Ext 2

NORTH SYDNEY BOYS HIGH SCHOOL

2014
ASSESSMENT TASK 2

Mathematics
Extension 2

General Instructions

- Reading time – **5** minutes
- Working time – **55** minutes
- A table of standard integrals is provided
- Write using blue or black pen
- Board approved calculators may be used
- Questions 1 – 4 to be answered on multiple choice answer sheet provided
- All necessary working should be shown for Questions 5 – 10 in the answer booklet provided
- Each new question is to be started on a **new page.**

Class Teacher:
(Please tick or highlight)
- O Ms Ziaziaris
- O Mr Lam
- O Mr Ireland

Student Number:

(To be used by the exam markers only.)

Question No	1-4	5	6	7	8	9	10	Total	Total
Mark	$\overline{4}$	$\overline{7}$	$\overline{4}$	$\overline{5}$	$\overline{13}$	$\overline{6}$	$\overline{4}$	$\overline{43}$	$\overline{100}$

Questions 1-4 - Answer on the multiple choice answer sheet provided.

1. If $\frac{x^2}{48}+\frac{y^2}{12}=1$ is the equation of an ellipse, then the equations of the directrices are :-

(A) $x=\pm 2$

(B) $x=\pm 8$

(C) $x=\pm 4$

(D) $x=\pm\frac{8\sqrt{3}}{\sqrt{5}}$

2. If $\frac{y^2}{16}-\frac{x^2}{25}=1$ is the equation of a hyperbola, then its eccentricity, e is:-

(A) $e=\frac{\sqrt{41}}{4}$

(B) $e=\frac{\sqrt{41}}{5}$

(C) $e=\sqrt{41}$

(D) $e=\frac{5\sqrt{41}}{4}$

3. $\int 3\cos\frac{1}{3}x\,dx=$

(A) $\sin\frac{1}{3}x+C$

(B) $-\sin\frac{1}{3}x+C$

(C) $-9\sin\frac{1}{3}x+C$

(D) $9\sin\frac{1}{3}x+C$

4. On a hyperbola, the distance between its vertices is 8 units and the distance between its foci is 10 units. The acute angle between its asymptotes will be

(A) $36^\circ 52'$

(B) $73^\circ 44'$

(C) 45°

(D) $75^\circ 34'$

CONTINUED...

Questions 5-10 – Answer in the booklet provided.

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5. (a) $\int (x+1)\sec^2(x^2+2x)dx$ **2**

(b) $\int \sin^2 3x dx$ **2**

(c) The region under the curve $y = \tan x$ between $x = \dfrac{\pi}{6}$ and $x = \dfrac{\pi}{3}$

is rotated about the x-axis. What is the volume of the solid of revolution formed. **3**

6. Find the equations of the two tangents to the ellipse $16x^2 + 25y^2 = 400$

which are parallel to the line $y = x + 2$. **4**

7. (a) Write $\dfrac{4x^2 - 5x - 7}{(x-1)(x^2+x+2)}$ in the form $\dfrac{A}{x-1} + \dfrac{Bx+C}{x^2+x+2}$. **3**

(b) Hence, evaluate $\int \dfrac{4x^2 - 5x - 7}{(x-1)(x^2+x+2)}dx$ **2**

CONTINUED...

8. The hyperbola $16x^2 - 9y^2 = 144$ has foci $S(5,0)$ and $S'(-5,0)$.

The directrices are $x = \frac{9}{5}$ and $x = -\frac{9}{5}$.

(a) State the equation of each asymptote of the hyperbola. **1**

(b) Sketch the hyperbola and indicate on your diagram the foci, directrices, and asymptotes. **2**

(c) By differentiation, find the gradient of the tangent to the hyperbola at $P(3\sec\theta, 4\tan\theta)$. **2**

(d) Show that the tangent to the hyperbola at P has equation $4x = (3\sin\theta)y + 12\cos\theta$. **2**

(e) Given that $0 < \theta < \frac{\pi}{2}$, show that Q, the point of intersection of the tangent to the hyperbola at P and the nearer directrix, has y coordinate $\frac{12 - 20\cos\theta}{5\sin\theta}$. **2**

(f) Calculate the gradients of SP and SQ. **2**

(g) Determine whether $\angle PSQ$ is a right angle. **2**

9 . Consider the equation $\frac{x^2}{3-\lambda} + \frac{y^2}{5-\lambda} = 1$

(a) Determine the real values for which the above equation defines an

(i) Ellipse **1**

(ii) Hyperbola **2**

(b) Sketch the curve for $\lambda = 4$, showing the branches and vertices only. **1**

(c) Describe how the shape of the curve changes as λ increases from 3 towards 5. **1**

(d) What is the limiting position of the curve as λ approaches 5. **1**

10. Prove by mathematical induction that $\sqrt{2+\sqrt{2+\sqrt{2+\ldots\ldots.\sqrt{2}}}} = 2\cos\frac{\pi}{2^{n+1}}$ for $n \geq 1$

where there are n lots of $2's$ on the left hand side. **4**

(eg. $\sqrt{2+\sqrt{2}} = 2\cos\frac{\pi}{2^{2+1}}$)

END OF EXAMINATION

Answer sheet for Section I

Mark answers to Section I by fully blackening the correct circle, e.g "●"

STUDENT NUMBER: ....

Class (please ✔)

○ 12M4A – Ms Ziaziaris ○ 12M4B – Mr Lam ○ 12M4C – Mr Ireland

1 – Ⓐ Ⓑ Ⓒ Ⓓ
2 – Ⓐ Ⓑ Ⓒ Ⓓ
3 – Ⓐ Ⓑ Ⓒ Ⓓ
4 – Ⓐ Ⓑ Ⓒ Ⓓ

STANDARD INTEGRALS

$\int x^n \, dx$	$= \frac{1}{n+1} x^{n+1} + C,$	$n \neq -1; \quad x \neq 0$ if $n < 0$
$\int \frac{1}{x} \, dx$	$= \ln x + C,$	$x > 0$
$\int e^{ax} \, dx$	$= \frac{1}{a} e^{ax} + C,$	$a \neq 0$
$\int \cos ax \, dx$	$= \frac{1}{a} \sin ax + C,$	$a \neq 0$
$\int \sin ax \, dx$	$= -\frac{1}{a} \cos ax + C,$	$a \neq 0$
$\int \sec^2 ax \, dx$	$= \frac{1}{a} \tan ax + C,$	$a \neq 0$
$\int \sec ax \tan ax \, dx$	$= \frac{1}{a} \sec ax + C,$	$a \neq 0$
$\int \frac{1}{a^2 + x^2} \, dx$	$= \frac{1}{a} \tan^{-1} \frac{x}{a} + C,$	$a \neq 0$
$\int \frac{1}{\sqrt{a^2 - x^2}} \, dx$	$= \sin^{-1} \frac{x}{a} + C,$	$a > 0, -a < x < a$
$\int \frac{1}{\sqrt{x^2 - a^2}} \, dx$	$= \ln\left(x + \sqrt{x^2 - a^2}\right) + C,$	$x > a > 0$
$\int \frac{1}{\sqrt{x^2 + a^2}} \, dx$	$= \ln\left(x + \sqrt{x^2 + a^2}\right) + C$	

NOTE: $\ln x = \log_e x, \; x > 0$

EXTENSION 2 - 2014 HSC ASSESSMENT TASK 2 SOLN'S

(1) $a^2 = 48$

$a = 4\sqrt{3}$

$b^2 = 12$

$b = 2\sqrt{3}$

$e^2 = 1 - \frac{b^2}{a^2}$

$e^2 = 1 - \frac{12}{48}$

$e = \frac{\sqrt{3}}{2}$

Dir : $x = \pm \frac{a}{e}$

$x = \pm \frac{4\sqrt{3}}{\frac{\sqrt{3}}{2}}$

$\boxed{x = \pm 8 \text{ — B}}$

(2) $e^2 = 1 + \frac{a^2}{b^2}$

$e^2 = 1 + \frac{25}{16}$

$\boxed{e = \frac{\sqrt{41}}{4} \text{ — A}}$

(3) $\int 3 \cos \frac{1}{3}x \, dx$

$= 3 \int \cos \frac{1}{3} x \, dx$

$= 3 \times 3 \sin \frac{1}{3} x + C$

$= \boxed{9 \sin \frac{1}{3} x + C \text{ — D}}$

(4) $a = 4$

$ae = 5$

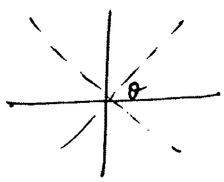
$e = \frac{5}{4}$

$e^2 = 1 + \frac{b^2}{a^2}$

$\therefore b = 3$

Asymptotes :

$y = \pm \frac{b}{a} x$



$\tan\theta = \frac{b}{a}$

$\tan\theta = \frac{3}{4}$ $\therefore \theta = 36°52'$

$\boxed{\text{Angle between asymptotes } 73°44' \text{ — B}}$

5. a) $\int (x+1)\sec^2(x^2+2x)\,dx$

$= \frac{1}{2}\int (2x+2)\sec^2(x^2+2x)\,dx$ — 1 mark

$= \frac{1}{2}\,\frac{\sec^3(x^2+2x)}{3} + C$

$= \frac{\sec^3(x^2+2x)}{6} + C$ — 1 mark

b) $\int \sin^2 3x\,dx$

$\cos 2x = 1 - 2\sin^2 x$

$\sin^2 x = \frac{1-\cos 2x}{2}$

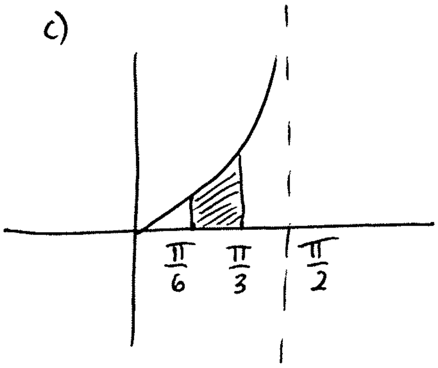
$\sin^2 3x = \frac{1-\cos 6x}{2}$

$= \frac{1}{2}\int (1-\cos 6x)\,dx$ — 1 mark to convert to $\cos 6x$

$= \frac{1}{2}\left[x - \frac{\sin 6x}{6}\right] + C$

$= \frac{x}{2} - \frac{\sin 6x}{12} + C$ — 1 mark to integrate

c)



$V = \pi\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \tan^2 x\,dx$

1 mark to set up integral

$= \pi\int_{\pi/6}^{\pi/3} (\sec^2 x - 1)\,dx$

1 mark to integrate

$= \pi\left[\tan x - x\right]_{\pi/6}^{\pi/3}$

$= \pi\left[\tan\frac{\pi}{3} - \frac{\pi}{3} - \tan\frac{\pi}{6} + \frac{\pi}{6}\right]$

$= \pi\left[\sqrt{3} - \frac{1}{\sqrt{3}} - \frac{\pi}{6}\right]$ c. units.

1 mark for answer (unsimplified is ok)

6. Let the tangents have eq'n

$y = mx + c$

But $m = 1$

$\therefore y = x + c$ — ①   1 mark

$16x^2 + 25y^2 = 400$ — ②

Sub ① into ②

$16x^2 + 25(x+c)^2 = 400$

$16x^2 + 25x^2 + 50xc + 25c^2 = 400$

$41x^2 + 50xc + 25c^2 - 400 = 0$

For tangency, $\Delta = 0$.   1 mark for tangency

ie. $(50c)^2 - 4(41)(25c^2 - 400) = 0$.

$2500c^2 - 164(25c^2 - 400) = 0$

$2500c^2 - 4100c^2 + 65600 = 0$

$-1600c^2 = -65600$

$c^2 = 41$ 1 mark

$c = \pm\sqrt{41}$

$\therefore$ Eq'n s of tangents are $y = x \pm \sqrt{41}$ 1 mark.

7. a) $\dfrac{4x^2-5x-7}{(x-1)(x^2+x+2)} = \dfrac{A}{x-1} + \dfrac{Bx+C}{x^2+x+2}$

$= \dfrac{A(x^2+x+2)}{(x-1)(x^2+x+2)} + \dfrac{(Bx+C)(x-1)}{(x-1)(x^2+x+2)}$

$= \dfrac{Ax^2+Ax+2A+Bx^2-Bx+Cx-C}{(x-1)(x^2+x+2)}$

$= \dfrac{(A+B)x^2+(A-B+C)x+2A-C}{(x-1)(x^2+x+2)}$ 1 mark

$\therefore A+B=4$ — ①

$A-B+C=-5$ — ②

$2A-C=-7$ — ③

Sub $B=4-A$ into ②

$A-(4-A)+C=-5$

$A-4+A+C=-5$

$2A+C=-1$ — ④

③ + ④

$4A=-8$

$A=-2$ — 1 mark getting to A.

$\therefore C=-1-2(-2)$

$C=3$

$B=6$ } 1 mark for B, C

$\therefore \dfrac{4x^2-5x-7}{(x-1)(x^2+x+2)} = \dfrac{-2}{x-1} + \dfrac{6x+3}{x^2+x+2}$

b) $\int \left(\dfrac{-2}{x+1} + \dfrac{6x+3}{x^2+x+2}\right) dx$

$= -2\ln|x+1| + 3\int \dfrac{2x+1}{x^2+x+2}\, dx$ 1 mark

$= -2\ln|x+1| + 3\ln|x^2+x+2| + C$ 1 mark

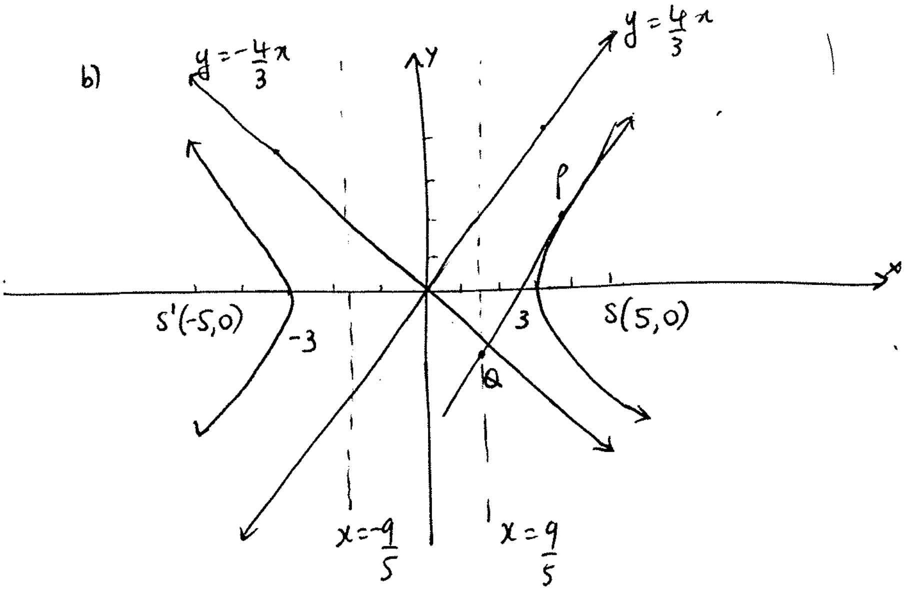
8. $16x^2 - 9y^2 = 144$

$$\frac{x^2}{9} - \frac{y^2}{16} = 1$$

a) $y = \pm \frac{b}{a}x$

$$\boxed{y = \pm \frac{4}{3}x}$$ — 1mark

b)



2 { — 1mark for vertices & foci
— 1 mark for directrices/asympt.

c) $\frac{2x}{9} - \frac{2y}{16}\frac{dy}{dx} = 0$

$$\frac{dy}{dx} = \frac{2x}{9} \times \frac{16}{2y}$$

$$\frac{dy}{dx} = \frac{16x}{9y}$$ — 1 mark for dy/dx

At $(3\sec\theta, 4\tan\theta)$

$$\boxed{\frac{dy}{dx} = \frac{4\sec\theta}{3\tan\theta} = \frac{4}{3\sin\theta}}$$ — 1mark for subⁿ of pt.

2 }

d) Eqⁿ of tangent

$$y - 4\tan\theta = \frac{4}{3\sin\theta}(x - 3\sec\theta)$$ — 1 mark for subⁿ

$$3y\sin\theta - 12\tan\theta\sin\theta = 4x - 12\sec\theta$$

$$3y\sin\theta - 12\tan\theta\sin\theta + \frac{12}{\cos\theta} = 4x$$

$$3y\sin\theta + \frac{12 - 12\sin^2\theta}{\cos\theta} = 4x$$ — 1 mark for answer

$$3y\sin\theta + 12\cos\theta = 4x$$

e) $4x = 3y\sin\theta + 12\cos\theta$ — ①

$x = \frac{9}{5}$ — ②

Solve simult. to obtain Q.

Sub ② into ①

$4\left(\frac{9}{5}\right) = 3y\sin\theta + 12\cos\theta$ ——— 1 mark for sub^n

$\frac{36}{5} - 12\cos\theta = 3y\sin\theta$

$y = \dfrac{\frac{36}{5} - 12\cos\theta}{3\sin\theta}$

$y = \dfrac{36 - 60\cos\theta}{15\sin\theta}$

$y = \dfrac{12 - 20\cos\theta}{5\sin\theta}$ ——— 1 mark for answer

f) $m_{SP} = \dfrac{4\tan\theta}{\sec\theta - 5}$ ——— 1 mark for m_{SP}

$m_{SQ} = \dfrac{\frac{12 - 20\cos\theta}{5\sin\theta}}{\frac{9}{5} - 5}$

$= \dfrac{12 - 20\cos\theta}{-16\sin\theta}$ ——— 1 mark for m_{SQ}

$= \dfrac{5\cos\theta - 3}{4\sin\theta}$

g) $m_{SP} \times m_{SQ} = \dfrac{4\tan\theta}{3\sec\theta - 5} \times \dfrac{5\cos\theta - 3}{4\sin\theta}$

$= \dfrac{5\tan\theta\cos\theta - 3\tan\theta}{\sin\theta(3\sec\theta - 5)}$

$= \dfrac{5\sin\theta - \frac{3\sin\theta}{\cos\theta}}{\left(\frac{3}{\cos\theta} - 5\right)\sin\theta}$ ——— 1 mark for changing to sin/cos.

$= \dfrac{5\sin\theta\cos\theta - 3\sin\theta}{(3 - 5\cos\theta)\sin\theta}$ ——— 1 mark for obtaining -1

$= \dfrac{\sin\theta(5\cos\theta - 3)}{(3 - 5\cos\theta)\sin\theta} = -1$

Since $m_{SP} \times m_{SQ} = -1$ then $\angle PSQ = 90°$.

9. $\dfrac{x^2}{3-\lambda} + \dfrac{y^2}{5-\lambda} = 1$

a) (i) Ellipse

$3-\lambda > 0$ and $5-\lambda = 0$

ie. $\lambda < 3$ and $\lambda < 5$

$\therefore \boxed{\lambda < 3}$

(ii) Hyperbola

$3-\lambda > 0$ and $5-\lambda < 0$

ie $\lambda < 3$ and $\lambda > 5$

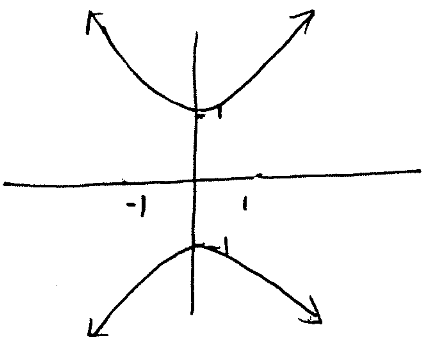
not possible.

OR. $3-\lambda < 0$ and $5-\lambda > 0$

ie. $\lambda > 3$ and ie. $\lambda < 5$

$\therefore \boxed{3 < \lambda < 5}$

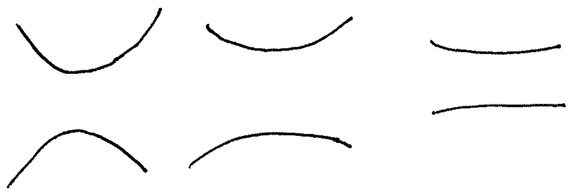
b)



$\dfrac{x^2}{-1} + \dfrac{y^2}{1} = 1$

ie. $y^2 - x^2 = 1$

c)



The branches become flatter almost parallel lines. ie. Concavity of branches decreases.

d) Curve approaches 2 parallel lines which converge to $y=0$.

10. $\sqrt{2+\sqrt{2+\sqrt{2+\cdots\sqrt{2}}}} = 2\cos\frac{\pi}{2^{n+1}}$ for $n \geq 1$

Step1 : Show true for $n=1$

$$LHS = \sqrt{2}$$

$$RHS = 2\cos\frac{\pi}{2^{1+1}}$$

$$= 2\cos\frac{\pi}{4}$$

$$= 2 \times \frac{1}{\sqrt{2}}$$

$$= \sqrt{2}$$

$LHS = RHS$ $\therefore$ True for $n=1$

Step2 : Assume true for $n=k$

ie. $\underbrace{\sqrt{2+\sqrt{2+\sqrt{2+\cdots\sqrt{2}}}}}_{k\ 2's} = 2\cos\frac{\pi}{2^{k+1}}$

Step3 : Prove true for $n=k+1$

$$\sqrt{2+\sqrt{2+\sqrt{2+\cdots+\sqrt{2}}}} = 2\cos\frac{\pi}{2^{k+2}}$$

From assumption add 2 to bothsides and take square root to obtain $k+1$ lots of 2's on LHS.

$$\sqrt{2+\sqrt{2+\sqrt{2+\cdots+\sqrt{2}}}} = \sqrt{2+2\cos\frac{\pi}{2^{k+1}}}$$

$$= \sqrt{2\left(1+\cos\frac{\pi}{2^{k+1}}\right)}$$

Let $\theta = \frac{\pi}{2^{k+1}}$

$$\therefore LHS = \sqrt{2(1+\cos\theta)}$$

$$= \sqrt{2}\sqrt{1+\cos\theta}$$

$$= \sqrt{2}.\sqrt{2}\cos\tfrac{1}{2}\theta$$

$$= 2\cos\tfrac{1}{2}.\tfrac{\pi}{2^{k+1}}$$

$$= 2\cos\frac{\pi}{2^{k+2}}$$

$$= RHS.$$

$\therefore$ True for $n=k+1$.

$\cos\theta = \sqrt{\frac{1+\cos 2\theta}{2}}$

$\therefore \cos\frac{1}{2}\theta = \sqrt{\frac{1+\cos\theta}{2}}$

$\sqrt{2}\cos\frac{1}{2}\theta = \sqrt{1+\cos\theta}$

Step4 : By Principle of Mathematical Induction true for all $n \geq 1$.