North Sydney Boys High School

MATHEMATICS (EXTENSION 2)

2015 HSC Course Assessment Task 2

Thursday March 5, 2015

General instructions

- Working time – 55 minutes. (plus 5 minutes reading time)
- Write using blue or black pen. Where diagrams are to be sketched, these may be done in pencil.
- Board approved calculators may be used.
- Attempt **all** questions.
- At the conclusion of the examination, bundle the booklets used in the correct order within this paper and hand to examination supervisors.

SECTION I

- Mark your answers on the answer grid provided (on page 6)

SECTION II

- Commence each new question on a new page. Write on both sides of the paper.
- All necessary working should be shown in every question. Marks may be deducted for illegible or incomplete working.

STUDENT NUMBER: .... **# BOOKLETS USED:** .....

Class (please ✔)

◯ 12M4A – Mr Lam ◯ 12M4B – Mr Lin ◯ 12M4C – Mr Ireland

Marker's use only.

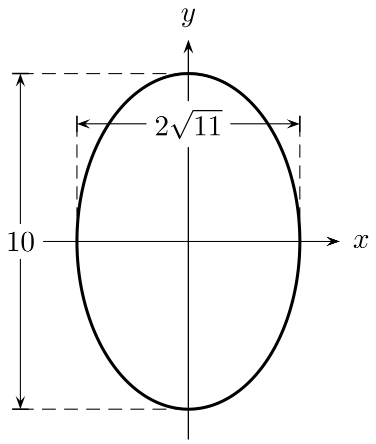
QUESTION	1-5	6	7	8	Total	%
MARKS	$\overline{5}$	$\overline{11}$	$\overline{13}$	$\overline{12}$	$\overline{41}$	

Section I

5 marks
Attempt Question 1 to 5
Mark your answers on the answer grid provided.

Questions **Marks**

1. A conic section is graphed as shown. The distance between the x intercepts is $2\sqrt{11}$ units, and the distance between the y intercepts is 10 units. Which of the following is the equation of the graph? **1**



(A) $\dfrac{x^2}{25} + \dfrac{y^2}{44} = 1$

(B) $\dfrac{x^2}{44} + \dfrac{y^2}{25} = 1$

(C) $\dfrac{x^2}{25} + \dfrac{y^2}{11} = 1$

(D) $\dfrac{x^2}{11} + \dfrac{y^2}{25} = 1$

2. Which of the following represents the equation of the chord of contact from $(5, -2)$ to the ellipse $\dfrac{x^2}{16} + \dfrac{y^2}{9} = 1$? **1**

(A) $\dfrac{x}{8} - \dfrac{5y}{16} = 1$ (B) $\dfrac{5x}{16} + \dfrac{2y}{9} = 1$ (C) $\dfrac{5x}{16} - \dfrac{2y}{9} = 0$ (D) $\dfrac{5x}{16} - \dfrac{2y}{9} = 1$

3. $S(4, 0)$ is a focus of the rectangular hyperbola $x^2 - y^2 = k$. Which of the following is the value of k? **1**

(A) $2\sqrt{2}$ (B) $4\sqrt{2}$ (C) 8 (D) 32

4. Which of the following is the equation of the conic represented by foci $(\pm 3, 0)$ and vertices $(\pm 2, 0)$? **1**

(A) $\dfrac{x^2}{5} + \dfrac{y^2}{4} = 1$ (B) $\dfrac{x^2}{4} + \dfrac{y^2}{5} = 1$ (C) $\dfrac{x^2}{4} - \dfrac{y^2}{5} = 1$ (D) $\dfrac{x^2}{5} - \dfrac{y^2}{4} = 1$

5. If $3x^2 + 2xy + y^2 = 2$, what is the gradient of the tangent at $x = 1$? **1**

(A) -2 (B) 0 (C) 4 (D) not defined

Section II

36 marks
Attempt Questions 6 to 8
Write your answers in the writing booklets supplied. Additional writing booklets are available.
Your responses should include relevant mathematical reasoning and/or calculations.

Question 6 (11 Marks) Commence a NEW page. **Marks**

(a) Consider the hyperbola $\mathcal{H}$ with equation $\dfrac{x^2}{9} - \dfrac{y^2}{16} = 1$.

i. Find the points of intersection of $\mathcal{H}$ with the x axis, as well as the eccentricity and foci of $\mathcal{H}$. **3**

ii. Write down the equations of the directrices and asymptote of $\mathcal{H}$. **2**

iii. Sketch $\mathcal{H}$, showing all of the above information. **2**

(b) i. Expand and simplify: $\left(1+\sqrt{2}\right)^2$. **1**

ii. A sequence a_n is defined by $a_0 = a_1 = 2$, and **3**

$$a_n = 2a_{n-1} + a_{n-2}$$

when $n \geq 2$.

Use mathematical induction to prove that

$$a_n = \left(1+\sqrt{2}\right)^n + \left(1-\sqrt{2}\right)^n \qquad \text{for all } n \geq 0$$

Question 7 (13 Marks) Commence a NEW page. **Marks**

(a) Describe in geometric terms, the locus of a point z which moves on the Argand diagram according to the equation **3**

$$|z-3| + |z+3| = 8$$

State the coordinates of the focus and equation of the directrices.

(b) The hyperbola $\mathcal{B}$ has equation $xy = 4$.

i. Sketch $\mathcal{B}$, indicating on your diagram the positions and coordinates of all points at which $\mathcal{B}$ intersects the axes of symmetry. **2**

ii. Show that the equation of the tangent to $\mathcal{B}$ at $P\left(2t, \dfrac{2}{t}\right)$ where $t \neq 0$, is **2**

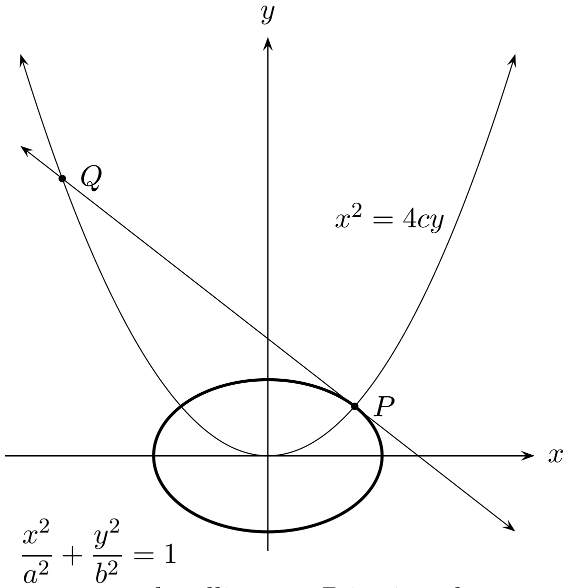
$$x + t^2 y = 4t$$

iii. If $s \neq 0$ and $s^2 \neq t^2$, show that the tangents to $\mathcal{B}$ at P and $Q\left(2s, \dfrac{2}{s}\right)$ intersect at $M\left(\dfrac{4st}{s+t}, \dfrac{4}{s+t}\right)$. **3**

iv. Suppose that in the previous part, the parameter $s = -\dfrac{1}{t}$. Show that the locus of M is a straight line through, but excluding the origin. **3**

Question 8 (12 Marks) Commence a NEW page. **Marks**

The parabola $x^2 = 4cy$ intersects the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ at the point $(2cp, cp^2)$ in the first quadrant where $a > b > 0$ and $c > 0$. The tangent to the ellipse at P meets the parabola again at $Q\left(2cq, cq^2\right)$.



(a) Show that the tangent to the ellipse at P is given by **3**

$$\frac{2cpx}{a^2} + \frac{cp^2y}{b^2} = 1$$

(b) If this tangent meets the parabola at $\left(2ct, ct^2\right)$, show that $\dfrac{p^2t^2}{b^2} + \dfrac{4pt}{a^2} - \dfrac{1}{c^2} = 0$ and deduce that, considered as a quadratic in t, this equation has roots p and q. **2**

(c) If PQ subtends a right angle at the origin, show that $pq = -4$ and deduce that **3**

$$\frac{1}{b^2} = \frac{1}{a^2} + \frac{1}{(4c)^2}$$

(d) Using the information from parts (b) and (c) or otherwise, show that if PQ subtends a right angle at the origin, then $p = 2e$, where e is the eccentricity of the ellipse. **4**

End of paper.

STANDARD INTEGRALS

$\int x^n \, dx$	$= \frac{1}{n+1} x^{n+1} + C,$	$n \neq -1; \quad x \neq 0$ if $n < 0$
$\int \frac{1}{x} \, dx$	$= \ln x + C,$	$x > 0$
$\int e^{ax} \, dx$	$= \frac{1}{a} e^{ax} + C,$	$a \neq 0$
$\int \cos ax \, dx$	$= \frac{1}{a} \sin ax + C,$	$a \neq 0$
$\int \sin ax \, dx$	$= -\frac{1}{a} \cos ax + C,$	$a \neq 0$
$\int \sec^2 ax \, dx$	$= \frac{1}{a} \tan ax + C,$	$a \neq 0$
$\int \sec ax \tan ax \, dx$	$= \frac{1}{a} \sec ax + C,$	$a \neq 0$
$\int \frac{1}{a^2 + x^2} \, dx$	$= \frac{1}{a} \tan^{-1} \frac{x}{a} + C,$	$a \neq 0$
$\int \frac{1}{\sqrt{a^2 - x^2}} \, dx$	$= \sin^{-1} \frac{x}{a} + C,$	$a > 0, -a < x < a$
$\int \frac{1}{\sqrt{x^2 - a^2}} \, dx$	$= \ln \left(x + \sqrt{x^2 - a^2} \right) + C,$	$x > a > 0$
$\int \frac{1}{\sqrt{x^2 + a^2}} \, dx$	$= \ln \left(x + \sqrt{x^2 + a^2} \right) + C$	

NOTE: $\ln x = \log_e x, \; x > 0$

Answer sheet for Section I

Mark answers to Section I by fully blackening the correct circle, e.g "●"

STUDENT NUMBER: ....

Class (please ✔)

○ 12M4A – Mr Lam ○ 12M4B – Mr Lin ○ 12M4C – Mr Ireland

1 – Ⓐ Ⓑ Ⓒ Ⓓ

2 – Ⓐ Ⓑ Ⓒ Ⓓ

3 – Ⓐ Ⓑ Ⓒ Ⓓ

4 – Ⓐ Ⓑ Ⓒ Ⓓ

5 – Ⓐ Ⓑ Ⓒ Ⓓ

Suggested Solutions

Section I

(Lam) **1.** (D) **2.** (D) **3.** (C) **4.** (C) **5.** (D)

Section II

Question 6 (Ireland)

(a) i. (3 marks)

$$\frac{x^2}{9} - \frac{y^2}{16} = 1$$

Semi major axis is 3, hence

$$V\,(\pm 3, 0)$$

Eccentricity:

$$\begin{aligned} b^2 &= a^2\left(e^2-1\right) \\ 16 &= 9\left(e^2-1\right) \\ e^2 - 1 &= \frac{16}{9} \\ \therefore e^2 &= \frac{16}{9} + 1 = \frac{25}{9} \\ e &= \frac{5}{3} \end{aligned}$$

Foci:

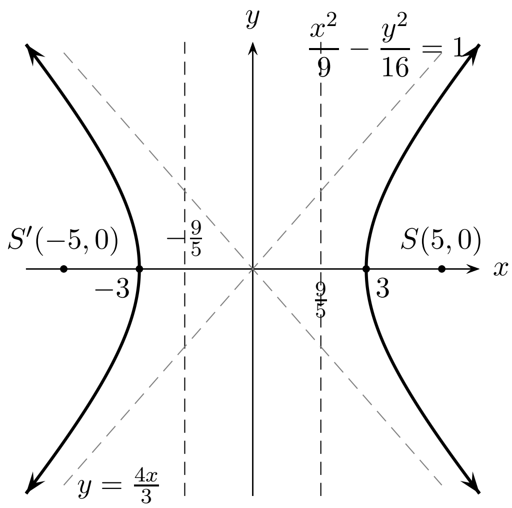
$$S\,(\pm ae, 0) = \left(\pm 3 \times \frac{5}{3}, 0\right) = (\pm 5, 0)$$

ii. (2 marks)

$$\text{Directrices:} \quad x = \pm\frac{a}{e} = \pm\frac{3}{\frac{5}{3}} = \pm\frac{9}{5}$$

$$\text{Asymptotes:} \quad y = \pm\frac{b}{a}x = \pm\frac{4x}{3}$$

iii. (2 marks)



(b) i. (1 mark)

$$\left(1+\sqrt{2}\right)^2 = 1+2\sqrt{2}+2 = 3+2\sqrt{2}$$

ii. (3 marks)
Let $P(n)$ be the proposition

$$a_n = \left(1+\sqrt{2}\right)^n + \left(1-\sqrt{2}\right)^n$$

whenever $n \geq 0$, with $a_0 = a_1 = 2$ and

$$a_n = 2a_{n-1} + a_{n-2}$$

- Base cases: $P(0)$ and $P(1)$
  - $P(0)$:
  $$a_0 = \left(1+\sqrt{2}\right)^0 + \left(1-\sqrt{2}\right)^0 = 2$$
  - $P(1)$:
  $$\begin{aligned} a_1 &= \left(1+\sqrt{2}\right)^1 + \left(1-\sqrt{2}\right)^1 \\ &= 1+\sqrt{2}+1-\sqrt{2} = 2 \end{aligned}$$

  Hence $P(1)$ and $P(2)$ are true.
- Inductive hypotheses: assume $P(3)$, $P(4)$, …, $P(k)$ and $P(k+1)$ are all true, i.e.
  - $P(k)$:
  $$a_k = \left(1+\sqrt{2}\right)^k + \left(1-\sqrt{2}\right)^k$$
  - $P(k+1)$:
  $$a_{k+1} = \left(1+\sqrt{2}\right)^{k+1} + \left(1-\sqrt{2}\right)^{k+1}$$

  Examine $P(k+2)$:

$$\begin{aligned} a_{k+2} &= 2a_{k+1} + a_k \\ &= 2\left[\left(1+\sqrt{2}\right)^{k+1} + \left(1-\sqrt{2}\right)^{k+1}\right] + \left(1+\sqrt{2}\right)^k + \left(1-\sqrt{2}\right)^k \\ &= \left(1+\sqrt{2}\right)^k\left(2\left(1+\sqrt{2}\right)+1\right) + \left(1-\sqrt{2}\right)^k\left(2\left(1+\sqrt{2}\right)+1\right) \\ &= \left(1+\sqrt{2}\right)^k\left(3+2\sqrt{2}\right) + \left(1-\sqrt{2}\right)^k\left(3-2\sqrt{2}\right) \\ &= \left(1+\sqrt{2}\right)^k\left(1+\sqrt{2}\right)^2 + \left(1-\sqrt{2}\right)^k\left(1-\sqrt{2}\right)^2 \\ &= \left(1+\sqrt{2}\right)^{k+2} + \left(1-\sqrt{2}\right)^{k+2} \end{aligned}$$

Hence $P(n)$ is true by induction.

Question 7 (Lin)

(a) (3 marks)

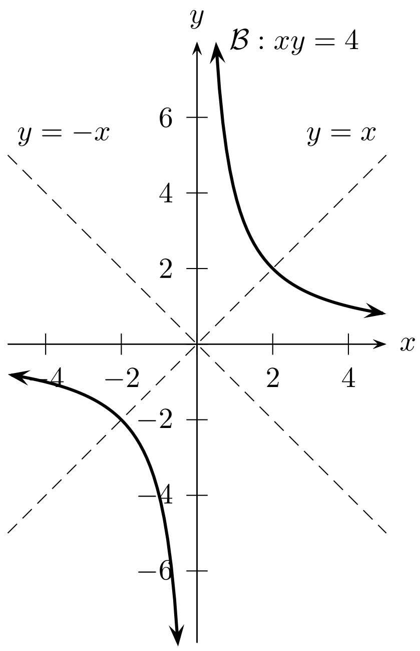
✓ [1] for each of description of locus, foci and directrices.

$$|z-3|+|z+3|=8$$
$$\Downarrow$$
$$PS+PS'=2a$$
$$\therefore S(\pm 3,0) \qquad a=4$$

Equation of directrices: found from eccentricity.

$$ae=3$$
$$4e=3$$
$$\therefore e=\frac{3}{4}$$
$$\therefore x=\pm\frac{a}{e}=\frac{4}{\frac{3}{4}}=\frac{16}{3}$$

(b) i. (2 marks)



Intersection with axes of symmetry:

$$\begin{cases} y=\frac{4}{x} \\ y=x \end{cases}$$
$$\therefore -x=\frac{4}{x}$$
$$x^2=4$$
$$x=\pm 2 \qquad y=\pm 2$$

ii. (2 marks)

Equation of tangent at $\left(2t,\frac{2}{t}\right)$:

$$y=4x^{-1}$$
$$\frac{dy}{dx}=-4x^{-2}\Big|_{x=2t}=\frac{-4}{4t^2}=-\frac{1}{t^2}$$

Apply the point-gradient formula,

$$\frac{y-\frac{2}{t}}{x-2t}=-\frac{1}{t^2}$$
$$t^2y-2t=-x+2t$$
$$x+t^2y=4t$$

iii. (3 marks)

At S, the equation of the tangent is $x+s^2y=4s$. Where the tangents from P And Q intersect,

$$\begin{cases} x+t^2y=4t & (1) \\ x+s^2y=4s & (2) \end{cases}$$

(1) − (2):

$$y\left(t^2-s^2\right)=4t-4s$$
$$\cancel{y(t-s)}(t+s)=4\cancel{(t-s)}$$
$$\therefore y=\frac{4}{s+t}$$

Substitute into (1) to find x:

$$x+t^2\left(\frac{4}{s+t}\right)=4t$$
$$x+\frac{4t^2}{s+t}=\frac{4t(s+t)}{s+t}$$
$$x=\frac{4st+4t^2-4t^2}{s+t}=\frac{4st}{s+t}$$

Hence the coordinates of M are

$$M\left(\frac{4st}{s+t},\frac{4}{s+t}\right)$$

iv. (3 marks)

✓ [1] for each of x_M and y_M

✓ [1] for $y=-x$ and justification for $x\neq 0$.

Spse $s=-\frac{1}{t}$. Hence $t\neq 0$ and $s\neq 0$ and $x\neq 0$.

$$\begin{cases} x_M=\frac{4st}{s+t}=\frac{4\times -\frac{1}{t}\times t}{-\frac{1}{t}+t}=-\frac{4}{t-\frac{1}{t}} \\ y_M=\frac{4}{s+t}=\frac{4}{t-\frac{1}{t}} \end{cases}$$

Hence $y_M = -x_M$ and the locus of M is on the line $y = -x$, excluding $(0,0)$ as $t \neq 0$.

Question 8 (Lam)

(a) (3 marks)

- Equation of tangent to ellipse:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

Differentiating to obtain gradient at (x_0, y_0):

$$\frac{d}{dx}\left(\frac{x^2}{a^2} + \frac{y^2}{b^2}\right) = 0$$
$$\frac{2x}{a^2} + \frac{2y}{b^2}\frac{dy}{dx} = 0$$
$$\frac{dy}{dx} = -\left.\frac{b^2 x}{a^2 y}\right|_{\substack{x=x_0\\y=y_0}}$$
$$= -\frac{b^2 x_0}{a^2 y_0}$$

Point-gradient formula

$$\frac{y - y_0}{x - x_0} = -\frac{b^2 x_0}{a^2 y_0}$$
$$a^2 y y_0 - a^2 {y_0}^2 = -b^2 x x_0 + b^2 {x_0}^2$$
$$\underbrace{b^2 x x_0 + a^2 y y_0}_{\div a^2 b^2} = \underbrace{b^2 {x_0}^2 + a^2 {y_0}^2}_{\div a^2 b^2}$$
$$\frac{x x_0}{a^2} + \frac{y y_0}{b^2} = \frac{{x_0}^2}{a^2} + \frac{{y_0}^2}{b^2}$$

As (x_0, y_0) is a point on the ellipse, then it satistfies $\dfrac{x_0}{a^2} + \dfrac{y_0}{b^2} = 1$:

$$\therefore \frac{x x_0}{a^2} + \frac{y y_0}{b^2} = 1$$

- As $x_0 = 2cp$, $y_0 = cp^2$ (a point on the parabola also), the equation of the tangent is

$$\frac{2cpx}{a^2} + \frac{cp^2 y}{b^2} = 1$$

(b) (2 marks)
Solving equation of tangent to ellipse & equation of parabola simultaneously:

$$\begin{cases} \frac{2cpx}{a^2} + \frac{cp^2 y}{b^2} = 1 & (1) \\ x^2 = 4cy & (2) \end{cases}$$

Rearrange (2) and substitute into (1):

$$y = \frac{x^2}{4c}$$
$$\frac{2cpx}{a^2} + \frac{\not{c}p^2\left(\frac{x^2}{4\not{c}}\right)}{b^2} = 1$$

At $x = 2ct$,

$$\frac{2cp(2ct)}{a^2} + \frac{cp^2\,(2ct)^2}{4b^2} = 1$$
$$\frac{4c^2pt}{a^2} + \frac{\not{4}p^2t^2c^2}{\not{4}b^2} = 1$$

Divide throughout by c^2,

$$\frac{4pt}{a^2} + \frac{p^2t^2}{b^2} = \frac{1}{c^2}$$
$$\therefore \frac{p^2t^2}{b^2} + \frac{4pt}{a^2} - \frac{1}{c^2} = 0 \qquad (\spadesuit)$$

Quadratic in t will have 2 roots if it were to intersect the parabola twice. Those points of intersection are at $P(2cp, cp^2)$ and $Q(2cq, cq^2)$. At those points, the parameter $t = p$ or $t = q$. Hence the roots to the quadratic are p and q.

(c) (3 marks)

- Gradient of OP: $m_{OP} = \frac{cp^2 - 0}{2cp - 0} = \frac{p}{2}$. Similarly,

$$m_{OQ} = \frac{q}{2}$$

- As PQ subtends a right angle at O, then $OP \perp OQ$:

$$m_{OP} \times m_{OQ} = -1$$
$$\frac{p}{2} \times \frac{q}{2} = -1$$
$$\therefore pq = -4$$

- From the previous quadratic in t, if $t = q$ then

$$\frac{p^2q^2}{b^2} + \frac{4pq}{a^2} - \frac{1}{c^2} = 0$$
$$\frac{16}{b^2} + \frac{-16}{a^2} - \frac{1}{c^2} = 0$$
$$\therefore \frac{1}{b^2} = \frac{1}{a^2} + \frac{1}{16c^2}$$
$$\therefore \frac{1}{b^2} = \frac{1}{a^2} + \frac{1}{(4c^2)} \qquad (\ddagger)$$

(d) (4 marks)

- Given $pq = -4$, and p, q being roots to the quadratic in t (♠), then the product of roots is -4:

$$pq = \frac{-\frac{1}{c^2}}{\frac{p^2}{b^2}} = -4$$
$$\therefore -\frac{b^2}{c^2p^2} = -4$$
$$\frac{b^2}{4c^2} = p^2$$
$$\therefore p = \frac{b}{2c}$$

- Also, $b^2 = a^2\left(1 - e^2\right)$:

$$\therefore \frac{b^2}{a^2} = 1 - e^2$$

- Take equation (‡) and multiply throughout by b^2:

$$1 = \frac{b^2}{a^2} + \frac{b^2}{(4c)^2}$$
$$\cancel{1} = \left(\cancel{1} - e^2\right)\frac{b^2}{16c^2}$$
$$\therefore e^2 = \frac{b^2}{16c^2}$$
$$e = \frac{b}{4c} = \frac{1}{2} \times \frac{b}{2c} = \frac{1}{2}p$$
$$\therefore p = 2e$$