NORTH SYDNEY GIRLS HIGH SCHOOL

EXTENSION 2 MATHEMATICS

HSC ASSESSMENT TASK

TERM 1 2007

Name: ________________________ **Class:** ________________

Instructions:

- Time allowed: 60 minutes + 2 minutes reading time.
- This task is worth 20% of your HSC assessment
- Attempt all three questions
- Show all necessary working
- Marks may be deducted for incomplete or poorly arranged work
- Start each question on a NEW PAGE
- Write on one side of the page only

Question1 (15 marks) **Marks**

(a) Given the complex number $z = 3 - 3i$ write down in exact form

(i) $|z|$ **1**

(ii) $\arg z$ **1**

(b) Express in the form $x + iy$ where x and y are real

(i) $(7 + 3i)(\overline{4 - i})$ **1**

(ii) $\left(\dfrac{2 - 5i}{4 - 3i}\right)$ **2**

(c) Evaluate $\sqrt{9 + 40i}$ **2**

(d) Find real numbers for a and b if $z = 3 + i$ satisfies $z + \dfrac{a}{z} = b$ **3**

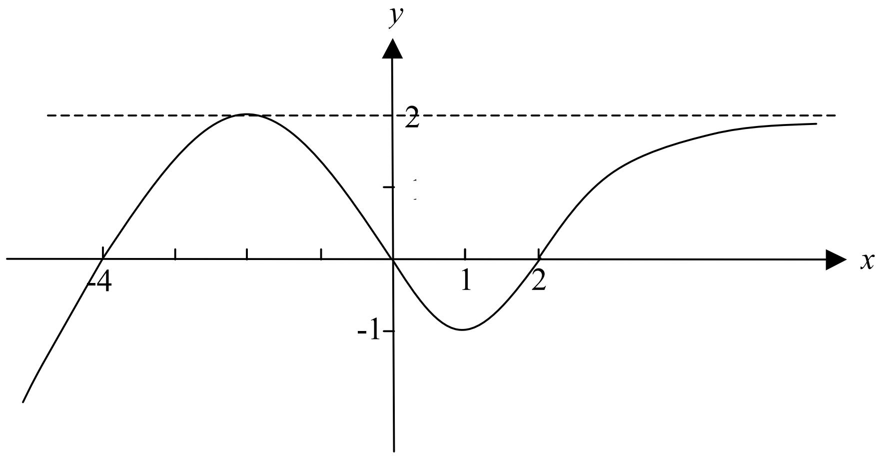
(e) Sketch the region on the Argand plane where **2**

$$|z - 3| < 3 \quad \text{and} \quad 0 < \arg z < \tfrac{\pi}{4}$$

(f) Describe in both algebraic and geometric terms the set of points z in the Argand plane such that $z^2 + (\bar{z})^2 = 0$ **3**

Question 2 (15 marks) **Marks**

(a) The graph of $y = f(x)$ is shown below



Draw neat sketches of the following *on the answer sheet provided*, showing clearly any stationary points and asymptotes

(i) $y = \dfrac{1}{f(x)}$ **2**

(ii) $y = f(\frac{x}{2})$ **2**

(iii) $y = [f(x)]^2$ **2**

(iv) $y = f(x) + |f(x)|$ **2**

(v) $y = 2^{f(x)}$ **2**

(b) Consider the function $f(x) = x(x-3)^2$

(i) Sketch the graph of the curve showing the coordinates of any turning points and the intercepts with the coordinate axes. **2**

(ii) Hence draw a neat separate sketch of $y^2 = f(x)$ **2**

(iii) Describe the nature of the curve $y^2 = f(x)$ at $x = 0$. **1**

Question 3. (15 marks) **Marks**

(a) z is a variable complex number such that $\dfrac{z-1}{z+i}$ is real.
Prove that z lies on a straight line and find its equation. **3**

(b) In the Argand diagram the points P, Q and R represent the complex numbers z_1, z_2 and $z_2 + i(z_2 - z_1)$ respectively.

(i) Explain why ΔPQR is a right-angled triangle. **2**

(ii) Find in terms of z_1 and z_2 the complex number represented by S if $PQRS$ is a rectangle. **2**

(c)

(i) Use De Moivre's theorem to find the solution ω to the equation $z^5 = i$ which has the smallest positive argument. **2**

(ii) Using an Argand diagram or otherwise write down the other four roots of $z^5 = i$ in mod/arg form. **2**

(iii) Deduce that $\omega^{-3} + \omega^{-7} + \omega + \omega^5 + \omega^9 = 0$ **1**

(d) Use De Moivre's theorem to show that **3**

$$(2 + 2i\tan\theta)^n + (2 - 2i\tan\theta)^n = \frac{2^{n+1}\cos n\theta}{\cos^n\theta}$$

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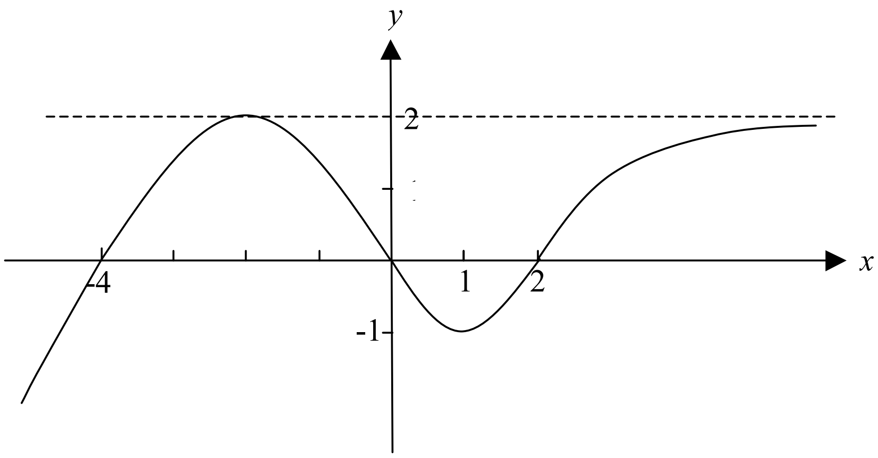
END OF TEST

Name: Class:

(i)



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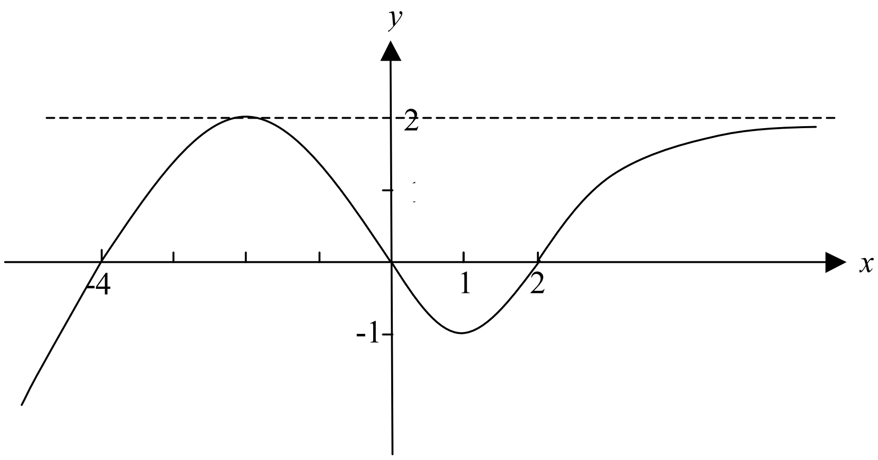

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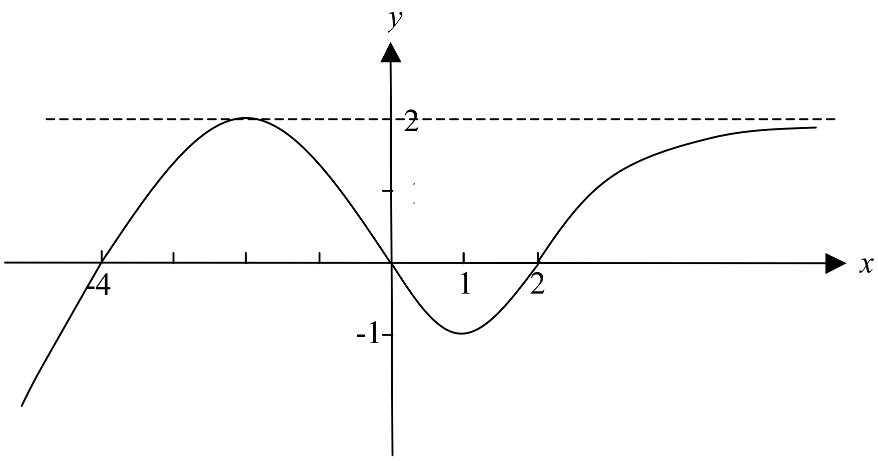

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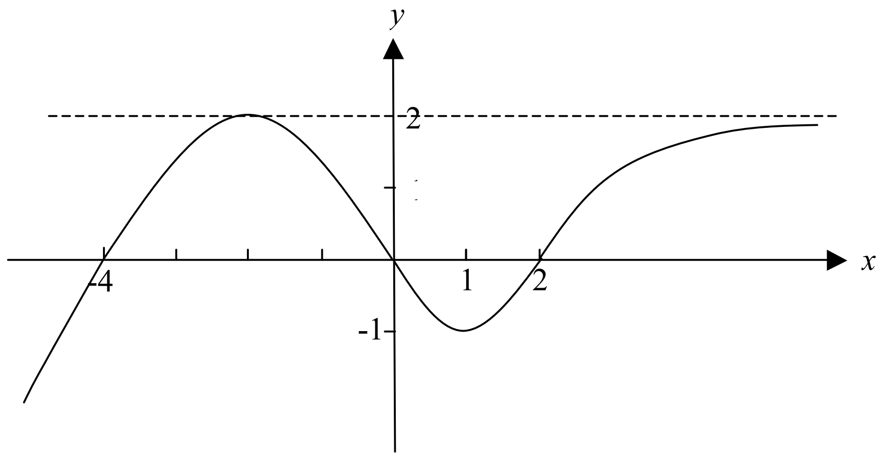
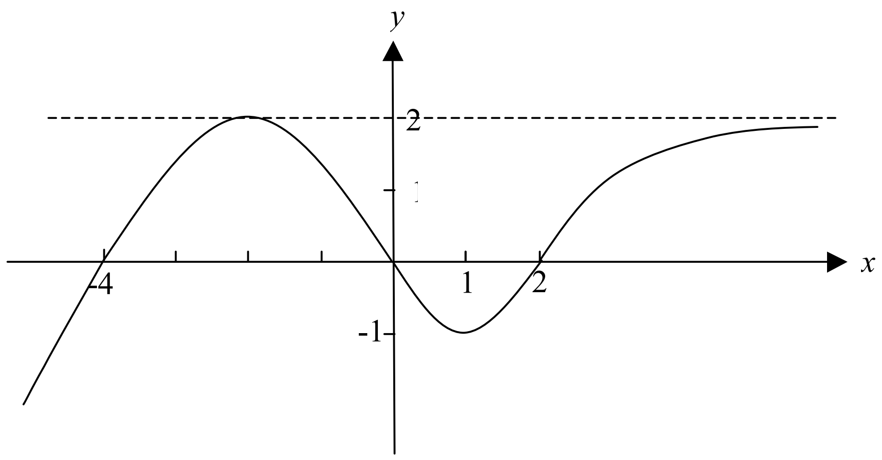


(v)



Question 1

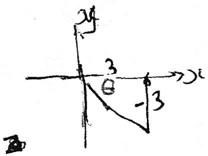
(a) $z = 3-3i$

(i) $|z| = \sqrt{3^2+(-3)^2}$

$= \sqrt{18}$

$= 3\sqrt{2}$

(ii)



$\arg z = -\frac{\pi}{4}$

(b)(i) $(\overline{z+3i})(\overline{4-i})$

$= (7+3i)(4+i)$

$= 28+7i+12i-3$

$= 25+19i$

(ii) $\left(\frac{2-5i}{4-3i}\right)\left(\frac{4+3i}{4+3i}\right)$

$= \frac{8+6i-20i+15}{16+9}$

$= \frac{23-14i}{25}$

$= \frac{23}{25} - \frac{14}{25}i$

(c) let $z = x+iy = \sqrt{9+40i}$

$(x+iy)^2 = 9+40i$

$x^2-y^2 = 9 \quad 2xy = 40$

$x^2-y^2 = 9 \quad xy = 20$

$x = 5, y = 4$

ie $5+4i$

(d) $z + \frac{a}{z} = b$

$(3+i) + \frac{a}{3+i} = b$

$(3+i)^2 + a = b(3+i)$

$9+6i-1+a = 3b+ib$

$(8+a)+6i = 3b+ib$

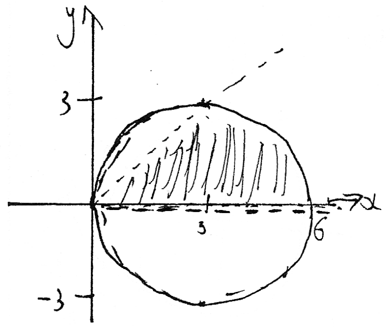
$\therefore b = 6$

$8+a = 3b$

$8+a = 18$

$a = 10$

(e) $|z-3| < 3$ and $0 < \arg z < \frac{\pi}{4}$



(f) $z^2 + (\bar{z})^2 = 0$

let $z = x+iy$

$(x+iy)^2 + (x-iy)^2 = 0$

$x^2+2xyi-y^2+x^2-2xyi-y^2 = 0$

$2x^2-2y^2 = 0$

$x^2-y^2 = 0$

$(x-y)(x+y) = 0$

Pair of lines $y = \pm x$

See sheet Q2(a)

Question 2(b)

(i) $y = x(x-3)^2$

$y' = (x-3)^2 + x \cdot 2(x-3)$

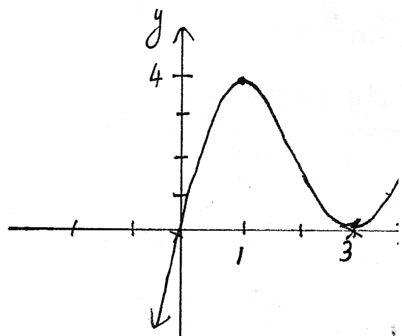
$= (x-3)[x-3+2x]$

$= (x-3)(3x-3)$

$= 3(x-1)(x-3)$

$x=1, y=4$

$x=3, y=0$



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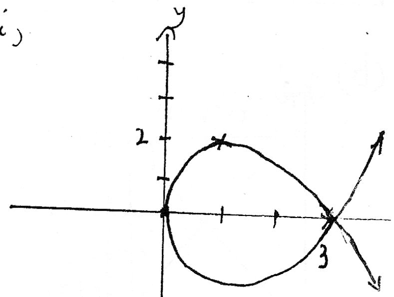


(iii) Vertical tangent.

Question 3

(a) $\text{Im}\left(\dfrac{z-1}{z+i}\right)=0 \quad [z \neq -i]$

let $z = x+iy$

$$\left[\frac{(x-1)+iy}{x+i(y+1)}\right]\times\left[\frac{x-i(y+1)}{x-i(y+1)}\right]$$

$$=\frac{x(x-1)+ixy-i(x-1)(y+1)+y(y+1)}{x^2+(y+1)^2}$$

$\text{Im} = 0$

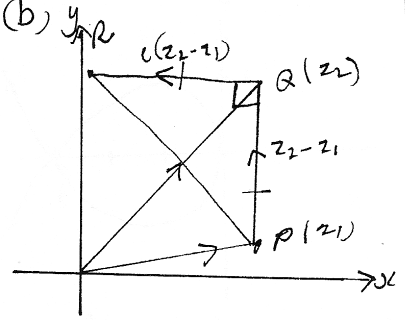
$$\frac{xy-(x-1)(y+1)}{x^2+(y+1)^2}=0$$

$xy-(xy+x-y-1)=0$

$-x+y+1=0$

$\boxed{x-y-1=0}$ ie line.

(b)



$i(z_2-z_1)$ is vector $\overrightarrow{z_2 z_1}$ rotated $90°$ anticlockwise

$z_2+i(z_2-z_1)$ is (z_2-z_1) $90°$ anticlockwise added to vector Oz_2

$\therefore \triangle OQR$ right angled

(c) (i) $z^5=i$

let $z=\cos\theta+i\sin\theta$

$\therefore (\cos\theta+i\sin\theta)^5=i$

$\cos5\theta+i\sin5\theta=0+i$

$\left.\begin{array}{l}\cos5\theta=0\\ \sin5\theta=1\end{array}\right\}$

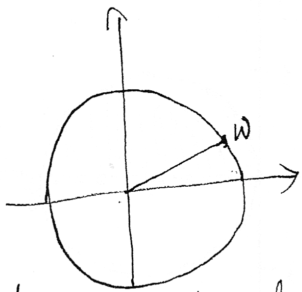
$\therefore 5\theta=\frac{\pi}{2}\ \ldots$

$\theta=\frac{\pi}{10}$

$\therefore \omega=\cos\frac{\pi}{10}+i\sin\frac{\pi}{10}$

Smallest argument.

(ii)



roots evenly spaced around unit circle

ie arguments $\frac{\pi}{10}+\frac{2\pi}{5}=\frac{5\pi}{10}=\frac{\pi}{2}$

$\frac{5\pi}{10}+\frac{4\pi}{10}=\frac{9\pi}{10}$

$\frac{9\pi}{10}+\frac{4\pi}{10}=\frac{13\pi}{10}$

$\frac{13\pi}{10}+\frac{4\pi}{10}=\frac{17\pi}{10}$

ie $z_1=\text{cis}\frac{\pi}{10}=\omega$

$z_2=\text{cis}\frac{5\pi}{10}=\omega^5$

$z_3=\text{cis}\frac{9\pi}{10}=\omega^9$

$z_4=\text{cis}\frac{13\pi}{10}=\text{cis}\left(\frac{-7\pi}{10}\right)=\omega^{-7}$

$z_5=\text{cis}\frac{17\pi}{10}=\text{cis}\left(-\frac{3\pi}{10}\right)=\omega^{-3}$

(iii) Sum of roots $=\frac{-b}{a}=0$

$\therefore \omega+\omega^5+\omega^9+\omega^{-7}+\omega^{-3}=0$

$\omega^{-7}+\omega^{-3}+\omega+\omega^5+\omega^9=0$

(d)

$(2+2i\tan\theta)^n+(2-2i\tan\theta)^n$

$$=\left(2+2i\frac{\sin\theta}{\cos\theta}\right)^n+\left(2-2i\frac{\sin\theta}{\cos\theta}\right)^n$$

$$=2^n\left[\frac{\cos\theta+i\sin\theta}{\cos\theta}\right]^n+2^n\left[\frac{\cos\theta-i\sin\theta}{\cos\theta}\right]^n$$

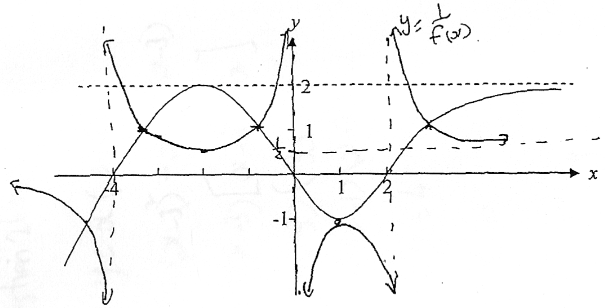
$$=2^n\left[\frac{\cos n\theta+i\sin n\theta+\cos n\theta-i\sin n\theta}{\cos^n\theta}\right]$$

$$=\frac{2^n\cdot 2\cos n\theta}{\cos^n\theta}$$

$$=\frac{2^{n+1}\cos n\theta}{\cos^n\theta}$$

Name: Q2(a) Solutions. Class:

(i)



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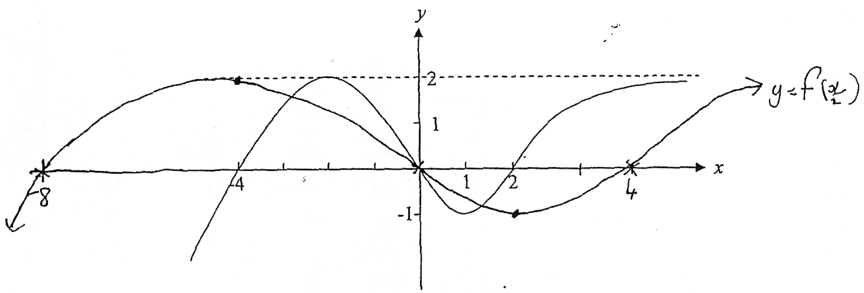
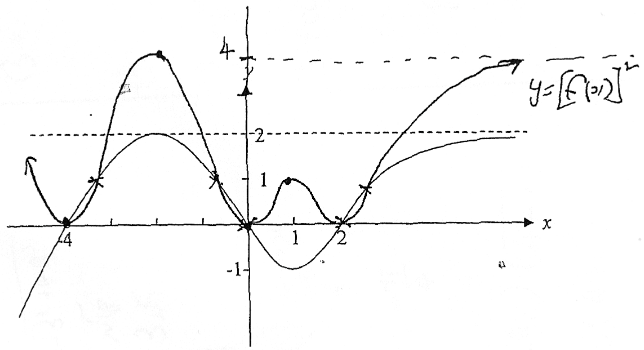
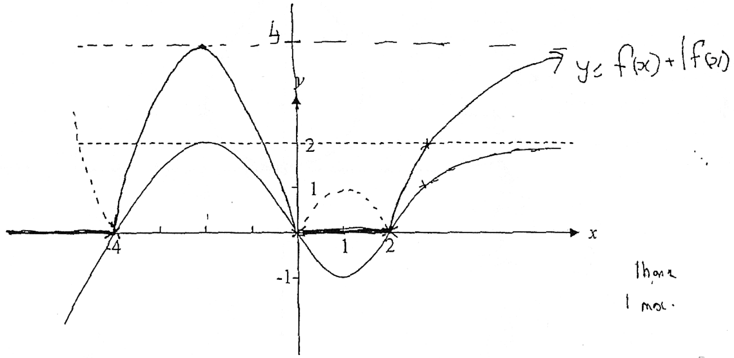


(iii)



(iv)



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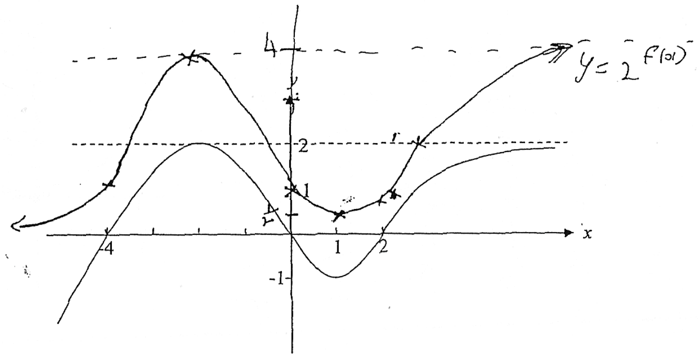
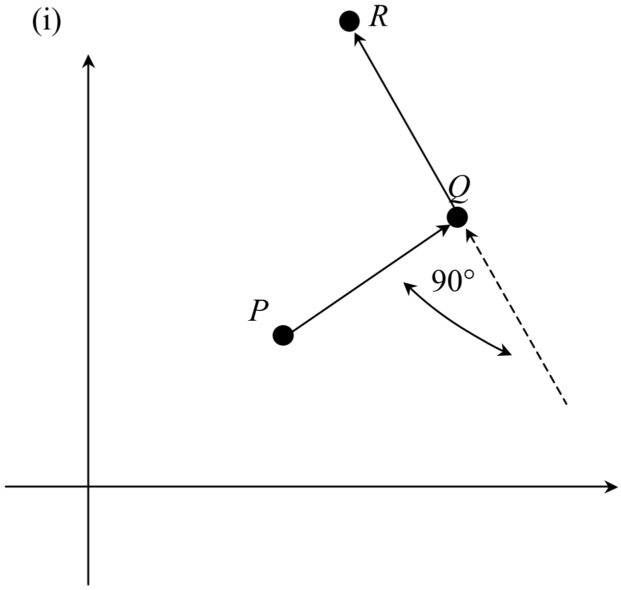


Q3 (b) (i)



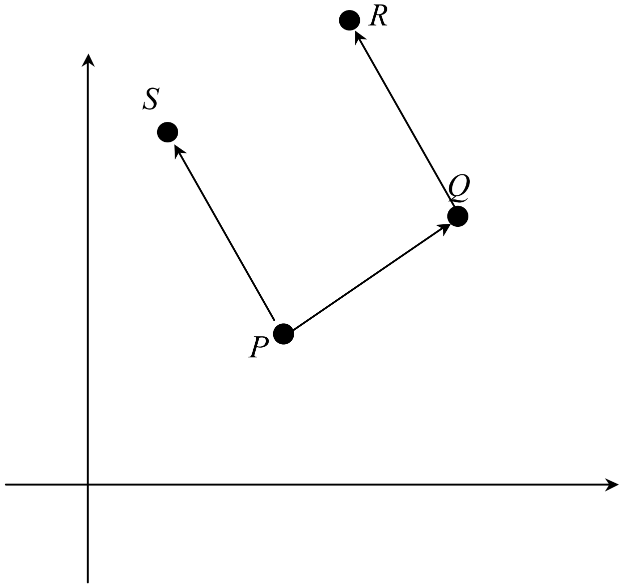
$\overrightarrow{PQ} = z_2 - z_1$

$\overrightarrow{RQ} = z_2 + i(z_2 - z_1) - z_2 = i(z_2 - z_1)$

$\therefore \overrightarrow{RQ}$ is an anti-clockwise rotation of $\overrightarrow{PQ}$ by $90°$

So there is a right angle at Q making ΔPQR right angled and isosceles

(ii)



Let S be the point that represents the number z_3, so that $PQRS$ is a rectangle (square)

$\therefore \overrightarrow{SP} = \overrightarrow{RQ}$ (opposite sides are equal and parallel)

$\therefore z_3 - z_1 = i(z_2 - z_1)$

$\therefore z_3 = z_1 + i(z_2 - z_1)$