NORTH SYDNEY GIRLS HIGH SCHOOL

YEAR 12 – TERM 1 ASSESSMENT

MATHEMATICS EXTENSION 2

2008

TIME ALLOWED: 60 minutes
Plus 2 minutes reading time

NAME ______________________________ CLASS _____________________

Q 1 / 16	Q 2 / 14	Q 3 / 11	Total / 41

INSTRUCTIONS:

- Start each question on a NEW PAGE
- Hand in each question separately, including a sheet for non-attempts
- Show all necessary working
- Write on one side of the page only
- Marks may be deducted for incomplete or poorly arranged work

Question 1 (16 marks)

a) Let $z = 2 + i$ and $w = 1 - 2i$. Find, in the form $x + iy$,

(i) $z + w$ **1**

(ii) $\frac{z}{w}$ **2**

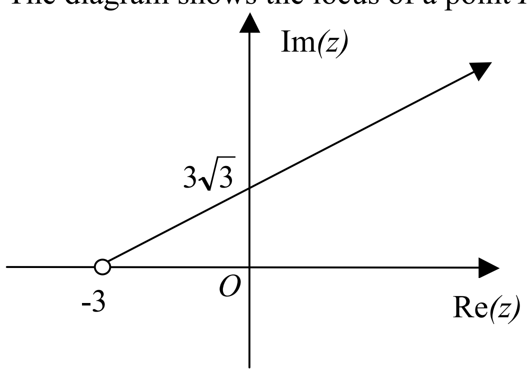
(iii) $\text{Im}(z + \overline{w})$ **1**

b) (i) For the complex number $z = \sqrt{3} + i$ find $|z|$ and $\arg z$. **2**
Hence express z in the form $r(\cos\theta + i\sin\theta)$.

(ii) Find $\overline{z}$ and z^2 giving your answers in the form $a + ib$. **2**

(iii) Use De Moivre's Theorem to find a positive integer *m* such that **3**
$(\sqrt{3} + i)^n - (\sqrt{3} - i)^n = 0$.

c) The diagram shows the locus of a point *P* representing the complex number *z*.



(i) Find the equation of the locus of *P* in terms of *z*. **1**

(ii) Find the modulus and argument of *z* when $|z|$ takes its least value. **2**

(iii) Hence find *z* in the form $a + ib$ for which $|z|$ is least. **2**

Question 2 (14 marks)

a) (i) On the same diagram, draw a neat sketch of the locus specified by each of the following: **2**

(α) $|z+1-3i|=2$

2

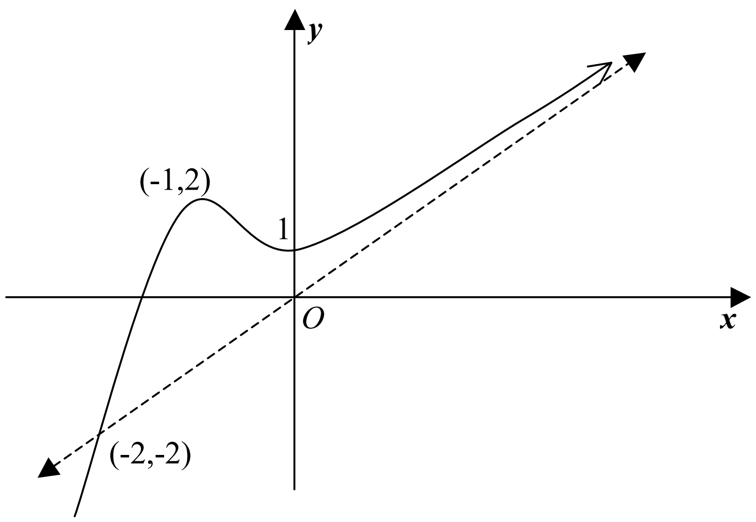
(β) $|z-5i|=|z+i|$

(ii) Hence shade the region of points z which satisfies simultaneously **1**

$$|z+1-3i|\le 2 \quad \text{and} \quad |z-5i|\ge|z+i|$$

b) On the same number plane, sketch the graph of $y=\dfrac{1}{x-3}$ and hence the graph of $y=\dfrac{1}{\sqrt{x-3}}$, clearly indicating any asymptotes and intercepts. **3**

c) The diagram shows the graph of $y=f(x)$. The line $y=x$ is an asymptote.



Draw neat sketches of the following *on the answer sheet provided,* showing clearly any asymptotes and important features.

2

(i) $f(|x|)$

2

(ii) $\dfrac{1}{f(x)}$

2

(iii) $x\,f(x)$

Question 3 (11 marks)

a) If the domain of the function *f* given by $f(x)=\dfrac{1}{1-x^2}$ is restricted to $\{x: |x|>1\}$, which one of the following describes the range of *f*? Briefly justify your answer. **2**

(A) $-\infty < f(x) < -1$

(B) $-\infty < f(x) < 0$

(C) $-\infty < f(x) < 1$

(D) $-1 < f(x) < \infty$

(E) $0 < f(x) < \infty$

b) The graph of $y^2 = x^2 + 9$

I has the *x* axis as an axis of symmetry
II has the *y* axis as an axis of symmetry
III has point symmetry about the origin

Which of the above statements is true? Justify your answer with a sketch. **2**

(A) I only (B) II only (C) III only (D) I and II only (E) I, II and III

c) Sketch the roots of $x^6 - 1 = 0$ on the Argand diagram. **2**

d) The points *P* and *Q* represent the complex numbers $4-2i$ and $6+4i$ respectively.

(i) Find the complex number represented by vector $\overrightarrow{PQ}$ **1**

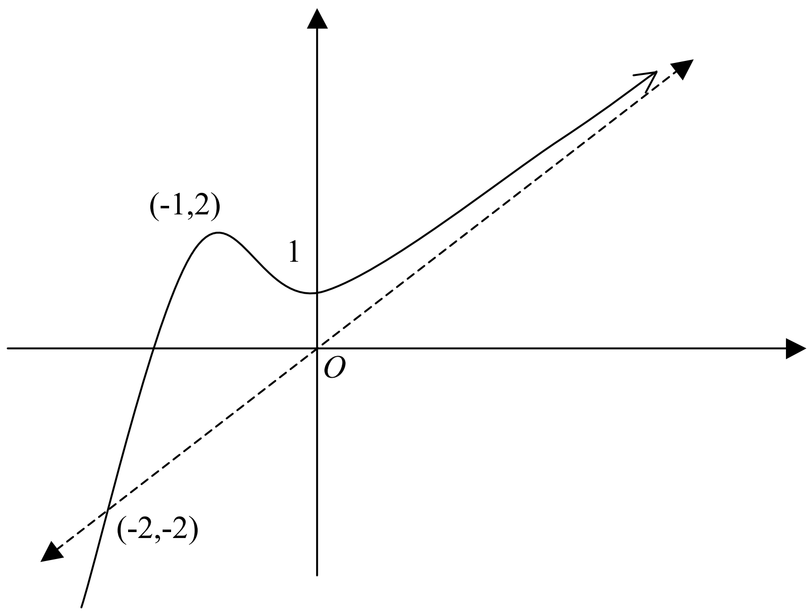
(ii) The points *P, Q* and *S* form an equilateral triangle. Find a possible value of the complex number represented by *S*. **3**

(iii) The points *P, O, Q* and *R* (named in cyclic order, *O* being the origin) form a parallelogram. Find the complex number represented by *R*. **1**

End of paper

Answer Sheet for Question 2 c) **Name: ________________Class: _____**

(i)



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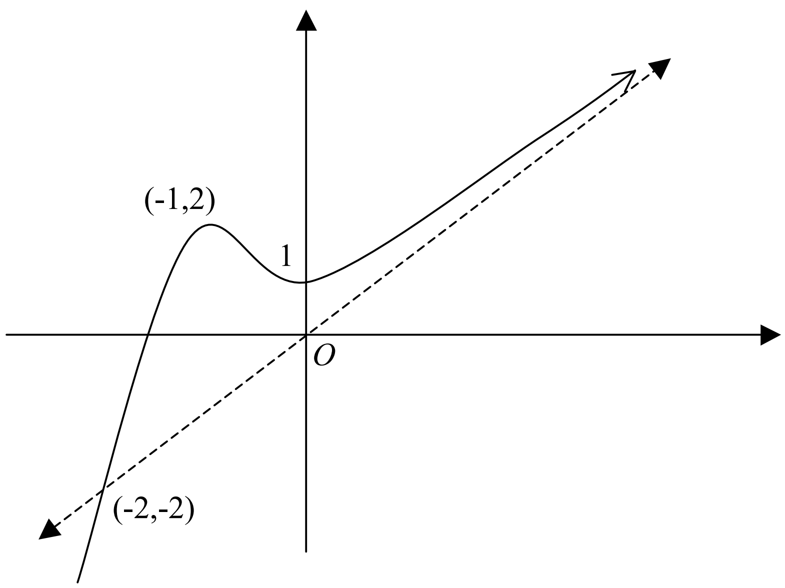
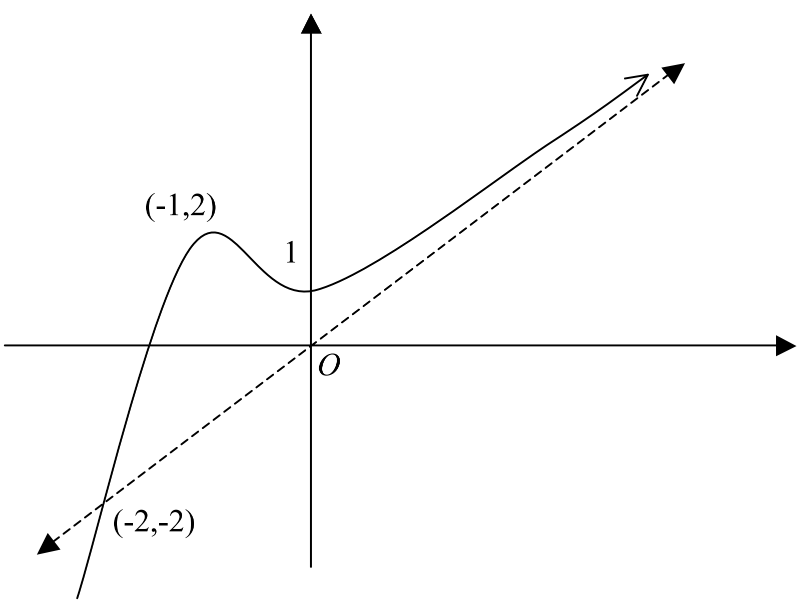


(iii)



Solutions

Question 1

(a) (i) $z+w=2+i+1-2i=3-i$

(ii) $\frac{z}{w}=\frac{2+i}{1-2i}\times\frac{1+2i}{1+2i}=\frac{2+i+4i-2}{5}=i$

(iii) $\mathrm{Im}(z+\overline{w})=\mathrm{Im}(2+i+1+2i)=3$

(b) (i) $z=\sqrt{3}+i$

$|z|=2, \arg z=\frac{\pi}{6}$

$\therefore z=2\left(\cos\frac{\pi}{6}+i\sin\frac{\pi}{6}\right)$

(ii) $\overline{z}=\sqrt{3}-i$

$z^2=\left[2\left(\cos\frac{\pi}{6}+i\sin\frac{\pi}{6}\right)\right]^2$

$=4\left(\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}\right)$ [By de Moivre's Thm]

$=4\left(\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)$

$=2+2i\sqrt{3}$

(iii) $\left(\sqrt{3}+i\right)^m-\left(\sqrt{3}-i\right)^m=0$

$\therefore\left[2\left(\cos\frac{\pi}{6}+i\sin\frac{\pi}{6}\right)\right]^m-\left[2\left(\cos\left(-\frac{\pi}{6}\right)+i\sin\left(-\frac{\pi}{6}\right)\right)\right]^m=0$

$\therefore 2^m\left[\cos\frac{m\pi}{6}+i\sin\frac{m\pi}{6}-\cos\left(-\frac{m\pi}{6}\right)-i\sin\left(-\frac{m\pi}{6}\right)\right]=0$

$\therefore\cos\frac{m\pi}{6}--\cos\left(\frac{m\pi}{6}\right)+2i\sin\frac{m\pi}{6}=0$

$\therefore\sin\frac{m\pi}{6}=0$

$\therefore\frac{m\pi}{6}=n\pi+(-1)^n\times\sin^{-1}(0)$

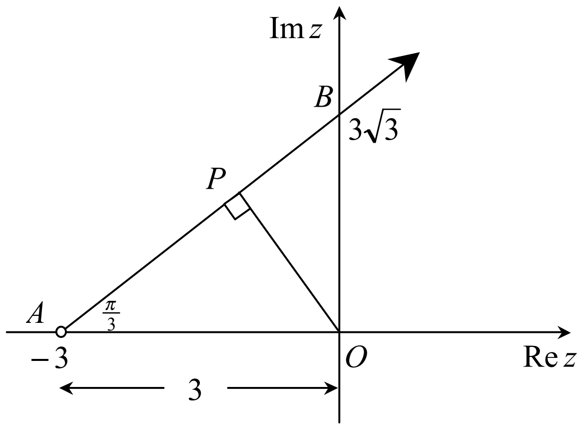
$\therefore m=6n$

ie m is a multiple of 6

(c) (i) $\arg(z+3)=\theta$ where $\tan\theta=\sqrt{3}$

$\therefore\arg(z+3)=\frac{\pi}{3}$

(ii)



Let P represent z where $OP = \min|z|$ and so $OP \perp AB$

$$\therefore \sin\frac{\pi}{3} = \frac{OP}{3}$$

$$\therefore OP = 3\sin\frac{\pi}{3} = \frac{3\sqrt{3}}{2}$$

$$\therefore \min|z| = \frac{3\sqrt{3}}{2}$$

$$\angle POX = \frac{5\pi}{6} \Rightarrow \arg z = \frac{5\pi}{6}$$

(iii)
$$z_{\min|z|} = \frac{3\sqrt{3}}{2}\left(\cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6}\right)$$

$$= \frac{3\sqrt{3}}{2}\left(-\frac{\sqrt{3}}{2} + \frac{i}{2}\right)$$

$$= -\frac{9}{4} + \frac{3i\sqrt{3}}{4}$$

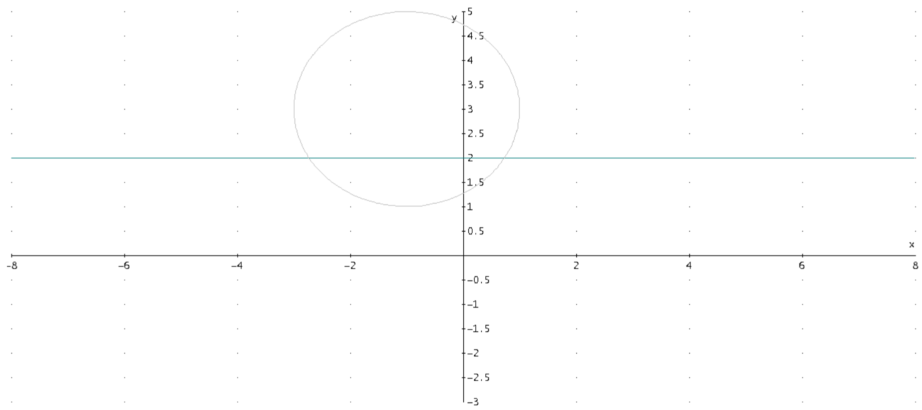
Question 2

(a) (i) $|z+1-3i| = 2 \Rightarrow |z-(-1+3i)| = 2$

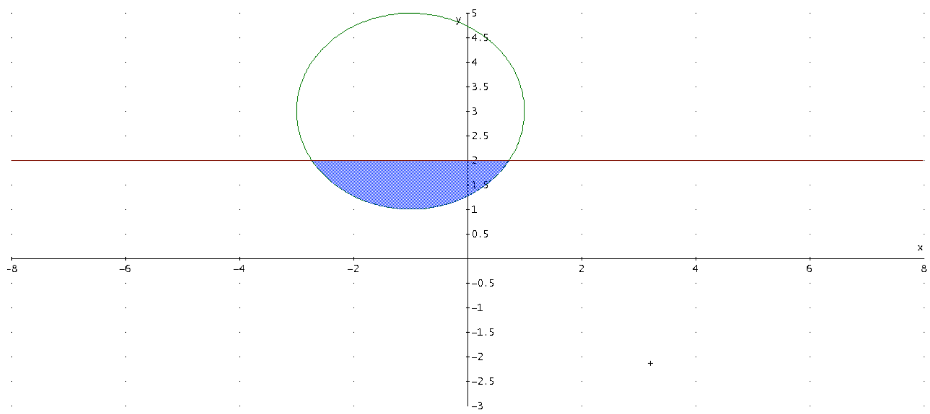
This is a circle of radius 2 centred at $z = -1+3i$

$|z-5i| = |z+i|$

This is the perpendicular bisector of $z = 5i$ and $z = -i$ ie $z = 2$

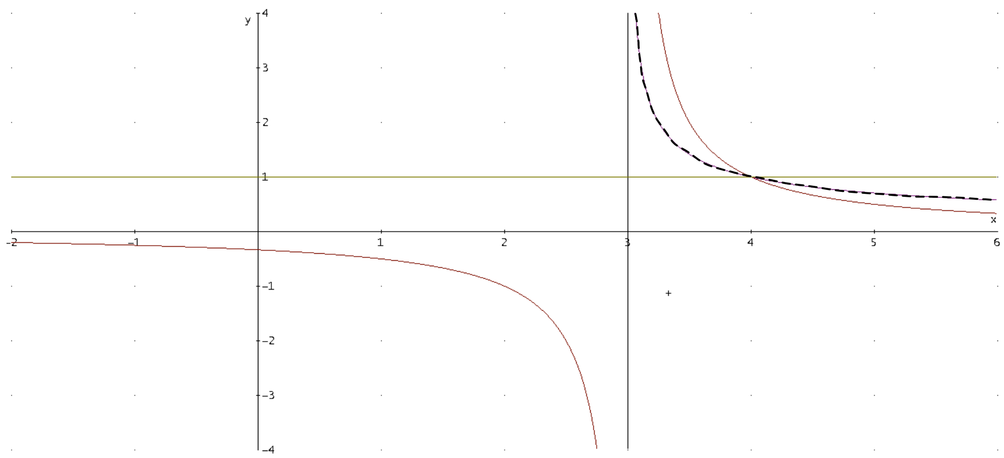


(ii)



(b) The curve drawn with dashes is $y = \dfrac{1}{\sqrt{x-3}}$

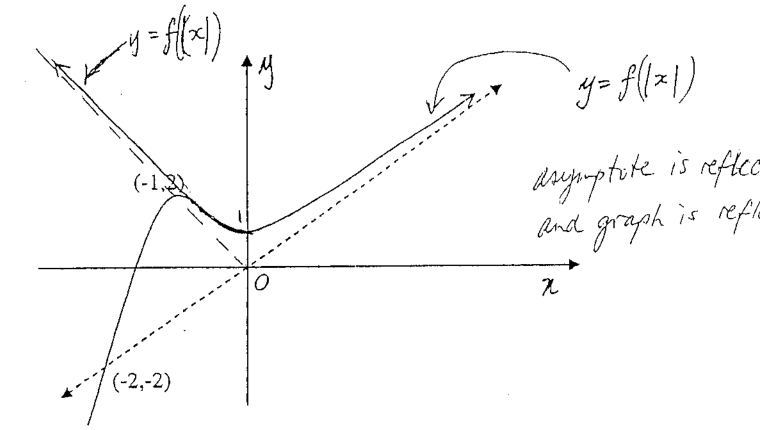
The x – axis is a horizontal asymptote



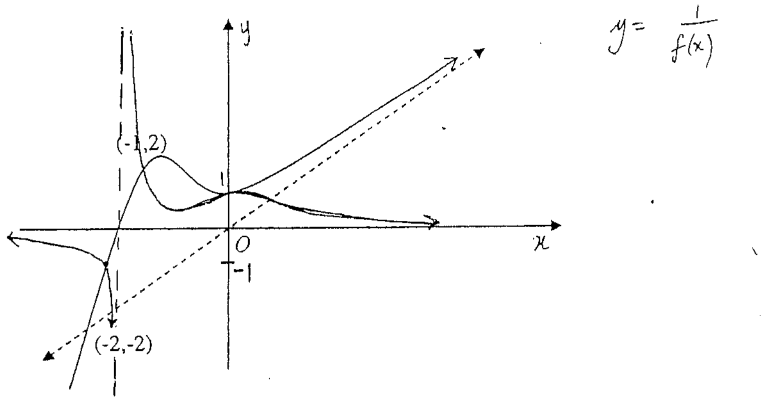
(c)

Answer Sheet for Question 2 c) Name: ______________

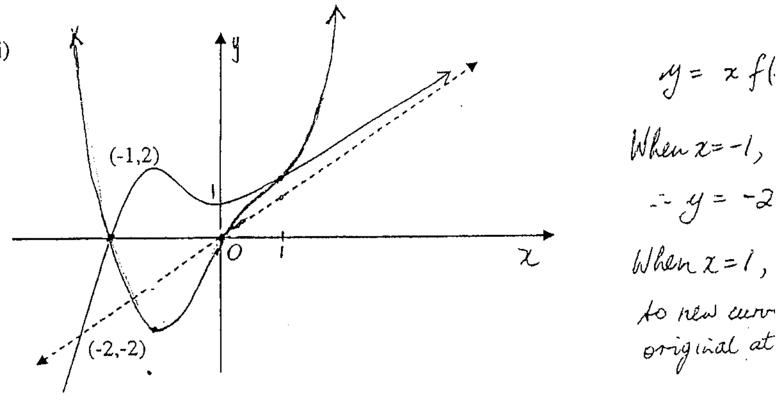
(i)



(ii)



(iii)

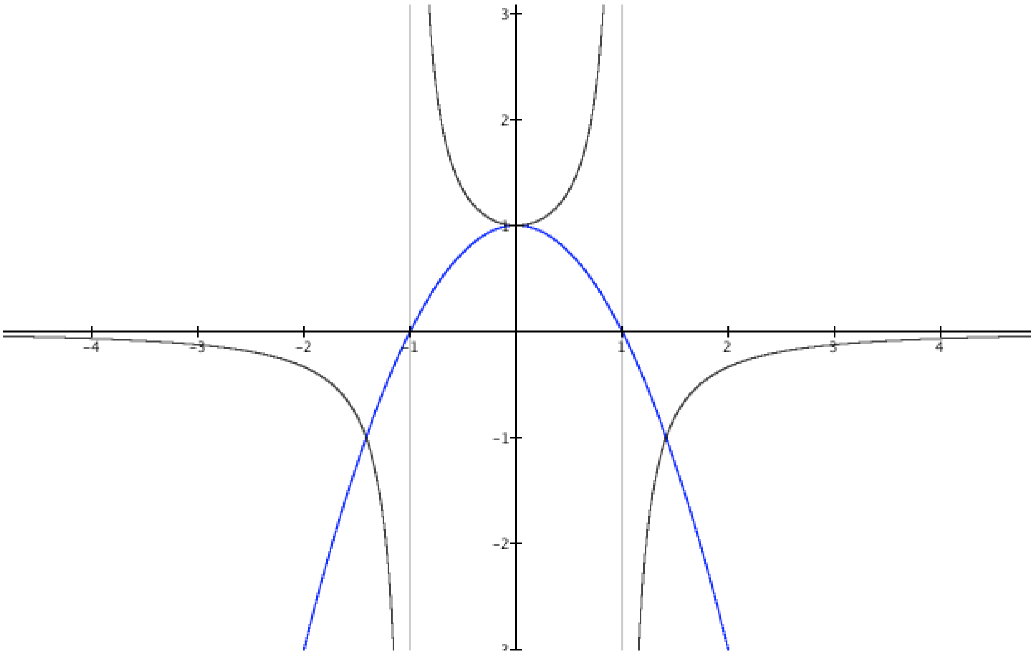


Question 3

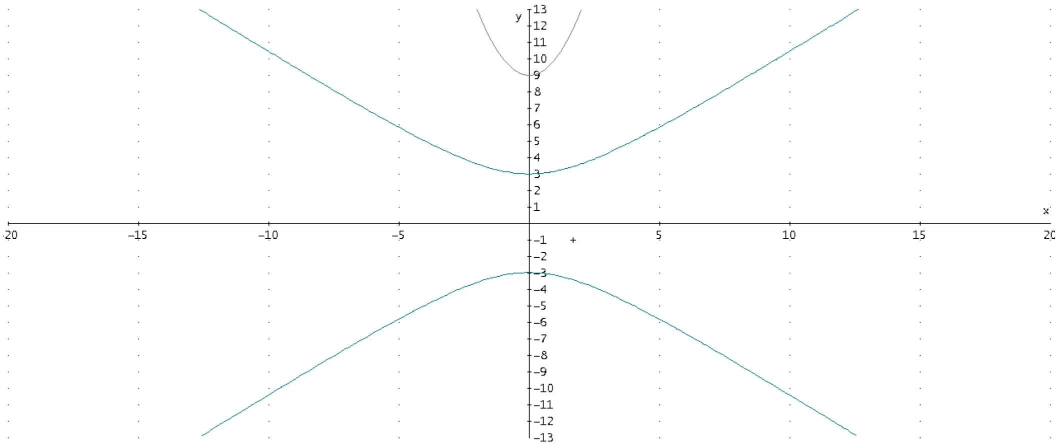
(a) $f(x)=\dfrac{1}{1-x^2}$

For $|x|>1$, then $-\infty < f(x) < 0$

$\therefore$ **B**



(b)



From the graph $y^2 = x^2 + 9$ is symmetric in the y and x axes, and is also point symmetric about the origin. So I, II and III ie **E**

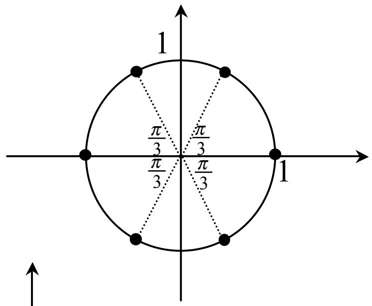
(c) $x^6 - 1 = 0$

$\therefore x^6 = 1 = 1\text{cis}(0 + 2k\pi) = \text{cis}(2k\pi),\ k \in \mathbb{Z}$

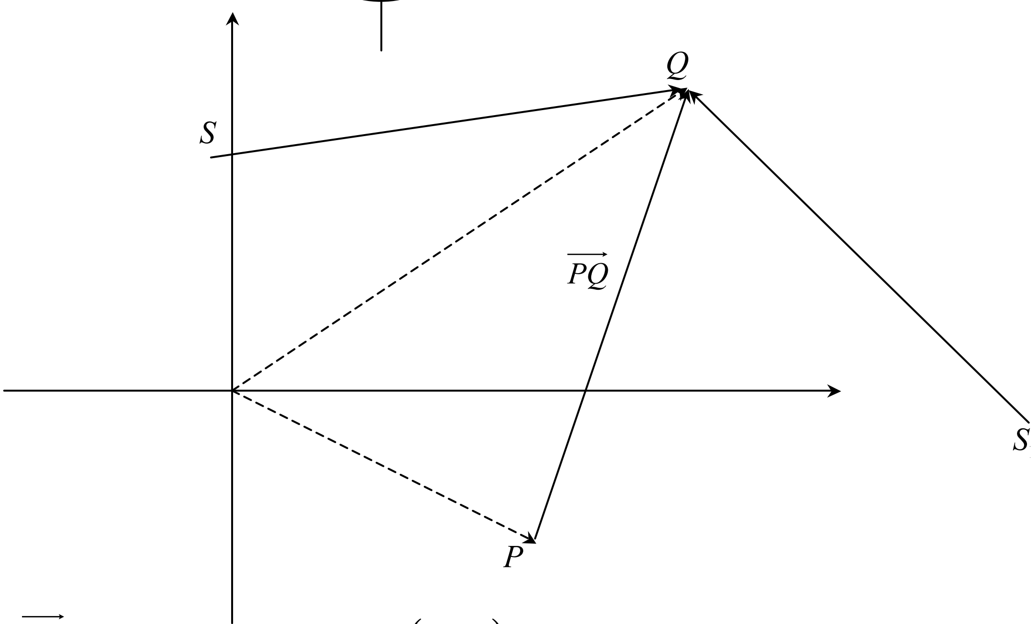
$\therefore x = \text{cis}\left(\frac{2k\pi}{6}\right) = \text{cis}\left(\frac{k\pi}{3}\right),\quad 0 \le k \le 5$

$\therefore x = 1, \text{cis}\left(\frac{\pi}{3}\right), \text{cis}\left(\frac{2\pi}{3}\right), \text{cis}(\pi), \text{cis}\left(\frac{4\pi}{3}\right) = \text{cis}\left(-\frac{\pi}{3}\right), \text{cis}\left(\frac{5\pi}{3}\right) = \text{cis}\left(-\frac{2\pi}{3}\right)$

So the roots all lie on the unit circle and are equally spaced around it, by $60°$.
The non-real roots occur in conjugate pairs



(d)



(i) $\overrightarrow{PQ}$ is represented by $6 + 4i - (4 - 2i) = 2 + 6i$

(ii) Two points can be found S_1 or S_2, by rotating $\overrightarrow{PQ}$ by $60°$ clockwise or anti-clockwise.
This gets the vectors $\overrightarrow{S_1Q}$ and $\overrightarrow{S_2Q}$.

$\overrightarrow{S_1Q}$: $(2+6i)\text{cis}60° = (2+6i)\left(\frac{1}{2} + \frac{i\sqrt{3}}{2}\right)$

$= (1+3i)\left(1 + i\sqrt{3}\right) = 1 - 3\sqrt{3} + i\left(3 + \sqrt{3}\right)$

$\overrightarrow{S_2Q}$: $(2+6i)\text{cis}(-60°) = (2+6i)\left(\frac{1}{2} - \frac{i\sqrt{3}}{2}\right)$

$= (1+3i)\left(1 - i\sqrt{3}\right) = 1 + 3\sqrt{3} + i\left(3 - \sqrt{3}\right)$

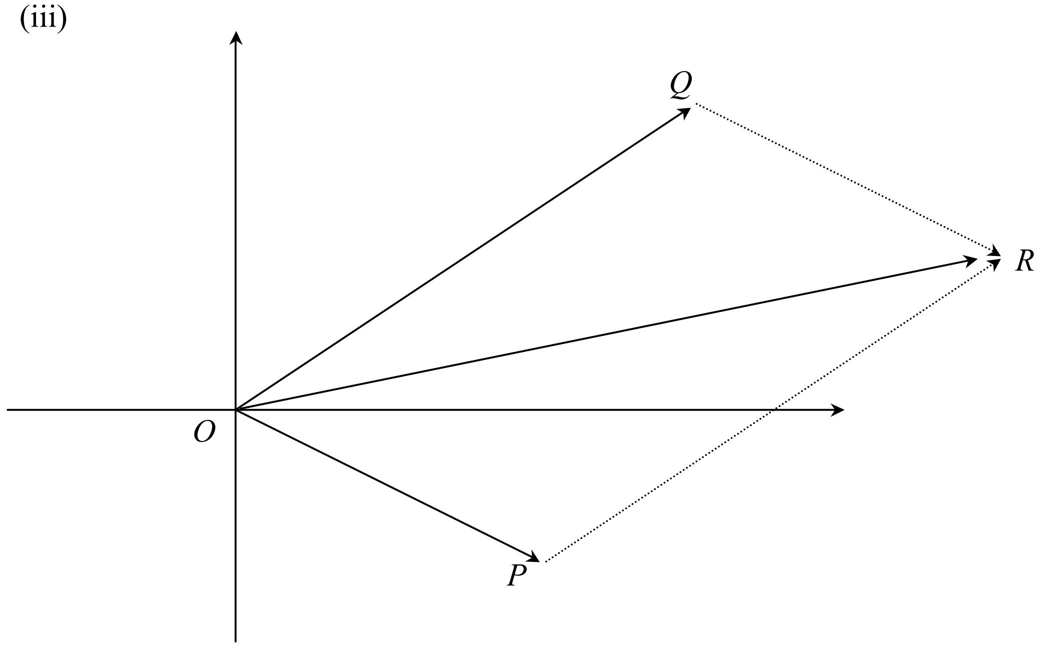
$\overrightarrow{OS_1} = \overrightarrow{OQ} - \overrightarrow{S_1Q}$

So S_1 is represented by $6+4i-\left[1-3\sqrt{3}+i\left(3+\sqrt{3}\right)\right]=5+3\sqrt{3}+i\left(1-\sqrt{3}\right)$

$\overrightarrow{OS_2} = \overrightarrow{OQ} - \overrightarrow{S_2Q}$

So S_2 is represented by $=6+4i-\left[1+3\sqrt{3}+i\left(3-\sqrt{3}\right)\right]=5-3\sqrt{3}+i\left(1+\sqrt{3}\right)$

(iii)



$\overrightarrow{OR} = \overrightarrow{OQ} + \overrightarrow{QR} = \overrightarrow{OQ} + \overrightarrow{OP}$

So R corresponds to $6+4i+4-2i=10+2i$