# NORTH SYDNEY GIRLS HIGH SCHOOL

# EXTENSION 2 MATHEMATICS

# HSC ASSESSMENT TASK

# TERM 1 2006

**Instructions**:

- Time allowed: 70 minutes + 2 minutes reading time.
- This task is worth 18% of your HSC assessment
- Attempt all four questions
- Show all necessary working
- Marks may be deducted for incomplete or poorly arranged work
- Start each question on a NEW PAGE
- Write on one side of the page only

**Question1** ( 10 marks) **Marks**

(a) Given the complex number $z = 1 - \sqrt{3}\, i$ write down

(i) $|z|$ 1

(ii) $\arg z$ 1

(iii) $z^3$ 1

(b) (i) Solve the equation $z = \sqrt{5 - 12i}$ 2

(ii) Given that $w = \dfrac{1 \pm \sqrt{5-12i}}{2+2i}$ is purely imaginary find $w^{400}$ 2

(c) (i) On the same Argand diagram, sketch the locus of 2

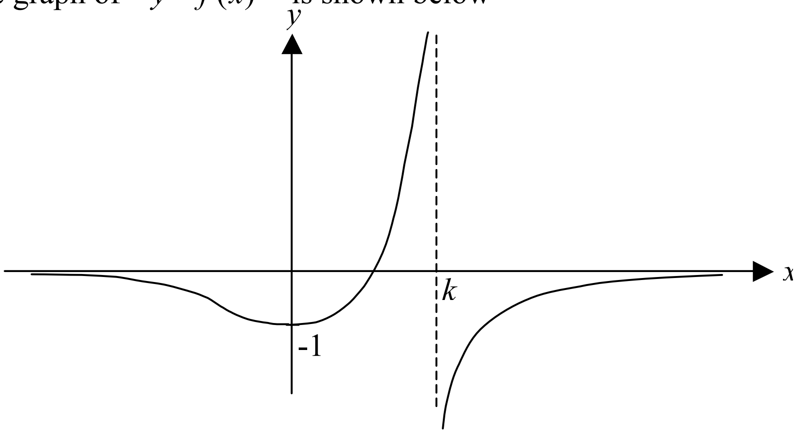
$|z-3| = 1$ and $|z| = |z-2i|$

(ii) Hence write down the complex number represented 1

by the intersection of the two loci.

**Question 2** (10 marks) Marks

The graph of $y = f(x)$ is shown below




Draw neat sketches of the following

(i) $y = f(x - k)$ 2

(ii) $y = f(-x)$ 2

(iii) $y = f(|x|)$ 2

(iv) $y = [f(x)]^2$ 2

(v) $y = e^{f(x)}$ 2

**Question 3.** ( 10 marks) Marks

(a) A conic is defined by the equation $\dfrac{x^2}{19-k}+\dfrac{y^2}{7-k}=1$

(i) Determine the values of $k$ for which the equation defines an ellipse . 1

(ii) Consider the above conic when $k=3$. Write down the value of the eccentricity, the coordinates of the foci and the equations of the directrices. 3

(iii) Sketch the above ellipse ( $k=3$ ) showing all important features. 2

(iv) Describe how the shape of the ellipse changes as $k$ increases in value from 3 to 7. 1

(b) Write down the equations of the asymptotes to the hyperbola $\dfrac{x^2}{a^2}-\dfrac{y^2}{b^2}=1$. 3

Hence deduce that the eccentricity $e=\sqrt{2}$ if these asymptotes are perpendicular to each other.

**Question 4.** (10 marks )

(a) The rectangle *OABC* lies in the Argand plane where *OC* is twice *OA*.
Vertex *O* corresponds to the origin.
The vertex *A* corresponds to the complex number *w*.
Express in terms of *w*, the complex number that corresponds to

(i) the point *C* 1

(ii) the point *D*, the point of intersection of the diagonals. 2

(b) $z$ is a variable complex number represented by the point *P*. 4
Describe and sketch the locus of *P* if $|z-4i|=\mathrm{Im}(z)$

(c) Let $z=\cos(\frac{\pi}{4})+i\sin(\frac{\pi}{4})$ and $w=\cos(\frac{\pi}{3})+i\sin(\frac{\pi}{3})$. Evaluate $zw$ 3
and hence give the exact value of $\cos\left(\frac{7\pi}{12}\right)$

**Question 5.** ( 10 marks) Marks

(a) Draw a neat sketch of $y = |x-4| + |x|$ 2

(b) On the same set of axes draw a neat sketch of 4
$y = \cos x$ and $y = \ln x$ for $0 \le x \le 2\pi$ .

Hence draw a neat sketch of $y = \cos x \ln x$ for $0 \le x \le 2\pi$ .

(c) Draw a neat sketch of $y = x^2\left(1-x^2\right)$ showing all intercepts 4
and the coordinates of any stationary points.
Hence sketch the relation $y^2 = x^2\left(1-x^2\right)$

**Question 6.** (10 marks)

(a) Explain why the set of points $z$ in the Argand plane defined 4
by $|z+5| - |z-5| = 8$ defines part of a conic.
Sketch this locus showing essential features.

(b) Consider the two series, $C$ and $S$ where
$C = 1 + \cos\theta + \cos 2\theta + \ldots\ldots\ldots\ldots \cos(n-1)\theta$ and
$S = \sin\theta + \sin 2\theta + \ldots\ldots\ldots\ldots + \sin(n-1)\theta$

(i) Write down $C + iS$ in terms of $\theta$ 1

(ii) If $z = \cos\theta + i\sin\theta$, express $C + iS$ as a series in terms of $z$ . 3

Hence show that $C + iS = \dfrac{1-z^n}{1-z}$ $, z \ne 1$

(iii) Show that 2

$$1 + \cos\theta + \cos 2\theta + \ldots\ldots\ldots\ldots \cos(n-1)\theta = \frac{\cos\left[(n-1)\theta\right] - \left(\cos\theta + \cos n\theta\right) + 1}{2 - 2\cos\theta}$$

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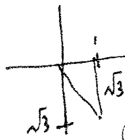
**End of paper**

# Solutions

**Question 1** (19 marks)

(a) (i) $|z| = \sqrt{1+3} = 2$ (1)




(ii) $\arg z = \frac{-\pi}{3}$ (1)

(iii) $z^3 = \left[2\left(\cos\left(\frac{-\pi}{3}\right) + i\sin\left(\frac{-\pi}{3}\right)\right)\right]^3$

$= 8\left[\cos(-\pi) + i\sin(-\pi)\right]$

$= -8$ (2)

(b) (i) let $z = x+iy$

$\therefore x^2 - y^2 = 5, \quad 2xy = -12$

$\therefore x = 3, y = -2$

$\therefore \sqrt{5-12i} = 3-2i$ (3)

(ii) $w = \dfrac{3 \pm \sqrt{5-12i}}{2}$

$= \dfrac{3 \pm (3-2i)}{2}$

$= \dfrac{6-2i}{2}, \dfrac{2i}{2}$

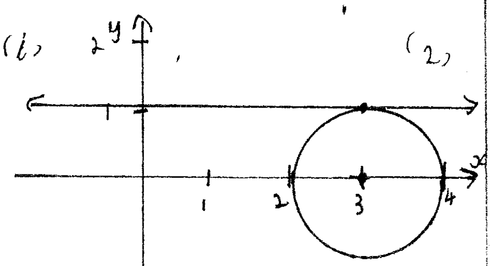
$= 3-i, \; i$

But w imaginary

$\therefore w = i$

$\therefore w^{400} = i^{400} = 1$ (2)

(c) (i) (2)




(ii) $z = 3+i$ (1)

(d) $|z - 4i| = \text{Im}(z)$

let $z = x+iy$

$|x + (y-4)i| = y$

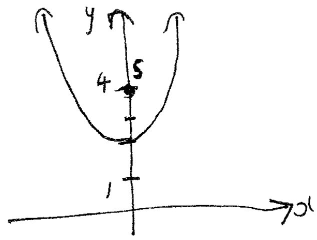
$\sqrt{x^2 + (y-4)^2} = y$

$x^2 + y^2 - 8y + 16 = y^2$

$x^2 = 8y - 16$

$x^2 = 8(y-2)$

Parabola, concave up Vertex $(0,2)$ $a=2$




(4)

(e) $z = \cos\frac{\pi}{4} + i\sin\frac{\pi}{4} = \frac{1}{\sqrt{2}}(1+i)$

$w = \cos\frac{\pi}{3} + i\sin\frac{\pi}{3} = \frac{1}{2}(1+\sqrt{3}i)$

$zw = \cos\left(\frac{\pi}{4}+\frac{\pi}{3}\right) + i\sin\left(\frac{\pi}{4}+\frac{\pi}{3}\right)$

$= \frac{1}{2\sqrt{2}}(1+i)(1+\sqrt{3}i)$

$= \frac{1}{2\sqrt{2}}(1 + i + \sqrt{3}i - \sqrt{3})$

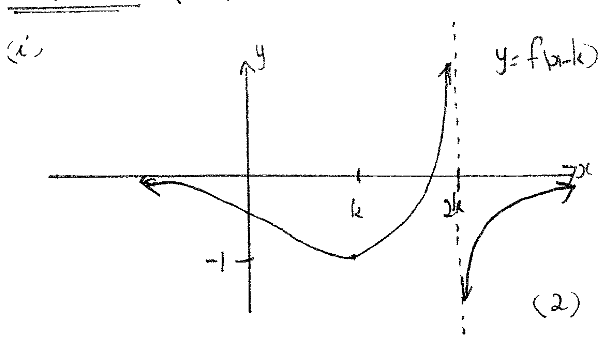
$\therefore \cos\frac{7\pi}{12} + i\sin\frac{7\pi}{12} = \frac{1-\sqrt{3}}{2\sqrt{2}} + \frac{(1+\sqrt{3})i}{2\sqrt{2}}$

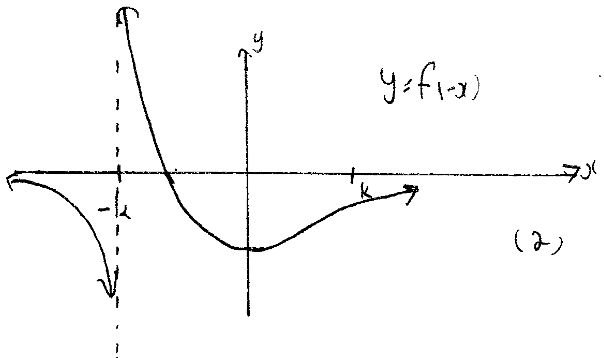
Comparing real parts

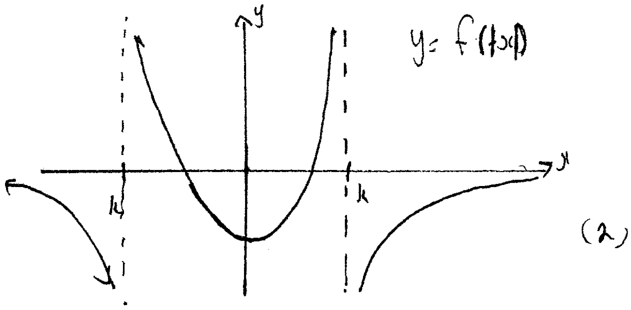
$\cos\frac{7\pi}{12} = \frac{1-\sqrt{3}}{2\sqrt{2}}$ (3)

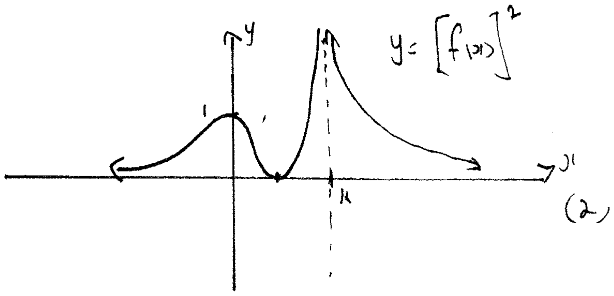
Question 2 (18 marks)

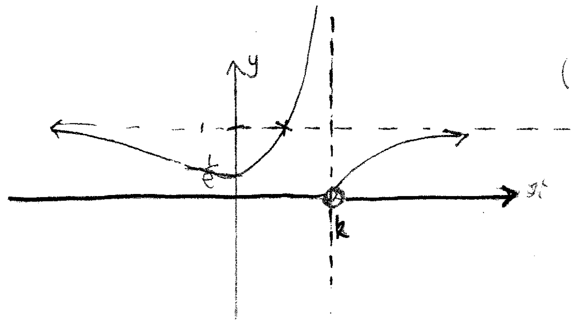
(a)




(b) $y = x^2(1-x^2)$ Even

Intercepts $x = 0, \pm 1$

$y = x^2 - x^4$

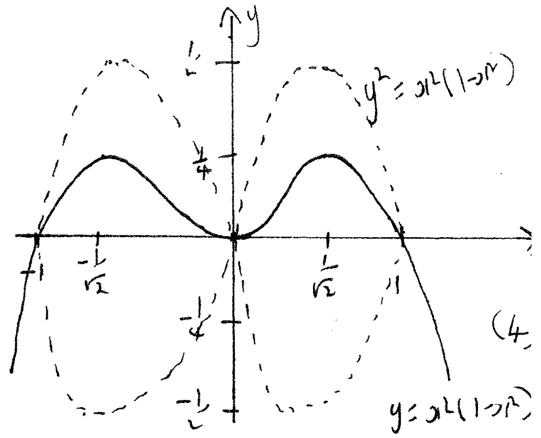
$y' = 2x - 4x^3 = 2x(1-2x^2)$

$y'' = 2 - 12x^2$

S.P's $x = 0,\ y = 0 \quad y'' > 0 \ \therefore \min$

$x = \frac{1}{\sqrt{2}},\ y = \frac{1}{4} \quad y'' < 0 \quad \max$

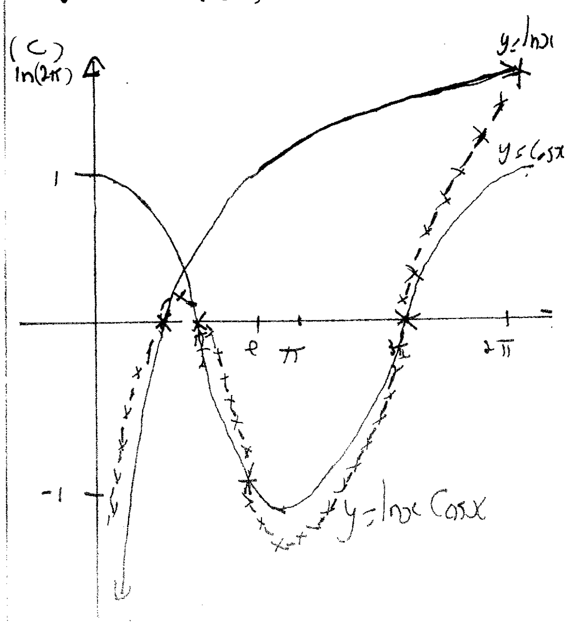
$x = -\frac{1}{\sqrt{2}},\ y = \frac{1}{4} \quad y'' < 0 \quad \max.$




$y^2 = x^2(1-x^2)$

$y = \pm\sqrt{x^2(1-x^2)}$

(c)



Question 3 (20 marks)

(a) (i) $19-k>0$ and $7-k>0$

$k<19$ and $k<7$ ie $\underline{k<7}$ (1)

(ii) $k=3 \rightarrow \frac{x^2}{16}+\frac{y^2}{4}=1$

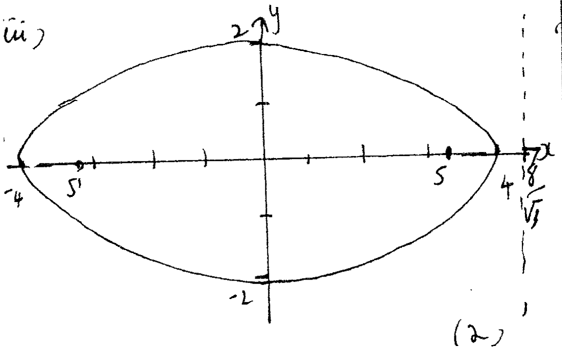
$a=4, b=2 \therefore 1-e^2=\frac{b^2}{a^2}=\frac{1}{4}$

$e=\frac{\sqrt{3}}{2}$

$\therefore$ foci $S(\pm 2\sqrt{3}, 0)$

directrices $x=\pm\frac{a}{e}=\frac{\pm 4}{\sqrt{3}/2}=\pm\frac{8}{\sqrt{3}}$ (3)

(iii)




(2)

(iv) as $k\to 7$ $a\to\sqrt{12}$, $b\to 0$

gets flatter $\to$ line interval $SS'$. (1)

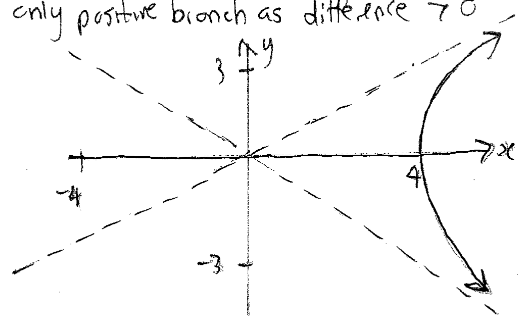
b) Hyperbola since the difference in the distances to two fixed points is a constant.

Focus $(\pm 5, 0)$ $2a=8 \therefore a=4$

$\therefore ae=5 \therefore e=\frac{5}{4} \therefore b=3$

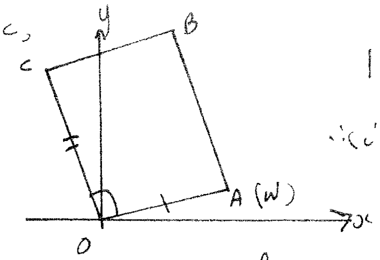
$\frac{x^2}{16}-\frac{y^2}{9}=1$ asymptotes $y=\pm\frac{3}{4}x$

only positive branch as difference $>0$




(4)

(c) (i)




$|OC|=2|OA|$

$\therefore$ (i) $C(2iw)$ (1)

(ii) D midpoint of AC ie $D\left(\frac{2iw+w}{2}\right)$

or $w(i+\frac{1}{2})$ (2)

(d) (i) $C+iS=1+(\cos\theta+i\sin\theta)+\cos 2\theta+i\sin 2\theta+\ldots$

$\ldots\cos[(n-1)\theta]+i\sin[(n-1)\theta]$

(ii) $\therefore C+iS=1+(\cos\theta+i\sin\theta)^1+(\cos\theta+i\sin\theta)^2+\ldots$ (1)

$=1+z+z^2+\ldots\ldots z^{n-1}$

- g.p. $a=1, r=z, n=n$

$\therefore C+iS=\frac{1(1-z^n)}{1-z}=\frac{1-z^n}{1-z}, \quad z\neq 1$ (3)

(iii) $C+iS=\frac{1-(\cos\theta+i\sin\theta)^n}{1-(\cos\theta+i\sin\theta)}$

$=\frac{[(1-\cos n\theta)-i\sin n\theta]}{[(1-\cos\theta)-i\sin\theta]}\times\frac{[(1-\cos\theta)+i\sin\theta]}{[(1-\cos\theta)+i\sin\theta]}$

$=\frac{(1-\cos\theta)(1-\cos n\theta)+\sin n\theta\sin\theta-i\sin\theta(1-\cos n\theta)-i\sin n\theta(1-\cos\theta)}{1-2\cos\theta+\cos^2\theta+\sin^2\theta}$

Comparing real parts

$C=\frac{(1-\cos\theta)(1-\cos n\theta)+\sin n\theta\sin\theta}{2-2\cos\theta}$

$=\frac{1-\cos\theta-\cos n\theta+\cos n\theta\cos\theta+\sin n\theta\sin\theta}{2-2\cos\theta}$

$1+\cos\theta+\cos 2\theta+\ldots+\cos(n-1)\theta$

$=\frac{\cos(n\theta-\theta)-(\cos\theta+\cos n\theta)}{2-2\cos\theta}$

$=\frac{\cos[(n-1)\theta]-(\cos\theta+\cos n\theta)}{2-2\cos\theta}$