of roots,

$$\begin{aligned}\alpha\beta + \alpha\gamma + \beta\gamma &= \alpha\left(\beta + \gamma\right) + \beta\gamma \\ &= \alpha^2 + \beta\gamma \\ &= \frac{p^2}{4} + \beta\gamma = \frac{q}{1} \\ \therefore \frac{p^2}{4} + \beta\gamma &= q\end{aligned}$$

- Using the product of roots,

$$\begin{aligned}\alpha\beta\gamma &= \left(-\frac{p}{2}\right)\left(q - \frac{p^2}{4}\right) \\ &= -\frac{p}{2}\left(q - \frac{p^2}{4}\right) = -r \\ \therefore -\frac{p}{2}\left(q - \frac{p^2}{4}\right) &= -r \\ \underbrace{-\frac{pq}{2} + \frac{p^3}{8}}_{\times 8} &= \underset{\times 8}{-r} \\ \therefore p^3 - 4pq + 8r &= 0\end{aligned}$$

(c) (3 marks)

✓ [1] for identifying conjugate root.

✓ [1] for other significant progress towards answer.

✓ [1] for final answer.

If $-2-i$ is a zero to $P(x) = x^4 + 6x^3 + 14x^2 + 14x + 5$, then $-2+i$ is also a zero.

- Sum of roots:

$$(-2-i) + (-2+i) + \alpha + \beta = -\frac{b}{a} = -6$$
$$\alpha + \beta = -2$$

- Product of roots:

$$(-2-i)(-2+i) \times \alpha\beta = \frac{e}{a} = 5$$
$$5\alpha\beta = 5$$
$$\alpha\beta = 1$$

$$\begin{cases} \alpha + \beta = -2 \\ \alpha\beta = 1 \end{cases}$$
$$\therefore \alpha = -1 \qquad \beta = -1$$

Roots are -1, -1, $-2-i$, $-2+i$.

(d) i. (2 marks)

✓ [1] correctly differentiating.

✓ [1] for final proof.

Let $P(x) = (x-\alpha)^2 Q(x)$

$$u = (x-\alpha)^2 \qquad v = Q(x)$$
$$u' = 2(x-\alpha) \qquad v' = Q'(x)$$
$$\begin{aligned} P'(x) &= uv' + vu' \\ &= (x-\alpha)^2 Q'(x) + 2(x-\alpha)Q(x) \\ &= (x-\alpha)\big((x-\alpha)Q'(x) + Q(x)\big) \\ &= (x-\alpha)Q_3(x) \end{aligned}$$

Hence $x = \alpha$ is also a root of $P'(x) = 0$.

ii. (1 mark)

If $y = mx$ is a tangent to $y = 3 - \frac{1}{x^2}$, then

$$mx = 3 - \frac{1}{x^2}$$

has a double root.

$$mx^3 = 3x^2 - 1$$
$$mx^3 - 3x^2 + 1 = 0$$

Hence $mx^3 - 3x^2 + 1 = 0$ has a double root.

iii. (3 marks)

If $P(x)$ has a double root at $x = \alpha$, then $x = \alpha$ is also a root to $P'(x)$:

$$P'(x) = 3mx^2 - 6x$$
$$P'(\alpha) = 3m\alpha^2 - 6\alpha = 03\alpha\,(m\alpha - 2) = 0$$
$$\therefore \alpha = 0, \frac{2}{m}$$

As $P'(0) \neq 0$, $x = \frac{2}{m}$ is the double root.

$$P\left(\frac{2}{m}\right) = 0$$
$$\therefore m\left(\frac{2}{m}\right)^3 - 3\left(\frac{2}{m}\right)^2 + 1 = 0$$
$$\frac{8}{m^2} - \frac{12}{m^2} + 1 = 0$$
$$\frac{4}{m^2} = 1$$
$$\therefore m = \pm 2$$
$$\therefore y = \pm 2x$$

(e) i. (2 marks)

Given $\cos 5\theta = 16\cos^5\theta - 20\cos^3\theta + 5\cos\theta$, then letting $x = \cos\theta$ will result in the polynomial

$$16x^5 - 20x^3 + 5x = 0$$

Hence to solve the polynomial, solve $\cos 5\theta = 0$:

$$\cos 5\theta = 0$$
$$5\theta = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \frac{7\pi}{2}, \frac{9\pi}{2}$$
$$\therefore \theta = \frac{\pi}{10}, \frac{3\pi}{10}, \frac{\pi}{2}, \frac{7\pi}{10}, \frac{9\pi}{10}$$
$$\therefore x = 0, \pm\cos\frac{\pi}{10}, \pm\cos\frac{3\pi}{10}$$

ii. (2 marks) Using pairs of roots (the only terms remaining are the ones which do not involve the zero)

$$16x^5 - 20x^3 + 5x = 0$$
$$\begin{aligned} &\left(\cos\frac{\pi}{10}\right)\left(-\cos\frac{\pi}{10}\right) \\ &\quad + \left(\cos\frac{3\pi}{10}\right)\left(-\cos\frac{3\pi}{10}\right) \\ &= -\cos^2\frac{\pi}{10} - \cos^2\frac{3\pi}{10} = \frac{c}{a} = -\frac{20}{16} \end{aligned}$$
$$\therefore \cos^2\frac{\pi}{10} + \cos^2\frac{3\pi}{10} = \frac{5}{4}$$